Logique & Analyse **224** (2013), 425–438

BIMODAL FRAGMENTS OF CONTINGENCY LOGICS

CLAUDIO PIZZI

Abstract

The paper aims at identifying the modal fragments of systems of contingency logic whose language includes a propositional constant τ. It turns out that the language of such systems allows defining two necessity operators of different strength, $\Box$ and O. It is proved that in the weakest contingency system K$\Delta\tau$w the τ-free fragment containing $\Box$-wffs is K and that an analogous result holds for O-wffs; that in K$\Delta\tau$ the fragment for both is KD; that in KT$\Delta\tau$ the fragment containing $\Box$-wffs is KT and the one containing O-wffs is KD. Such results are derived from the central result of the paper, which concerns the bimodal fragments of K$\Delta\tau$w, K$\Delta\tau$, KT$\Delta\tau$, here called K$\Box$O, KD$\Box$O and KT$\Box$O. The axiomatic bases for such systems consist in the unions of the axioms of the related monomodal fragments extended with the bridge axiom $\Box p \supset \mathrm{O}p$.

§1. The present paper aims at providing a detailed treatment of a research program started in Pizzi [2007] and partially developed in Pizzi [2013]. In Pizzi [2007] a system called K$\Delta\tau$w was viewed as a minimal system for the operator of non-contingency or absoluteness (symbolized by Δ) if the Δ-based language is extended with a suitably axiomatized propositional constant τ and with a contingency operator ∇ introduced by the standard definition $\nabla \mathrm{A} =_{\mathrm{Df}} \neg\Delta \mathrm{A}$. The following is an axiomatization of K$\Delta\tau$w where the constant τ is axiomatized by only one axiom (KΔ4).[1]

KΔ0. All the tautologies of the propositional calculus PC

KΔ1. $\Delta p \equiv \Delta\neg p$

KΔ2. $(\Delta p \wedge \Delta q) \supset \Delta(p \wedge q)$

KΔ3. $(\Delta p \wedge \nabla(\neg p \vee r)) \supset \Delta(p \vee q)$

KΔ4. $\Delta\tau \supset \Delta p$

[1] The problem of assigning an intuitive meaning to τ may be the object of some non-trivial philosophical discussion, but lies outside the scope of this paper. As a suggestion for the reader, τ may be interpreted in K$\Delta\tau$w as "what I believe to be true is true" and in K$\Delta\tau$ as "the moral code is respected".

Rules:
(US) Uniform Substitution
(MP) $\vdash$ A, $\vdash$ A $\supset$ B $\to \vdash$ B
(ΔNec) $\vdash$ A $\to \vdash \Delta$A
(ΔEq) $\vdash$ A $\equiv$ B $\to \vdash \Delta$A $\equiv \Delta$B

The system which is obtained from K$\Delta\tau$w by omitting the axiom KΔ4 is called KΔ in Kuhn [1995] and is the minimal system of contingential logic written in a language without propositional constants.

In Pizzi [2007] it is proved that K$\Delta\tau$w is sound and complete with respect to the class of K$\Delta\tau$w-models, where a K$\Delta\tau$w-model is any 4-ple M = <W, W^τ, R, V> which is defined as follows:
(i) $W \neq \varnothing$
(ii) $W^\tau \subseteq W$
(iii) R is *weakly τ-forked*, which means
($^\circ$) $\forall x(\neg\exists y x\mathrm{R}y \vee \exists y_1 \exists y_2((x\mathrm{R}y_1 \wedge y_1 \in \mathrm{W}^\tau) \wedge (x\mathrm{R}y_2 \wedge y_2 \notin \mathrm{W}^\tau)))$
(iv) V is defined as usual for truth-functional wffs and for the other wffs it is defined by the following two clauses:
(iv1) $\mathrm{V}(\tau, x) = 1$ iff $x \in \mathrm{W}^\tau$
(iv2) $\mathrm{V}(\Delta\mathrm{A}, x) = 1$ iff, for every y such that $x\mathrm{R}y$ and every y' s.t. $x\mathrm{R}y'$, $\mathrm{V}(\mathrm{A}, y) = \mathrm{V}(\mathrm{A}, y')$
Obviously in place of (iv2) we could have
(iv2$'$) $\mathrm{V}(\nabla\mathrm{A}, x) = 1$ iff, for some y such that $x\mathrm{R}y$ and some y' s.t. $x\mathrm{R}y'$, $\mathrm{V}(\mathrm{A}, y) \neq \mathrm{V}(\mathrm{A}, y')$

Let us now extend K$\Delta\tau$w with the following two definitions:
[Def$\Box$] $\Box\mathrm{A} =_{\mathrm{Df}} \Delta\mathrm{A} \wedge \Delta(\tau \supset \mathrm{A})$
[DefO] $\mathrm{OA} =_{\mathrm{Df}} \Delta(\tau \supset \mathrm{A})$

Note that the dual operators $\Diamond$A ($=_{\mathrm{Df}} \neg\Box\neg$A) and PA ($=_{\mathrm{Df}} \neg$O$\neg$A) turn out to be equivalent to $\nabla\mathrm{A} \vee \nabla(\tau \wedge \mathrm{A})$ and $\nabla(\tau \wedge \mathrm{A})$ respectively.

The decision procedure for K$\Delta\tau$w presupposes the definitional equivalence of K$\Delta\tau$w with a system, K$\Box\tau$w, which is the $\Box$-based normal system K extended with the following axiom:
($\Box\tau$w) $(\Box\tau \vee \Box\neg\tau) \supset \Box p$
and with the definition
[DefΔ] $\Delta\mathrm{A} =_{\mathrm{Df}} \Box\mathrm{A} \vee \Box\neg\mathrm{A}$[2]

The proof of the equivalence is provided in Pizzi [2007]. Given that ($\Box\tau$w), i.e. $(\Box\tau \vee \Box\neg\tau) \supset \Box p$, is equivalent to $\Box\tau \equiv \Box\neg\tau$, the system

[2] This definition is not the only definition of absoluteness granting the translation of the two systems. In Pizzi [2013] the proposed definition is $\Delta\mathrm{A} =_{\mathrm{Df}} ((\Diamond\tau \wedge \Diamond\neg\tau) \wedge (\Box\mathrm{A} \vee \Box\neg\mathrm{A})) \vee ((\Box\tau \vee \Box\neg\tau) \wedge \Box\mathrm{A})$. The advantage of the latter definition is that, if it is applied to axiom KΔ4: $\Delta\tau \supset \Delta p$, it yields exactly the axiom of K$\Box\tau$w $(\Box\tau \vee \Box\neg\tau) \supset \Box p$, not the weaker wff $(\Box\tau \vee \Box\neg\tau) \supset (\Box p \vee \Box\neg p)$ which would be the output of DefΔ.

