Logique & Analyse **224** (2013), 405–423

EXPOUNDING ASSERTION AND EXPANDING NEGATIONS IN LOGIC

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This article considers two aspects of classical two-valued logic that deserve more attention than they receive: elaborating the details of asserting a proposition, that is, assigning to it one of the truth-values ‘true’ and ‘untrue’; and extending the negation of a proposition to modes when only a part of it is negated. These two modifications are applied to a trio of case studies in logic: the formulation of several paradoxes, the use of indirect proof-methods when the theorem in question is asserted, and the distinction between implication and inference. Attention is also drawn in places to some features of logic that are often treated rather casually; for example, specifying universes of discourse, and theories of truth. Relationships with mathematics are emphasised.

1. *Aims and jargon*

Take, for example, the proposition H := ‘the mathematician G. H. Hardy plans to publish an edition of his papers’. What effects follow from asserting that H is true, or that it is untrue? And if we deny H, are we denying the plan or the publishing, or both? Answers to these questions are sought for propositions in general, including their bearing upon mathematics, where connections with logic are particularly close.

First we must indicate the use of certain technical terms, of which the principal ones are rendered in sans serif. The word ‘logic’ or phrase ‘a logic’ is confined to some systematic account of methods of forming propositions, making deductions with and from them, and working with their truth-values; ‘logics’ are several ‘logic’s. A logic may well be axiomatised, use notations, admit paradoxes, and/or link closely to some branches of mathematics.

Many logics are now available (modal, epistemic, many-valued, fuzzy, constructive, quantum, combinatory, infinitary, and so on); but mainly for reasons of space we consider only traditional bivalent logic, called ‘L2’. There the available truth-values of propositions are the pair ‘true’ and ‘untrue’; the law of excluded middle, ‘any proposition is either true *or* untrue’, is upheld to guarantee consistency.[1] The word '*or*' denotes the exclusive mode 'either/or' of disjunction, while 'or' is the inclusive mode 'and/or'.

The basic component of L2 is held to be the proposition, and its reference (or lack of reference) to some state of affirms as the dominant property; terms are subsidiary to propositions, not *vice versa*. L2 divides into the propositional calculus, in which a proposition R is an indivisible "atom", related to other propositions by logical connectives (negation, conjunction, disjunctions, implication, equivalence); and the functional calculus, where a proposition is dissected to reveal its propositional functions of one variable or relations of several variables (of individuals) x, y, . . . , such as 'x is longer than y' in geometry. Existential and universal quantification ('there exists', and 'for all' or 'for every') can range over propositions, functions and individuals. L2 is also involved in asserting truth-values and drawing inferences, as we see in the suite of case studies.

L2 is applied to settings. They are theories of *any* kind: statistical inference in medicine, the history of Swedish maypole dancing, the ethics of feudalism in the Middle Ages, Euclidian geometry, the behaviour of acids at low temperatures, . . . A theory might well entrain its own further settings; for example, physical optics could draw upon geometry, which in turn may appeal to the differential and integral calculus and the arithmetic of real numbers.

All propositions in L2 are in indicative mood, and in either active or passive voice. Excluded are questions and commands, which have their own logical features; more informal uses of a logic, such as logistics or the logic of the situation; and theories that have been called 'inductive logics', which belong to the philosophy of science and probability theory. However, they are all parts of logical knowledge, the umbrella term that refers to the totality of logics.

Collections of things will be used on occasion, especially ranges of significance of propositional functions and relations and universes of discourse of logical arguments. These are not assumed to be sets in the sense of set theory, but multisets to which members may belong more than once. They are needed in many situations, including elementary mathematical ones such as the prime factors of integers and the repeated zeroes of polynomial functions — yet hardly anyone notices [Grattan-Guinness 2011a].

Finally, quotation marks are used not only to quote, quasi-quote *or* name but also for the sake of clarity to act like brackets and enclose words or propositions that are being discussed.

[1] I note, with some bewilderment, the failure of mankind to recognise the important status of temporal logic as distinct from any version of L2 because, among several reasons, 'and' means 'and then', so that conjunction is not commutative in its propositional calculus. See [Øhrstrom and Hasle 2006] on its eventual arrival in the 1950s.