It can be shown, however, that in systems at least as strong as K$\Delta\tau$ (see page 7) the two definitions are equivalent.

K$\Box\tau$w is easily decidable thanks to a simple extension of the tableau method for K, so K$\Delta\tau$w turns out to be decidable via a translation of K$\Delta\tau$w-wffs into K$\Box\tau$w-wffs. The tableau procedure for K$\Box\tau$w provides a constructive proof of the completeness of K$\Box\tau$w, so indirectly one for K$\Delta\tau$w.

Now we take into consideration a system which is named K$\Box$O in Pizzi [2013]. K$\Box$O is a minimum normal bimodal logic extended with a bridge-axiom whose language has, beyond truth-functional operators, the two primitives $\Box$ and O. We stipulate that the other primitives of the language are $\perp$ and $\supset$, while $\neg$, $\wedge$, $\vee$ are defined as usually.

The axioms of K$\Box$O, to be subjoined to an axiomatic basis for the standard propositional calculus PC, are:
K. $\Box(p \supset q) \supset (\Box p \supset \Box q)$
O. $\mathrm{O}(p \supset q) \supset (\mathrm{O}p \supset \mathrm{O}q)$
$\Box$O. $\Box p \supset \mathrm{O}p$
Rules: Modus Ponens (MP), Uniform Substitution (US)
(Nec) $\vdash$ A $\rightarrow \vdash \Box$A
(Eq) Replacement of proved material equivalents

In what follows we are going to prove a fact which was formulated as a conjecture in Pizzi [2013], i.e. that K$\Box$O is the τ-free fragment of K$\Delta\tau$w, or in other words that K$\Box$O axiomatizes all and only the theorems of K$\Delta\tau$w + Def$\Box$ + DefO containing only truth-functional operators, $\Box$ and O.

The first step is to prove that K$\Box$O is included in K$\Delta\tau$w + Def$\Box$ + DefO, i.e. to prove what follows:

T1. *For every K$\Box$O-wff A, if A is a thesis of K$\Box$O, A is a thesis of K$\Delta\tau$w + Def$\Box$ + DefO.*

Proof. The proof is by induction on the length of proofs. In the basis case we prove that the translations of the three axioms obtained by applying Def$\Box$ and DefO are K$\Delta\tau$w-theorems. For the inductive step we prove that the rules preserve the given property. The proof of K: $\Box(p \supset q) \supset (\Box p \supset \Box q)$ may be found in Pizzi [2007], §2. The proof of $\mathrm{O}(p \supset q) \supset (\mathrm{O}p \supset \mathrm{O}q)$ is derived from $\Box(r \supset (p \supset q)) \supset (\Box(r \supset p) \supset \Box(r \supset q))$ (a trivial consequence of K), by replacing τ for r. $\Box p \supset \mathrm{O}p$ is equivalent to the K-theorem $\Box p \supset \Box(\tau \supset p)$. The induction step is trivial. [Q.E.D.]

Two steps are now essential to perform the proof.

The first step consists in proving the soundness and completeness of K$\Box$O w.r.t. the class of K$\Delta\tau$w-models, i.e. of 4-ples <W, R, R^τ, V> which are defined by the following properties:
(i) W $\neq \varnothing$
(ii) for every x, y in W, $x\mathrm{R}^\tau y$ implies $x\mathrm{R}y$
(iii) V is as in K$\Delta\tau$w-models for truth-functional wffs and for the other wffs is as follows:
(iiia) $\mathrm{V}(\Box\mathrm{A}, x) = 1$ iff, for every y s.t. $x\mathrm{R}y$, $\mathrm{V}(\mathrm{A}, y) = 1$
(iiib) $\mathrm{V}(\mathrm{OA}, x) = 1$ iff, for every y s.t. $x\mathrm{R}^\tau y$, $\mathrm{V}(\mathrm{A}, y) = 1$

We will call such models K☐O-models.

We now prove soundness and completeness of K☐O with respect to the class of K☐O-models.

T2. *For every K☐O-wff A, if A is a K☐O-thesis, A holds at all K☐O-models.*
Proof. Trivial induction on the length of proofs.

T3. *For every formula A, if A holds at all K☐O-models, A is a K☐O-thesis.*
Proof. The proof is provided by a suitable application of the Henkin method.

The canonical model build over K☐O is a 4-ple <W^+, R^+, $R^{\tau+}$, V^+> where

(i) $W^+ = \{w, w', w' \dots\}$ is the set of maximal K☐O-consistent sets

(ii) R^+, $R^{\tau+}$ are defined as follows:

(iia) wR^+w' iff for every w, w' in W^+, if $\Box$A belongs to w, A belongs to w′

(iib) $wR^{\tau+}w'$ iff for every w, w' in W^+ if OA belongs to w, A belongs to w′

(iii) $V(p, w) = 1$ if and only if p belongs to w.

The proof consists in proving that the canonical model is a K☐O-model. The only non-trivial step is proving that the canonical model over K☐O is such that $R^{\tau+} \subseteq R^+$. Suppose that $wR^{\tau+}w'$. Then $w' \supseteq \{A : OA \in w\}$. As $\Box A \supset OA \in w$ for all formulas A, it follows that $\{A : \Box A \in w\} \subseteq \{A : OA \in w\}$, and thus that $w' \supseteq \{A : \Box A \in w\}$ — i.e. wRw' as desired. [Q.E.D.]

The next step of the proof consists in associating to every K☐O-formula a K$\Delta\tau$w-formula *via* a mapping f defined as follows:

(i) $f(p) = p$

(ii) $f(\bot) = \bot$

(iii) $f(A \supset B) = f(A) \supset f(B)$

(iv) $f(OA) = \Delta(\tau \supset f(A))$

(v) $f(\Box A) = \Delta f(A) \wedge \Delta(\tau \supset f(A))$

This translation allows us to prove the following theorem.

T4. *For every K☐O-wff A, if A is a thesis of K☐O, $f(A)$ is a thesis of K$\Delta\tau$w.*
Proof. By induction on the length of proofs. The f-images of the axioms of K☐O are K$\Delta\tau$w-theorems and the rules of K$\Delta\tau$w preserve the K$\Delta\tau$w-theoremhood of the f-images. [Q.E.D.]

The central problem is how to prove the converse of T4. In order to build a proof we move from a K☐O-model M = <W, R, R^τ, V> and define on its base a *derived model* M* = <W*, W^τ*, R*, V*> in this way:

a) W* $=$ W
b) W$^\tau$* $= \{z \in$ W: for some x in W, xR$^\tau z\}$
c) xR*y iff xRy
d) If W$^\tau$* $\neq \varnothing$, there is a y' such that $y' \notin$ W$^\tau$* and xR*y'
e) V* is as V as wffs with truth-functional operators are concerned. For the other wffs:
e1) V*$(\Delta\mathrm{A}, x) = 1$ iff for every y such that xR*y and every y' such that xR*y', V*$(\mathrm{A}, y) =$ V*(A, y')
e2a) For all y such that $y \in$ W$^\tau$*, V*$(\tau, z) = 1$
e2b) For all y such that $y \notin$ W$^\tau$*, V*$(\tau, y) = 0$

Obviously, in place of (e1) we could have
e1$'$) $\mathrm{V}(\nabla\mathrm{A}, x) = 1$ iff, for some y such that $x\mathrm{R}y$ and some y' s.t. $x\mathrm{R}y'$, $\mathrm{V}(\mathrm{A}, y) \neq \mathrm{V}(\mathrm{A}, y')$

We may now prove two lemmas:
Lemma 5. *M* is a K$\Delta\tau$w-model.*

The conditions defining the K$\Delta\tau$w-models are satisfied by M*, as one can see from the following considerations.
(i) Since W* $=$ W and W $\neq \varnothing$, W* $\neq \varnothing$
(ii) W$^\tau$* $= \{z \in$ W: for some x in W, xR$^\tau z\}$ is included in W*. In fact, if W$^\tau$* is $\varnothing$, this property is vacuously satisfied. If W$^\tau$* is not $\varnothing$, given that R$^\tau \subseteq$ R and R $=$ R*, the members of W$^\tau$* must belong to W*.
(iii) in order to show that R* is *weakly τ-forked*, i.e. that
($^\circ$) $\forall x(\neg\exists y\, x\mathrm{R}^*y \vee \exists y_1\exists y_2((x\mathrm{R}^*y_1 \wedge y_1 \in \mathrm{W}^\tau) \wedge (x\mathrm{R}^*y_2 \wedge y_2 \notin \mathrm{W}^\tau)))$
we take into consideration an arbitrary x and consider two cases:
(1) R* $= \varnothing$. This means that no world y is R*-seen by x, so $\neg\exists y\, x$R*y and, quantifiying over x, we conclude that ($^\circ$) holds *a fortiori*.
(2) R* $\neq \varnothing$. In this case there is an y such that xR*y. There are two possible subcases.
(2a) $\mathrm{V}(\tau, y) = 1$. This means that y belongs to W$^\tau$* and that W$^\tau$* is not $\varnothing$. So by clause d) of the above definition of a derived model there is an y' such that $y' \notin$ W$^\tau$ and xR*y'. But W$^\tau$* $= \{z \in$ W: for some x in W, xR$^\tau z\}$ and, given that, for every x and y, xR$^\tau y$ implies xR*y, it follows that $\exists y_1\exists y_2((x\mathrm{R}^*y_1 \wedge y_1 \in \mathrm{W}^\tau{}^*) \wedge (x\mathrm{R}^*y_2 \wedge y_2 \notin \mathrm{W}^\tau{}^*))$, so *a fortiori* R* has the property ($^\circ$).
(2b) $\mathrm{V}(\tau, y) = 0$, so $y \notin$ W$^\tau$*. If W$^\tau$* is $\varnothing$, this means that $\neg\exists y\, x$R*y, so we are back to case 1). Otherwise there is at least a member y_1 of W$^\tau$*, so we are again in conditions to assert, as for (2a), that for any x $\exists y_1\exists y_2((x\mathrm{R}^*y_1 \wedge y_1 \in \mathrm{W}^\tau{}^*) \wedge (x\mathrm{R}^*y_2 \wedge y_2 \notin \mathrm{W}^\tau{}^*))$. R* is then weakly τ-forked.
(iv) As the properties of V* are concerned, the clauses defining V in K$\Delta\tau$w-models are satisfied by e) and e2) and vacuously satisfied by e2a) and e2b), since τ does not occur in the language of K$\Delta\tau$w. [Q.E.D.]

Now we may prove the second Lemma:

Lemma 6. *Let M be a K☐O-model and M* a model derived from M. Then, for every K☐O-wff A, $V(A, x) = 1$ in M iff $V^*(f(A), x) = 1$ in M*.*

Proof. By induction on the complexity of wffs. Let us suppose that the equivalence holds for any arbitrary K☐O-wff A. The non-trivial step is in proving that the equivalence holds for the wffs OA and ☐A.

(ia) Suppose $\mathrm{V}(\Box\mathrm{A}, x) = 1$ in the K☐O-model M. Then at every world y such that $x\mathrm{R}y$, $\mathrm{V}(\mathrm{A}, y) = 1$ and by Induction Hypothesis, $\mathrm{V}^*(f(\mathrm{A}), y) = 1$ in M*; since R = R*, this holds at every world y in W* such that $x\mathrm{R}^*y$, which means $\mathrm{V}^*(\Delta f(\mathrm{A}), x) = 1$ in M*.