2. *Aspect one: Assertion in logics and in its settings*

2.1. *A modest heyday*

The distinction between an asserted proposition and a non-asserted proposition often arises covertly but seldom overtly. For example, when a mathematician states a theorem, he implicitly asserts it to be true, or to be provable from the assumptions or axioms of the theory involved, and he may well have a (faulty?) proof. However, the consequences and even the details of *explicitly* making an assertion have rarely been explored.

Gottlob Frege led the way on stressing assertion in formal and symbolic logic, from the early 1890s onwards. He prefaced a symbolic proposition with a special compound sign '⊢' indicating the assertion ('|'), usually the affirmation, of its content ('-') [Textor 2010]. When Bertrand Russell studied Frege from 1902 he took over both assertion in both kinds and the sign, and he and A. N. Whitehead used them systematically in their logical writings.

However, Russell omitted assertion from his popular account [1919] of mathematical logic. This decision (or maybe just an oversight) foreshadowed a decline among logicians to individuate assertion. David Hilbert used the word 'assertion' early on in his textbook and monograph on logic but not with a special sign,[2] and not at all in his 'metamathematical' writings.

From the mid 1930s onwards logicians even changed the reference of '⊢R' from the assertion of R to the indication that it was a theorem (for example, [Quine 1951, 88], where the Whitehead-Russell use is 'somewhat obscure'). Among other later eminent authors, Alonzo Church did not have a sign for the assertion of a symbolic proposition R but 'displayed [R] on a separate line or lines as sufficient indication of its assertion' [1956, 24].

Many other logic textbooks and monographs give assertion little or even no attention. While they discuss truth-values, mostly it is devoted to determining the values of compound propositions as functions of their component propositions, usually by the truth-table method. If the word 'assertion' is used at all, it may well be in the different sense of emphasising features *of* a proposition, such as its claim that, say, a real number exists with some specific property.

2.2. *Diagnoses of the decline*

Presumably one cause of this decline was the growing recognition of the central role being given to metalogic, where asserted propositions are located. The metalogic ML2 of L2 gained most attention. It was urged partly

[2] See [Hilbert and Ackermann 1928] and [Hilbert and Bernays 1934], both esp. ch. 1.

through Hilbert, and especially Kurt Gödel’s study [1931] of incompletability of first-order arithmetic (Rudolf Carnap soon coined ‘metalogic’ because of its central role in that study) and Alfred Tarski (who introduced ‘metalanguage’). Indeed, a striking example of neglect is Tarski’s textbook on logic, where the opening sentence states that ‘Every scientific theory is a system of sentences which are accepted as true and which may be called LAWS or ASSERTED STATEMENTS (sometimes one says, for short, simply STATEMENTS)’ [1941, 3] and yet assertion is treated only in one passage of the chapter ‘On the deductive method’.

Another reason for the neglect of assertion is social practice. Strict speaking (or writing) requires us frequently to put ‘It is true that’, ‘I deny that’, or some such phrase into our discourses; but in the quick flow of contexts this would soon become very boring. So we explicitly assert rather infrequently; when we face some legal challenge, say, *or* seek to emphasise our political stance.

A further cause of neglect is that an asserted proposition is a *conjecture*, awaiting a test (for example, a scientific experiment) to see if it is refuted *or* corroborated *or* is not determinable in the domain of reference of the setting. There are various theories of truth (correspondence to the facts, belief, redundancy, pragmatic, ...), none of them capable on its own of covering the range of circumstances involved: all aspects of the physical, social and psychological worlds, mathematical proof-methods, and so on.[3] Truth theories and testing *use* a logic, but they *are* not logic; so logicians do not usually discuss them, and therefore not the acts of assertion that produces the testable propositions.[4] But this is very unsatisfying, as significant issues are at hand; even the choice of a logic for a setting is *itself* under test [Bartley 1980].

We only raise the issue here, and require just the usual asymmetry of denial between truth and untruth: that if a proposition R is true then every contradictory proposition is untrue, whereas if R is untrue then any competitor might be true, so that only true propositions are derivable from a true R whereas any proposition follows from an untrue R.