Also, $\mathrm{V}(\Box\mathrm{A}, x) = 1$ implies $\mathrm{V}(\mathrm{OA}, x) = 1$, by axiom ☐O of K☐O, so at every world y in W such that $x\mathrm{R}^\tau y$ $\mathrm{V}(\mathrm{A}, y) = 1$. Thus, by construction of W^τ* and Induction Hypothesis, if $y \in \mathrm{W}^\tau$*, $\mathrm{V}^*(f(\mathrm{A}), y) = 1$ and $\mathrm{V}^*(\tau \supset f(\mathrm{A}), y) = 1$. Since $\mathrm{V}(\tau, y) = 0$ for every y not in W^τ*, $\mathrm{V}^*(\tau \supset f(\mathrm{A}), y) = 1$ at every y in W*-W^τ*, so $\mathrm{V}^*(\tau \supset f(\mathrm{A}), y) = 1$ at every y in W*. So $\mathrm{V}^*(\Delta(\tau \supset f(\mathrm{A}), x) = 1$. Since we already established that $\mathrm{V}^*(\Delta f(\mathrm{A}), x) = 1$ in M*, by clause (iv) of the definition of f it follows that $\mathrm{V}^*(f(\Box\mathrm{A}), x) = 1$.

(ib) Suppose conversely that $\mathrm{V}^*(f(\Box\mathrm{A}), x) = 1$ in the derived model M*. Then, by clause (v) of the definition of f, $\mathrm{V}^*(\Delta(\tau \supset f(\mathrm{A})), x) = 1$ and $\mathrm{V}^*(\Delta f(\mathrm{A}), x) = 1$. The latter conclusion is compatible with two alternatives: that at every y s.t. $x\mathrm{R}^*y$ $\mathrm{V}^*(f(\mathrm{A}), y) = 0$ or that at every y s.t. $x\mathrm{R}^*y$ $\mathrm{V}(f(\mathrm{A}), y) = 1$.

According to the first alternative, by Induction Hypothesis we have $\mathrm{V}(\mathrm{A}, y) = 0$ at every y of M such that $x\mathrm{R}y$. So $\mathrm{V}(\Box\neg\mathrm{A}, x) = 1$. By the preceding result proved in (ia) this would imply, jointly with the supposition that $\mathrm{V}^*(f(\Box\mathrm{A}), x) = 1$, also $\mathrm{V}^*(f(\Box\neg\mathrm{A}), x) = 1$. This implies $\mathrm{V}^*(\Delta(\tau \supset f(\mathrm{A}), x)) = 1$ and $\mathrm{V}^*(\Delta(\tau \supset \neg f(\mathrm{A}), x)) = 1$, so by axiom KΔ2 $\mathrm{V}^*(\Delta((\tau \supset f(\mathrm{A})) \wedge (\tau \supset \neg f(\mathrm{A}))), x) = 1$ and by PC $\mathrm{V}^*(\Delta\neg\tau, x) = 1$: so, by KΔ1, $\mathrm{V}^*(\Delta\tau, x) = 1$.

But this is incompatible with the semantic condition described by the clause $\exists y_1\exists y_2((x\mathrm{R}^*y_1 \wedge y_1 \in \mathrm{W}^\tau{}^*) \wedge (x\mathrm{R}^*y_2 \wedge y_2 \notin \mathrm{W}^\tau{}^*))$, since by $\mathrm{V}^*(\Delta\tau, x) = 1$ τ should have value 1 or 0 everywhere in the accessibility sphere of x. Now since every KΔτw-model has the property described in (°), this means $\neg\exists y\ x\mathrm{R}^*y$. In this case, given that R = R*, we have also $\neg\exists y\ x\mathrm{R}\ y$ in M; so $\mathrm{V}(\Box\mathrm{A}, x) = 1$.

According to the second alternative, in every y s.t. $x\mathrm{R}^*y$ $\mathrm{V}(f(\mathrm{A}), y) = 1$. Thus by Induction Hypothesis we have $\mathrm{V}(\mathrm{A}, y) = 1$ at every y of M such that $x\mathrm{R}y$. So, even in this case, $\mathrm{V}(\Box\mathrm{A}, x) = 1$.

(iia) Suppose $\mathrm{V}(\mathrm{OA}, x) = 1$ for some x of the K☐O-model M. Then at every world y in M such that $x\mathrm{R}^\tau y$, $\mathrm{V}(\mathrm{A}, y) = 1$; so, by Induction Hypothesis, V*$(f(\mathrm{A}), y) = 1$ in the derived model M*. Since y is R^τ-accessible to x, $\mathrm{V}(\tau, y) = 1$ by clauses b) and e2) of the definition of M*, and V*$(\tau \supset f(\mathrm{A}), y) = 1$. W^τ* is by definition the set of the worlds z such that, for some x, in M $x\mathrm{R}^\tau z$, so at all worlds z in W^τ* V*$(\tau \supset f(\mathrm{A}), z) = 1$. But at every world y' in W*-W^τ* $\mathrm{V}(\tau, y') = 0$, so V*$(\tau \supset f(\mathrm{A}), y') = 1$ by the truth-functionality of $\supset$. Thus, for every y'' in M*, V*$(\tau \supset f(\mathrm{A}), y'') = 1$. Then V*$(\Delta(\tau \supset f(\mathrm{A})), x) = 1$, so V*$(f(\mathrm{OA}), x) = 1$ in M*.
(iib) Suppose conversely V*$(f(\mathrm{OA}), x) = 1$ in the model M* derived from M. This by definition of f is the same as V*$(\Delta(\tau \supset f(\mathrm{A})), x) = 1$.