2.3. *The principal properties of assertion*

While a logic can be developed in terms of unasserted propositions, as a “pure” calculus, its importance increases greatly when propositions are asserted in some setting [van Heijenoort 1967a]; so assertion needs to be elucidated. We state it as a pair of operations on an unasserted proposition in

[3] See, for example, [Haack 1978] and [Field 2008].

[4] See, for example, [Quine 1951, esp. pp. 3–5 and ch. 1]; [Rosser 1953, esp. ch. 2] and [Church 1956, esp. pp. 23–27].

indicative mood to produce two propositions in ML2. We take the Hardy example in the version

H := ‘Hardy plans to publish an edition of his papers’.

When it is asserted, a truth-value is assigned to it, in two asserted propositions:

its affirmation +H := ‘it is true that H’ and

its denial −H := ‘it is untrue that H’.

Some authors advocate affirmation as the sole kind of assertion (for example, Tarski above), and assume that the denial of H and the affirmation of its negation are metalogically equivalent:

DN := −H if and only if +(not-H).

We prefer to place denial on a par with affirmation for three reasons:

1) to encourage checking that DN actually does obtain in a given setting, especially when the conjectured truth-value of an assertion cannot be definitively established;

2) to imitate settings where denial is explicitly the form of an assertion, such as impossibility theorems in mathematics and science, and all sorts of asseverations and oaths made in legal contexts; and

3) to exploit the structurally similar properties of symmetry and duality that obtain between this pair of assertions and other duos in (M)L2 such as and/or, assert/unassert and true/untrue (indeed, for this reason the word ‘untrue’ is preferred to its synonym ‘false’).

2.4. *Other properties*

1) In keeping with the exploratory cast of this investigation, we admit paradoxes into L2 but deprive it of links to its metalogic ML2, which is also bivalent and is the place to consider the kinds of consistency, completeness and decidability that L2 does or does not possess.[5] Therefore ML2 admits paradoxes concerning asserted propositions parallel to those in L2 about unasserted propositions (section 5). No attempt is made to solve *or* to accommodate any logical paradoxes in either L2 or ML2.

2) We can assert only a proposition that is self-standing, not when it is subsidiary clause of another one; there is no property of sub-assertion. But we can assert a compound proposition, such as an implication: the distinction between unasserted and asserted propositions is not to be confused with that between categorical and hypothetical propositions.

3) ‘Is (un)true’ is the principal verb in an asserted proposition, to which every verb in R is subsidiary. This feature exemplifies the fact that there are

[5] I ignore the practice among mathematicians of naming as ‘paradoxes’ some true but surprising theorems (Saint Petersburg, birthday, d’Alembert’s, statics, and so on).

several ways of negating a proposition. Like assertion, it is rarely explored in logic texts; but it is the other aspect highlighted in this article.

3. *Aspect two: Modes of negating a proposition*

3.1. *(Un)awareness*

Syllogistic logicians were long aware of distinguishing a pair of contradictory propositions (if one is true, then the other is untrue) from a pair of contrary propositions (both can be true but not both untrue); so they took due note of different modes of negation. A good example is Augustus De Morgan, in his pioneer efforts to formulate the parts of syllogistic logic that for him underlay Euclid's *Elements*.[6]

One context in which modes of negation are recognised concerns quantification in the functional calculus, where, for example, 'every integer is not prime' is distinguished from 'not every integer is prime' and 'there does not exist a prime number' from 'there exists a non-prime number'. Among other settings, mathematical analysis contains an elaborate collection of definitions that draws upon these distinctions (usually implicitly): the various modes of (non-)(quasi-)uniform convergence of an infinite series of functions $\Sigma_r f_r(x)$ over an interval of values of x *or* in the neighbourhood of a value of x *or* at a value of x [Hardy 1918].

But otherwise post-syllogistic logic, especially mathematical logic from Frege onwards, normally deploys only the 'external' mode of negation, where an entire proposition R is negated. (For example, the discussion of DN in sub-section 2.3 above dealt with the external negation of H.) But when an adjectival or adverbial clause within R is 'internally' negated, then R has to be rephrased to draw only upon external negations, maybe of a pair of propositions. This form is often found in mathematics: an example drawn from the theory of functions just noted is the (untrue) proposition 'An infinite series of functions that is not non-uniformly convergent over a finite interval of values cannot be integrated term-by-term', which becomes 'If an infinite series of functions is not non-uniformly convergent over a finite interval of values, then it cannot be integrated term-by-term'.

However, we cannot so easily convert propositions within which a subproposition *or* a subsidiary infinitive verb is negated. Perhaps these situations have been ignored because they do not entrain new connectives; but this attitude both underrates their importance and is out of line with the recognition that all modes of all connectives, and also both propositional quantifiers,

[6] De Morgan [1847], vii (on Euclid), 3–6, 59–63 (on negations).

have their own scopes, which must feature in rules for well-forming propositions (for instance, the two modes of disjunction).

3.2. *Exegesis of the modes*

We take again
H := 'Hardy plans to publish an edition of his papers'.
H has no built-in negations and one verbal clause, and is to be distinguished from 'Hardy publishes that he plans an edition of his papers'. There are three modes of negation:

External: Hex := 'Hardy does not plan to publish an edition of his papers', symbolised 'not-H'. The whole proposition is negated, with the negation acting on the principal verb.

Internal: Hin := 'Hardy plans not to publish an edition of his papers', symbolised '*not*-H'. Some subsidiary verb or clause is negated.

External-internal: Hexin := 'Hardy does not plan not to publish an edition of his papers' (maybe expressing a feeling of uncertainty about the project), symbolised 'not-*not*-H'.
With this quartet is associated an octet of asserted propositions.

If only external negation were available, then we must resort to clumsy locutions to reformulate internal modes; for example, Hin by the ungainly conjunction 'Hardy plans an action A to happen' and A := 'not-(publish an edition of his papers)'. But this is very unsatisfactory; the distinction between logic and syntax needs a reappraisal [Lycan 2001], especially now when so many non-bivalent logics are available. The category of negation should be desimplified into negations.

3.3. *Further properties of negations*

1) There is no internal-external mode of negation ('Hinex'). The laws of well formation of a proposition forbid us from internally negating the external negation of a proposition R, for the latter negation limits rewording of R to text surrounding the principal verb; the rest of the proposition is immune. Akin to the lack of assertion of a sub-proposition, this property plays a key role in our analysis.

2) If R has only one verb, as in
h := 'Hardy publishes an edition of his papers',
then it has only one negation, which therefore is external; but it becomes internal in its assertions $\pm$h, 'it is (un)true that Hardy publishes an edition of his papers'. This effect is called 'demotion'.

3) If a proposition contains more than one predicate and/or subordinate clause, such as 'Hardy plans to publish an edition of his papers and to reprint

his textbook', then it takes more than one internal negation, each of which has its own external-internal mode.

4) A compound proposition can admit negations of either kind in any of its constituent sub-propositions: for example, in the implication 'if R then S' both sub-propositions R and S can be externally and internally negated.

4. *Three case studies of assertion and negations*

The rest of this article examines three situations in L2 that involve different modes of negating both asserted and unasserted propositions. After a reformulation of the propositional and some other paradoxes, we examine indirect proof-methods, especially proof by contradiction, when the theorem involved is asserted; the relationship between indirect and direct methods in general is studied. The final case study considers whether theorems in mathematics are construed as implications *or* (*or* or?) as inferences. The trio is presented as a sequence; no general logical conclusion is drawn from them.

To avoid obvious repetitions, sometimes the points and distinctions made are stated in terms only of unasserted propositions, but versions pertain also to asserted ones. For reason of space we continue normally to confine the study to propositions; but there are consequences concerning sentences (propositions expressed in some natural or semi-formal or programming language) and statements (utterances of sentences), which are the focus of many current discussions of logic, such as handling paradoxes.