Then at all worlds y of M* such that xR*y, V*$(\tau \supset f(\mathrm{A}), y) = 1$ or V*$(\tau \supset f(\mathrm{A}), y) = 0$. Let us consider separately the two alternatives.
(iib1) In the former alternative V*$(\tau, y) = 1$ implies V*$(f(\mathrm{A}), y) = 1$. This means that, if y is a world belonging to W^τ*, V*$(f(\mathrm{A}), y) = 1$. By Induction Hypothesis, then, in such y of M, $\mathrm{V}(\mathrm{A}, y) = 1$. By definition of W^τ* this means that at every z in M such that $x\mathrm{R}z$, $\mathrm{V}(\mathrm{A}, z) = 1$, so $\mathrm{V}(\mathrm{OA}, x) = 1$.
(iib2) The second alternative implies that V*$(\tau \supset f(\mathrm{A}), y) = 0$ at all worlds y of M* such that xR*y, this implying that at every such y V*$(\tau, y) = 1$ and V*$(f(\mathrm{A}), y) = 0$. But this is possible only when the R*-accessibility sphere is empty, i.e. $\neg\exists y\ x$R*y. Now, since R* = R, this means that in model M the relations R and R^τ (included in R) are both empty, so $\mathrm{V}(\mathrm{OA}, x) = 1$ in M. [Q.E.D.]

We have now simply to prove:
T7. *If $f(A)$ is a thesis of K$\Delta\tau$w, A is a thesis of K☐O.*
Proof. Suppose for a contradiction that A is not a thesis of K☐O. Thanks to the completeness of K☐O, then, at some world x of some K☐O-model M $\mathrm{V}(\mathrm{A}, x) = 0$. But, according to Lemma 2 above, there is a world x of the K$\Delta\tau$w-model M* such that V*$(f(\mathrm{A}), x) = 0$. So, by the completeness of K$\Delta\tau$w, $f(\mathrm{A})$ is not a thesis of K$\Delta\tau$w. [Q.E.D.]

T4 and T5 jointly yield the required result:
T8. *For every K☐O-wff A, A is thesis of K☐O iff $f(A)$ is a thesis of K$\Delta\tau$w.*

§1.1 A by-product of the preceding theorem T8 concerns the monomodal fragments of K$\Delta\tau$w + Def☐ + DefO, i.e. the fragments of this system containing only ☐ or only O beyond truth-functional connectives. As a matter of fact, such fragments are also fragments of the bimodal system K☐O, so the attention may be limited to this fragment of K$\Delta\tau$w. The reader can quickly find a proof of the following two theorems T9 and T10, where K is the well-known minimal normal system axiomatized as PC + K + $\vdash$ A $\rightarrow\vdash\Box$A, and KO is the system axiomatized as PC + O + $\vdash$ A $\rightarrow\vdash$ OA.

T9. *Let A be any wff containing only $\Box$ or truth-functional operators. Then A is a theorem of K$\Delta\tau$w + Def$\Box$ if and only if A is a K-theorem.*

(Sketch of the proof). We introduce two mappings, f° and g. f° is defined on K-language and is as the mapping f of page 4 with the following difference in clause (v):

(v*) $f^\circ(\Box\mathrm{A}) = \Box f^\circ(\mathrm{A}) \wedge \mathrm{O}f^\circ(\mathrm{A})$

Then, observing that $\Box p \equiv (\Box p \wedge \mathrm{O}p)$ is a K$\Box$O-theorem equivalent to the bridge Axiom $\Box$O, it is easy to prove the following theorem by induction on the length of proofs:

Lemma 10. *If A is A K-theorem, $f^\circ(A)$ is a K$\Box$O-theorem.*

The second mapping g is defined on K$\Box$O-language. Neglecting the trivial clauses for truth-functional wffs, the distinctive clauses are as follows:

(v**a) $g(\Box\mathrm{A}) = \Box g(\mathrm{A})$

(v**b) $g(\mathrm{OA}) = \Box g(\mathrm{A})$

Again by induction on the length of proofs we prove

Lemma 11. *If A is a K$\Box$O-theorem, $g(A)$ is a K-theorem.*

The last step of the proof aims to proving that there is a definitional equivalence between the two axiom systems:

Lemma 12. *For every K-wff A, A $\equiv gf^\circ(A)$ is a K-theorem.*

(The proof is by induction on the complexity of A)

A second result concerns the fragment containing only O-formulas.

T13. *Let A be any wff containing only O or truth-functional operators. Then A is a theorem of K$\Delta\tau$w + DefO if and only if A is a KO-theorem.*

The proof is along the lines of the preceding one in T9–T12 with the only difference that, in the light of the equivalence O$p \equiv$ O$p \vee \Box p$, the definition of f° is characterized by the following clause:

(v***) $f^\circ(\mathrm{OA}) = \mathrm{O}f(\mathrm{A}) \vee \Box f(\mathrm{A})$.

while for the definition of g the last clause is as in (v**a) and (v**b), with the symbol $\Box$ on the RHS replaced by O.

§2. The preceding bimodal system K$\Box$O has been derived as a fragment of a minimal monomodal contingency system extended with an axiom expressing the minimal properties of a propositional constant. It is natural to suppose that extending K$\Delta\tau$w with other axioms involving the same propositional constant, a bimodal fragment stronger than K$\Box$O could be identified.

The most simple result which can be found in this direction is the following. We extend K$\Delta\tau$w with the axiom

KΔ5. $\Delta\tau \supset \nabla p$

Clearly the conjunction of KΔ5 and KΔ4 is equivalent to an unique axiom which is $\Delta\tau \supset \bot$, which in its turn is equivalent to the simple
K$\nabla\tau$. $\nabla\tau$

The system K$\Delta\tau$w extended with KΔ5 or, alternatively, the system PC + KΔ1 – KΔ3 + K$\nabla\tau$, will be called K$\Delta\tau$. Since the axiom K$\nabla\tau$ may be proved to be underivable in K$\Delta\tau$w, the two systems K$\Delta\tau$ and K$\Delta\tau$w are distinct systems. Our claim is now that, preserving the preceding definition of the $\Box$, the bimodal fragment of K$\Delta\tau$ is K$\Box$O extended with the following couple of axioms:

D. $\Box p \supset \Diamond p$

DO. O$p \supset$ Pp

The new system will be called KD$\Box$O since it has the structure of a bimodal deontic logic.

The proof that KD$\Box$O is the τ-free fragment of K is obtained by a suitable modification of the preceding proof for K$\Box$O.

In order to avoid a tedious replication of the known schema of proof, we limit ourself to a sketch enhancing the changes to be introduced with respect to the preceding proof.