5. *Case study one: Framing logical paradoxes*

Logic is self-referential: when we apply L2 to itself, we find, among other features, the ancient propositional paradoxes. We examine the versions for asserted propositions; but, as stated in sub-section 2.5, we do not examine any solutions *or* accommodations of paradoxes and contradictions that have been proposed, usually for the versions using unasserted propositions.[7]

5.1. *The liar paradox* follows from the unasserted proposition

L := 'This proposition is untrue'.

Assume L to be true, and deduce that it is untrue; assume L to be untrue, and deduce that it is true; voilà. It has been studied often since at least Greek

[7] Self-reference is especially evident in the case of dialethic logic, where paradoxes are welcomed as propositions that are both true and untrue ([Priest 1995] and elsewhere).

antiquity, and indeed has become quite fashionable again;[8] a wide variety of variants is available [Smullyan 1978]. However, drawing upon a well-known distinction in the philosophy of language [Donnellan 1966], we note that L is not an assertion at all but only a self-attribution. So we consider the assertions in ML2

+L := 'It is true that this proposition is untrue' and

−L := 'It is untrue that this proposition is untrue'.

We obtain the paradox from the rule of self-contradiction. From +L derive +not-L, and so from the rule infer the denial −+L, or −L; effect the same upon −L and infer − −L, or +L: conjoin the two denials, and deduce that L is (true and untrue).

5.2. *The Cretans' paradox*

This is another well-studied ancient example. Involving universal propositional quantification, it can be stated as

C := 'every proposition possessing some non-paradoxical property Y is untrue, and this proposition possesses property Y',

or, for short, 'Every Y-proposition is untrue'. It is called 'Cretan' because in ancient times Y was 'asserted by a Cretan in a given time period'.

When C is asserted in ML2 the paradox disappears, for

+C := 'It is true that every Y-proposition is untrue'

is untrue since, among other reasons, it allows both any proposition P and not-P to be true; while

−C := 'It is untrue that every Y-proposition is untrue'

is true for corresponding reasons.

In both propositions, and also in the unasserted proposition C, an interesting example of self-reference arises when we instantiate the quantifier in C to deduce from it the liar proposition L; but we cannot infer that L is true, for the antecedent proposition C is untrue and so admits any proposition as consequent by the property of explosion. So L does not affect the respective truth-values of +C and −C. This is an instance of another feature of paradoxes that is often overlooked; namely, the effect that one of them may have upon others. Here is another example.

5.3. *The strengthened liar paradox*

This one has excited a wide range of reactions since its introduction in the mid 1960s.[9] It is usually stated as

[8] See, for example, [Field 2008] and [Beall 2007], and various other books written or edited by the latter author.

[9] See, for example, [Rieger 2001], and *passim* in [Beall 2007].

S := 'This proposition is not true',

where 'not true' is not a synonym for 'untrue' but involves some *or* all of the indefinitely iterable truth-states (as we call them) that arise from considering paradoxes: for example, 'true and untrue', 'neither true nor untrue' and 'neither true nor (neither true nor untrue)'. Thus for any truth-state U 'not U' means 'one of the other available states'. The adjective 'strengthened' reflects the ability of S to resist some of the solutions that apply to the liar paradox. As usual, the unasserted version based upon S has dominated the interest, but +S and $-$S are also available, and take the truth-state (not-true and not-untrue). The Cretans' paradox can be re-analysed similarly.

Also awaiting attention are the (un)asserted propositions that draw upon this repertoire of iterable truth-states. For two such states U and V, the schema is 'It is U that this proposition is V'; examples include 'This proposition is true and untrue' and 'It is not true that this proposition is untrue *or* (neither true nor untrue)'. The propositions in which U=V include $-$L. Most of these propositions do not readily have settings, but it is worth distinguishing those that are paradoxical from those that are not.

5.4. *On paradoxes involving sets and negations*

From the logical point of view the 'paradox of the greatest ordinal number' in transfinite arithmetic is not a paradox but a proof by contradiction of the theorem 'It is true that there exists no largest ordinal'; assume that it is the number O, and prove that both O + 1 $\leq$ O and O + 1 > O. (DN affirms 'It is true that either such a number exists or it does not exist'.) The same analysis pertains also to the two "paradoxes" of the greatest cardinal number, both the version that assumes that the sequence $\{\aleph_r\}$ of alephs that starts with $\aleph_0$ has a final member, and the one that makes the same claim about the sequence $\{(2^\wedge)^{r\wedge}\aleph_0\}, r > 1$ created by using the power-set construction iteratively to exponentiate $\aleph_0$.

Russell, Hardy's senior colleague at Trinity College Cambridge, bestowed paradoxhood upon these theorems, in his keenness to find paradoxes in the foundations of mathematical logic and set theory [Russell 1903, 362–368]. He created his own genuine paradox by applying the power-set construction to the set of all sets and concluding that 'It is true that the set X of all non-self-belonging sets self-belongs if and only if X does not self-belong' [1903, ch. 10]. But he did not analyse the version that obtains with asserted propositions: for this purpose, set up

B := 'X self-belongs' and not-B := 'X does not self-belong',

analyse the quartet of propositions $\pm$B and $\pm$not-B, and show that both B and not-B are (true and untrue).

6. *Case study two: On indirect proof-methods in mathematics*

6.1. *The proof-method by contradiction*

One of the oldest indirect proof − methods is that by contradiction; as Hardy put it, '*reductio ad absurdum*, which Euclid loved so much, is one of a mathematician's finest weapons' [1940, 94].[10] His remark bears especially upon [Euclid *Elements*, Book 10], which contains a remarkable range of theorems about commensurable and incommensurable magnitudes in geometry, often proved by this method.

The context of Hardy's remark was another famous ancient theorem, in arithmetic, which has also attracted many philosophers and mathematicians. In its asserted version it states

+Q := 'It is true that $\sqrt{2}$ is an irrational number',

that is, 'It is true that no positive co-prime integers a and b exist such that $\sqrt{2}$ = a/b'. The proof starts by affirming the internal negation, 'It is true that some positive co-prime integers a and b exist such that $\sqrt{2}$ = a/b' (after cancelling out the double negation); obtains a contradiction;[11] concludes that *any* proposition can be proved, but rejects this consequence by presuming that arithmetic is consistent; then deduces the theorem. However, the proof has delivered neither the double denial '− −Q' nor the double external negation 'not-not-Q', but the denial of the external negation of Q (the connective that was demoted to the internal negation of +Q), 'It is untrue that $\sqrt{2}$ is not an irrational number', symbolised '−(not-Q)'. To obtain +Q we must add DN, *or* some equivalent such as 'It is true that every number is either rational or irrational'; '$\sqrt{2}$ exists' is not quite enough, since it allows for the existence of infinitesimals.[12]

6.2. *Other indirect proof-methods*

A striking special version of this proof-method, sometimes called 'self-contradiction', occurs when the proof of theorem R creates as the contradiction the proposition (if not-R, then R). Whitehead and Russell [1910, *2·01] named *as* the proof-method by contradiction the unasserted proposition

[10] Hardy's opinion carries weight, not only because of his eminence but also his considerable familiarity with both the history and the philosophy of mathematics [Grattan-Guinness 2001].

[11] Prove $2b^2 = a^2$; then either deduce that both a and b are even integers after all, or assume the prime factorisation theorem and note that this equation equates an odd number of prime numbers with an even number of them.

[12] Note that the issues raised here differ from the rejection by intuitionist and constructivist mathematicians of the law of excluded middle. On the contrary, the law is retained but different doubts are raised.

'If (if R, then not-R), then not-R'.

Church called this a '*Special law*' of the method [1956, 142], the full version for him being

'If (if R, then S), then (if (if R, then not-S), then not-R)'

for any proposition S. Both authors presented the rules as implications, not as inferences; Russell notoriously muddled the two together, but Church is surprising.

An inferential version of this rule provides another analysis of the liar paradox. Firstly, deduce $-$L from +L, and interpret it as a self-contradictory inference, from which $-$+L follows; secondly, deduce +L from $-$L, from which $- -$L follows: conjoining,

'it is untrue that +L is untrue' and 'it is untrue that $-$L is untrue'.

A variation on this proof-method, Pierre de Fermat's proof by infinite descent of some theorem T(n) in number theory, assumes as theorem not-T(n) for some positive integer n and then impossibly "proves" it for an infinite descending sequence of positive integers smaller than n, thus forcing the conclusion not-not-T(n).