1) In the first place we have to prove the completeness of K$\Delta\tau$. The semantics for this system is provided by defining K$\Delta\tau$-models, which are not weakly τ-forked as K$\Delta\tau$w-models but simply τ-forked. In other words, the only difference is that it is now excluded that the accessibility sphere is empty: $\forall x \exists y_1 \exists y_2((x\mathrm{R}y_1 \wedge y_1 \in \mathrm{W}\tau) \wedge (x\mathrm{R}y_2 \wedge y_2 \notin \mathrm{W}\tau))$

2) The decision procedure for K$\Delta\tau$ presupposes the proof of the definitional equivalence beween K$\Delta\tau$ and K$\Box\tau$, which is K extended with the axiom $\Diamond\tau \wedge \Diamond\neg\tau$ and the standard definition of Δ. The completeness of K$\Delta\tau$ depends then on the completeness of the equivalent system K$\Box\tau$ (see Pizzi [2007]).

3) KD$\Box$O is included in K$\Delta\tau$ + Def$\Box$ + DefO. The proof is as in T3 at page 3, with the supplementary proof concerning the two new axioms D: $\Box p \supset \Diamond p$ and DO: O$p \supset$ Pp. Note that $\Box p \supset \Diamond p$ is equivalent to $\Diamond \mathsf{T}$, so also (by Def$\Box$), to $\nabla \mathsf{T} \vee \nabla(\tau \wedge \mathsf{T})$ and then to $\nabla\tau$, while O$p \supset$ Pp is equivalent to PT. Proceed then using DefO and truth-functional reasoning to show that PT is equivalent to $\nabla\tau$.

4) KD$\Box$O-models are like K$\Box$O-models with the only difference that R and R^τ are both serial, i.e. $\forall x \exists y_1\ x\mathrm{R}y_1$ and $\forall x \exists y_2\ x\mathrm{R}^\tau y_2$. KD$\Box$O is proved to be sound and complete w.r.t. the class of KD$\Box$O-models along the same lines exposed at page 3. In particular, it must be proved that the canonical model over KD$\Box$O is serial in the mentioned sense.

5) The derived model M* now is derived from KD$\Box$O-model M and is defined as at page 3. In place of Lemma 1 we have now to prove that M* is a K$\Delta\tau$-model, i.e. that R is τ-forked. The only difference, given that R* $=$ R and R is serial, is that the alternative R* $= \varnothing$ is not to be considered.

6) In place of Lemma 2 we have now to prove:
Lemma 14. *Let M be a KD□O-model and M* a model derived from M. Then, for every KD□O-wff A,* $V(A, x) = 1$ *in M iff V**$(f(A), x) = 1$ *in M*.*
The proof runs as for Lemma 6 of page 4 with the following difference. When we are under the supposition V*$(f(\Box\mathrm{A}), y) = 1$, we consider the alternative (ib), i.e. that V*$(f(\mathrm{A}), y) = 0$ for every y such that xR*y. This premise leads to the conclusion V*$(\Delta\tau, x) = 1$. But M* is a K$\Delta\tau$-model and the axiom $\nabla\tau$ receives value 1 in all such models. So it is impossible that V*$(\Delta\tau, x) = 1$.

A parallel argument holds for the supposition V*$(\mathrm{OA}, x) = 1$ and the alternative (iib2) implying that V*$(\tau \supset f(\mathrm{A}), x) = 0$ at all worlds y of M* such that xR*y; this means that at every y such that xR*y, V*$(\tau, y) = 1$ and V*$(f(\mathrm{A}), y) = 0$. But it is impossible that V*$(\tau, y) = 1$ since M* is τ-forked, which means that V*$(\tau, y) = 0$ at some y accessible to x.

Given this simplified proof of the preparatory Lemmas, it is straightforward to prove:
T15. *If f(A) is a thesis of K*$\Delta\tau$*, A is a thesis of KD□O.*

By an argument which parallels the one of pp. 5–6 we are able to prove that the □-fragment of K$\Delta\tau$ is equivalent to KD and that the O-fragment is equivalent to KO + DO, which we shall call here KDO. In other words:
T16. *Let A be a wff containing only □ and truth-functional operators. Then A is a theorem of K*$\Delta\tau$ *+ DefO if and only if A is a KD-theorem.*
T17. *Let A be a wff containing only O and truth-functional operators. Then A is a theorem of K*$\Delta\tau$ *+ DefO if and only if A is a KDO-theorem.*

REMARK. The metatheorems T9 and T16 have been already proved in §3 and §4 of Pizzi [2007], but with an utterly different method.

§3. An interesting question concerns contingential systems whose modal fragment is at least as strong as KT. A well-known result by Montgomery and Routley (1966) proves the definitional equivalence between KT + DefΔ and a contingential system extended with the definition
[Def′□] □A $=_{\mathrm{Df}}$ ΔA $\wedge$ A

The axiomatical basis proposed by Montgomery and Routley for the relevant contingential system is very elegant. This system, that they call T_1, is PC extended with two axioms
KΔ1. $\Delta p \equiv \Delta\neg p$
KΔ6. $p \supset (\Delta(p \supset q) \supset (\Delta p \supset \Delta q))$
and with the only rule NecΔ: $\vdash$ A $\rightarrow \vdash$ ΔA

We are here interested in relating the new definition Def′□ with our initial Def□, which is expressed in a language containing propositional constants. We need of course to express in contingential language the basic principle T: $\Box p \supset p$. This can be done by extending the minimal contingential system

KΔ with the following axiom:
TΔ. $(\Delta p \wedge \Delta(\tau \supset p)) \supset p$
The new system is called KT$\Delta\tau$. It is easy to see that K$\Delta\tau$ is included in KT$\Delta\tau$:

1) $(\Delta\tau \wedge \Delta(\tau \supset \tau)) \supset \tau$	TΔ, τ/p
2) $\Delta\tau \supset \tau$	1), $\vdash \Delta(\tau \supset \tau)$, Eq
3) $(\Delta\tau \wedge \Delta(\tau \supset \neg\tau)) \supset \neg\tau$	TΔ, $\neg\tau/p$, KΔ1
4) $\Delta\tau \supset \neg\tau$	3), $\vdash \Delta(\tau \supset \neg\tau) \equiv \Delta\tau$
5) $\nabla\tau$	2), 4), PC

KT$\Delta\tau$-models are K$\Delta\tau$-models which are reflexive:
($^{\circ\circ}$) $\forall x(x\mathrm{R}x \wedge \exists y_1 \exists y_2\ ((x\mathrm{R}^* y_1 \wedge y_1 \in \mathrm{W}^\tau) \wedge (x\mathrm{R}^* y_2 \wedge y_2 \notin \mathrm{W}^\tau))$

In other words, in K$\Delta\tau$-models R is reflexive and also τ-forked: we will say that it is τ-reflexive. Clearly every τ-reflexive model is *a fortiori* reflexive.