Other indirect proof-methods include the rule of contraposition,

'If R, then S; hence if not-S, then not-R'.

When not-S is affirmed and not-R inferred, this rule is the *modus tollendo tollens* rule of inference. The ancient proof-method known from the late 16th century onwards as 'consequentia mirabilis' [Angelelli 1975], is akin to Church's full law:

'(If R, then S) and (If not-R, then S) and (R *or* not-R); hence S'.

6.3. *On the relationship between indirect and direct proof-methods*

Another issue is whether or not a theorem is proved indirectly or directly in the first place. Georg Cantor's two proofs of the existence of transcendental numbers, commonly thought to be existential, are in fact so constructive that algorithms can be generated from them [Gray 1994].

The issue is addressed in general in a fine discussion [Hardy (M.) and Woodgold 2009] of Euclid's theorem on the number of prime numbers. To prove *directly* that any finite set S of prime numbers is not complete he exhibited a prime number n that did not belong to S ([*Elements*, Book 9, Proposition 20], with a proof confined to three given numbers). But his theorem has long been commonly re-interpreted as saying that the set F of all prime numbers is infinite; and his direct proof is converted into one by contradiction, in which we assume that F is a finite set and use the construction of n as the counter-example. This version is stated in many books on number theory and/or the history of mathematics, of which M. Hardy and Woodgold provide a remarkable bibliography (including [Hardy 1940, 92–94]).[13]

The authors do not explicitly discuss the role of negations; indeed, they accept as correct the proof of the irrationality of $\sqrt{2}$. But their discussion beautifully highlights the relationship between direct and indirect proof-methods, and the role of modes of negation both in stating theorems and methods of proving them.

Occasionally attempts have been made to convert indirect proofs into direct ones in a general way. Motivated by his dislike of the reliance of indirect proofs on untrue propositions, Bernard Bolzano offered a lengthy commentary in 1837, but his solution did not much advance much beyond seeing some role for contraposition.[14]

Much later the German logician Leopold Löwenheim was more successful in a paper [1946].[15] Firstly, he assumed whatever is required by the mathematical context, including definitions, by forming propositions of which the conjunction is A. Then he proved proposition B indirectly by adjoining not-B to A and deriving a proposition W 'which is known false', such as a contradiction [p. 126]. To convert this proof into a direct one, he drew upon assertion (in the verbal versions 'is (un)true') and contraposition. The indirect proof of B establishes

[(A and not-B) is true; hence W is true];
hence, by contraposition, [W is untrue; hence (A and not-B) is untrue];
but W is indeed untrue;
hence by *modus ponens*, [(A and not-B)) is true];
hence by a De Morgan law, [(not-A or B) is true];
but by definition of A, [not-A is untrue];
hence [B is true]

as desired, and moreover established directly.

Among various elaborations and alterative strategies Löwenheim advocated 'artificial devices', including 'numbers which eventually proved to be 0'; and his example was the conversion of the indirect proof of the infinitude of prime numbers into a direct proof of the theorem that 'the number n of all sets which are finite and contain all prime numbers is 0' [p. 139]. Thus his little-known paper belongs to the heritage of M. Hardy and Woodgold.

[13] M. Hardy and Woodgold also point out that that, after forming the product P(N) of the first N prime numbers in order of magnitude starting from 2, some authors claim that P(N) + 1 is a prime number; but the first counter-example arises with the product to 17.

[14] [Bolzano 1837, art. 530]; English translation in [1972, 385–392]. For discussion see Hölder [1929, 1930], and for meta-discussion [Mancosu 1996, 105–118].

[15] The work was done around 1917, but the original manuscript is lost [Thiel 1977, 243–244]; I cite an English translation made by Quine.

7. *Case study three: Implication and inference in logic and mathematics*

We treat here a very untidy department of logic, with material implication, logical implication, inference, presupposition, entailment, consequence and deducibility often muddled together, not least by logicians and mathematicians; in sub-section 6.2 we noted an example in Russell and Church. [Corcoran 1973b] lists a dozen senses of implication alone. We follow orthodoxy in regarding material and formal implication as belonging to L2 and associated with the conditional connective, and move up to ML2 and focus upon deduction, especially rules of inference.