The soundness of this system with respect to the class of such models is unproblematic.

T18. *If A is a KT$\Delta\tau$-thesis, A is KT$\Delta\tau$-valid.*

Proof. Since the axioms KΔ0–4 hold in all K$\Delta\tau$-models, a fortiori they hold in all KT$\Delta\tau$-models. So, given that the rules preserve the mentioned property, it is enough to prove that $(\Delta\mathrm{A} \wedge \Delta(\tau \supset \mathrm{A})) \supset \mathrm{A}$ holds in all KT$\Delta\tau$-models. Let us suppose by Reductio that, at some world x, $\mathrm{V}(\Delta\mathrm{A}, x) = 1$, $\mathrm{V}(\Delta(\tau \supset \mathrm{A}), x) = 1$ and $\mathrm{V}(\mathrm{A}, x) = 0$. Considering the assignment $\mathrm{V}(\Delta\mathrm{A}, x) = 1$ we have to consider three subcases:
1) A has value 1 at all y such that $x\mathrm{R}y$ (from which it follows that $\tau \supset$ A has also value 1, so the case in which $\tau \supset$ A has value 0 is to be excluded). At every τ-reflexive model we have that $x\mathrm{R}x$; so a contradiction follows since A receives value 1 and 0 at the same world. TΔ has then value 1 at all τ-reflexive models.
2) A has value 0 at all y such that $x\mathrm{R}y$ and $\tau \supset$ A has value 1 at any such y, so also at any y belonging to W^τ. So τ has also value 0 in y: contradiction.
3) A has value 0 at all y such that $x\mathrm{R}y$ and $\tau \supset$ A has value 0 in any such y, so τ has value 1 at all R-accessible y. But this is incompatible with the condition of being τ-forked, which implies that at least one R-accessible world is a $\neg\tau$-world. [Q.E.D.]

The converse of T18, i.e. the completeness of KT$\Delta\tau$ with respect to the given semantics, will not be treated in the present paper, although there is no reason to suspect that the system lacks this property.

Since we are in search of the bimodal fragment of contingential systems in $\Box$ and O, an obvious preliminary consideration is that KD$\Box$O is a subsystem of KT$\Delta\tau$, and that $\Box$A $\supset$ A, i.e. the translation of $(\Delta\mathrm{A} \wedge \Delta(\tau \supset \mathrm{A})) \supset \mathrm{A}$, is also a theorem of KT$\Delta\tau$. Our conjecture is now that the bimodal fragment of the latter system, here called KT$\Box$O, is provided by the system KD$\Box$O + $\Box p \supset p$.

Before proving the conjecture, it is of some interest to discuss the following problem. Given that the box $\Box$ received two different interpretations in the literature — ΔA $\wedge$ A and ΔA $\wedge$ $\Delta(\tau \supset$ A$)$ — which is the position in KT$\Delta\tau$ of the equivalence ΔA $\wedge$ A $\equiv$ ΔA $\wedge$ $\Delta(\tau \supset$ A$)$? As a matter of fact, even lacking of a proof of the completeness of KT$\Delta\tau$ we may prove what follows:

T19. $\Delta p \wedge p \equiv \Delta p \wedge \Delta(\tau \supset p)$ *is a KT$\Delta\tau$-thesis.*

Proof. The basic step is in proving that the following wffs are K$\Delta\tau$-valid:

(i) $((\Delta p \wedge \Delta(\tau \supset p)) \supset p) \supset ((\Delta p \wedge p) \supset \Delta(\tau \supset p))$

(ii) $((\Delta p \wedge \Delta(\tau \supset p)) \supset p) \supset ((\Delta p \wedge \Delta(\tau \supset p)) \supset p)$

(i). Let us consider Δp and p with value 1 at x and, by Reductio, V$(\Delta(\tau \supset p, x) = 0$. The assignment 1 to Δp is compatible with two alternatives. If the value of p is everywhere 1 in the accessibility sphere of x it cannot happen that $\tau \supset p$ is 0 at some accessible worlds, so $\Delta(\tau \supset p)$ should be 1 in x: contradiction. On the other hand, if p has everywhere value 0 in the sphere, it cannot happen that $\tau \wedge \neg p$ has somewhere value 1, what follows by the property of R being -forked: contradiction. (i) is then K$\Delta\tau$-valid.

(ii). (ii) is a substitution-instance of the tautology $p \supset p$.

Since both (i) and (ii) are K$\Delta\tau$-valid, they are K$\Delta\tau$-theses thanks to the completeness of K$\Delta\tau$, so *a fortiori* they are KT$\Delta\tau$-thesis. Since $(\Delta p \wedge \Delta(\tau \supset p)) \supset p$ is a KT$\Delta\tau$-thesis, by Modus Ponens we derive the two consequents, so the two implications $(\Delta p \wedge p) \supset (\Delta p \wedge \Delta(\tau \supset p))$ and $(\Delta p \wedge \Delta(\tau \supset p)) \supset (\Delta p \wedge p)$, so also their equivalence. [Q.E.D.]

A consequence of the proof of T19 is that the two definitions of the box provided by Def$\Box$ and Def$'\Box$ are equivalent in the new system KT$\Delta\tau$, but they are equivalent even in the weaker system K$\Delta\tau$.

REMARK. Note that O$p \supset p$ is inconsistent with KD$\Box$O and KT$\Box$O. In fact $\vdash \Delta(\tau \supset p) \supset p$ yields $\vdash \Delta(\tau \supset \tau) \supset \tau$, so by Modus Ponens $\vdash \tau$ and by NecΔ $\vdash \Delta\tau$, which is the negation of $\nabla\tau$.

The proof that the bimodal fragment of KTΔ is KT$\Box$O results from an easy adaptation of the one for KD$\Box$O. The relation R of the KT$\Box$O-models <W, R, R^τ, V> is as in KD$\Box$O but with the additional property of being reflexive. (Note however that the properties of R^τ are unvaried).

The mapping f is defined as at page 3.

The theorems that are expected to hold here are formulated as follows (the proof are simply suggested).

T20. *For every KT$\Box$O-wff A, if A is a thesis of KT$\Box$O, A is a thesis of KTΔ + Def$\Box$ + DefO.*

(As at page 3, adding that $\Box p \supset p$ is derived from axiom TΔ in KT$\Delta\tau$.)

T21. *If A is a KT□O-thesis, A holds at all KT□O-models.*
(As in T2 with the further proof that $\Box p \supset p$ holds at all KT□O-models, which belong to a subclass of reflexive models.)
T22. *If A holds at all KT□O-models, A is a KT□O-thesis.*
(The proof follows the lines of T3. The analysis of canonical model requires proving that the canonical model over KT□O is reflexive.)
T23. *If A is a thesis of KT□O, $f(A)$ is a thesis of KT$\Delta\tau$.*
(Trivial)

The derived model M* = <W*, W$^{\tau}$*, R*, V*> is defined as before. Since xR*y iff xRy, it follows that R* is reflexive.

Lemma 24. *M* is a KTΔO-model.*
(The only new problem is to show that R is τ-reflexive, i.e. that $\forall x(x\mathrm{R}x \wedge \exists y_1 \exists y_2((x\mathrm{R}y_1 \wedge y_1 \in \mathrm{W}^{\tau}) \wedge (x\mathrm{R}y_2 \wedge y_2 \notin \mathrm{W}^{\tau})))$.) Since R is reflexive, R* is such and *a fortiori* serial. This means that the accessibility sphere of every world x is never empty. The property of being τ-forked is common to other contingential models and does not introduce changes.
Lemma 25. *Let M be a KT□O-model and M* a model derived from M. Then, for every KT□O-wff A, $V(A, x) = 1$ in M iff V*$(f(A), x) = 1$ in M*.*
T26. *If $f(A)$ is a thesis of KT$\Delta\tau$, A is a thesis of KT□O.*

It remains to take into consideration the proofs concerning the monomodal fragments of the system KT$\Delta\tau$. The reader can reconstruct them along the lines of T12 and T13. However, the result here is not completely symmetric. In fact we have:
T27. *Let A be a wff containing only □ and truth-functional operators. Then A is a KT$\Delta\tau$-theorem if and only if A is a KT-theorem.*
T28. *Let A be a wff containing only O and truth-functional operators. Then A is a KT$\Delta\tau$-theorem if and only if A is a KDO-theorem.*

The third system KT$\Delta\tau$ has another peculiarity with respect to the weaker ones. Thanks to the Montgomery-Routley result and to theorem T19, it turns out that the τ-free fragment of KT$\Delta\tau$ containing only □ and truth-functional operators, i.e. KT, is definitionally equivalent to a system (i.e. Montgomery-Routley's T_1) containing only Δ and truth-functional operators.

One of the consequences of the mentioned result is that a system which is definitionally equivalent to KT□O and which will be named KTΔO is the following, written in the operators $\neg$, $\supset$, Δ, O.

1. $\Delta p \equiv \Delta \neg p$
2. $p \supset (\Delta(p \supset q) \supset (\Delta p \supset \Delta q))$
3. $p \supset (\Delta p \supset \mathrm{O}p)$
4. $\mathrm{O}(p \supset q) \supset (\mathrm{O}p \supset \mathrm{O}q)$
5. $\mathrm{O}p \supset \mathrm{P}p$

Rules: US, MP, NecΔ: $\vdash$ A $\rightarrow \vdash$ ΔA

If contingency and absoluteness are considered modal notions on the same level of the notions represented in the Aristotelian square of oppositions (and there are historical and theoretical reasons to defend this claim), the "mixed" system KTΔO may be considered in proper sense a *bimodal* system. Note that of course Op does not imply Δp nor p, so the operator O cannot be read as representing logical necessity: it might be read as a deontic operator but also, say, as an operator for temporal necessity or for physical necessity. In the latter case, defining a notion of physical contingency as

[Def∇^f] ∇^fA $=_{\mathrm{Df}}$ PA $\wedge$ P$\neg$A

might allow studying the interrelations between different notions of contingency in a direct and analytically interesting way.

Dipartimento di Scienze Sociali, Politiche e Cognitive
via Roma 56, 53100 Siena
Italy
E-mail: pizzic@msn.com

REFERENCES

Cresswell, M.J. [1988] Necessity and Contingency, *Studia Logica*, 47, 145–149.

Humberstone, I.L. [1992] The Logic of Non-Contingency, *Notre Dame JFL*, 36, 214–234.

Humberstone, I.L. [2002] The Modal Logic of Agreement and Non-Contingency, *Notre Dame JFL*, 43, 95–127.

Kuhn, T.S. [1995] Minimal Non-Contingency Logics, *Notre Dame JFL*, 36, 230–234.

Montgomery, R. and Routley, R. [1966] Contingency and non-Contingency bases for normal modal logics, *Logique et Analyse*, 9, 318–328.

Pizzi, C. [1999] Contingency Logics and Propositional Quantification, *Manuscrito* (Unicamp, Campinas) 22, 283–303.

Pizzi, C. [2007] Necessity and Relative Contingency, *Studia Logica*, 85, 305–410.

Pizzi, C. [2006] A Logic of Contingency with a Propositional Constant, 141–153, in E. Ballo e M. Franchella eds., *Logic and Philosophy in Italy*, Polimetrica, Monza.

Pizzi, C. [2013] Relative Contingency and Bimodality, *Logica Universalis*, 7, 113–123.