7.1. *Theorems as implications?*

In all branches of mathematics a theorem is often stated as an unasserted implication 'If S then T', where S is the conjunction of the assumed sufficient conditions required to prove theorem T, which is the last line of the deduction from S. Mathematicians are well aware of the logic involved in extending the theorem by weakening S to conditions that themselves imply S (for example, that T holds for every integer and not just for positive ones). They are less likely to notice that they (might well have) proved, at least implicitly, the corresponding *modus ponens* inference of asserted propositions 'It is true that S; it is true that if S then T; hence it is true that T'.[16] If the theory involved is axiomatised and X is the axiom system, then 'T is true' can be replaced by 'T follows logically from the axioms'.

If either situation arises then a proof gap emerges, which is filled by the 'deduction theorem', proved by Tarski and others around 1930 [1941, 125–130]: for unasserted propositions it states that 'if T is deducible from X, then (if X then T) is deducible'. Corresponding remarks hold for the (meta-) theorems 'If T then N' involving necessary conditions N, and 'T if and only if F' for necessary and sufficient conditions F. The Bourbakists, normally antipathetic to logic, did note the deduction theorem in their proofs; but many mathematicians would regard it as obvious. However, the obvious is not to be disdained: there is a history of proving "obvious" theorems in mathematics, such as the unique factorisation of integers in number theory, the intermediate value theorem in mathematical analysis, and the Jordan curve theorem in topology.

Finally, a task pursued in both logics and mathematics is the search for methods of shortening or simplifying deductions and proofs. [MacLane 1934] pioneered an attempt to systemise them, and now they constitute part

[16] This point is not made in two books on 'logic for mathematicians', [Rosser 1953] and [Hamilton 1988].

of the concerns of automated proof- and theorem-checking algorithms in logics, mathematics and computer science.

7.2. *Beware rules of inference!*

A proof-gap may arise, in any setting, when the *modus ponens* rule of inference is used to deduce a proposition which itself is an implication [McGee 1985]. Symbolically, for propositions R, S and T drawn from the setting the rule states that

It is true that R; It is true that, if R then (if S then T); hence
It is true that (if S then T).

The risk is that ‘if S then T’ may be true when it is governed by the first implication involving R but loses this status when on its inferred own. For example, let

R := ‘Hardy is pleased’,
S := ‘Hardy has not proved the Riemann hypothesis’ and
T := ‘Hardy has completed his book [1940]’.

The second proposition in the rule reads

‘if Hardy is pleased, then if Hardy has not proved the Riemann hypothesis then Hardy has completed his book [1940]’,

which seems to be plausible; but the second implication on its inferred own as the third proposition lacks credibility, for no information is furnished that links hypothesis with book.

The converse of this point bears upon another mathematical practice. Let S be the conjunction of explicit assumptions with which the prover of T is content. Now there may also be controversial ones (such as a parallel axiom in Euclidian geometry, or an axiom of choice in set theory) that need special emphasis, and others that he has not noticed (such as continuity principles with Euclid himself). Let these latter two kinds be conjoined as R; then the theorem actually takes the form ‘If R then (if S then T)’, which is the second proposition above when assertions are introduced.

The same points apply to the *modus tollendo tollens* rule of inference, which now reads:

It is true that R; It is true that if not-(if S then T) then not-R; hence it is true that (if S then T).

8. *Reconciliation*

This exposition of assertion and expansion in the modes of negations carries notable effects on logical knowledge, while the trio of case studies bear especially upon mathematics among the various settings. So maybe it can help reduce the long-standing cleft in which mathematicians regard logicians as fussing over obvious details, while logicians see mathematicians fumbling over refined subtleties [Corcoran 1973a]. Despite the recent increase in interactions of these disciplines with each other and with computer science and artificial intelligence, the cleft is still noticeable.

As for the distinction between logic and its settings themselves, I have adapted and adopted several of the features of this article to attempt to characterise logical knowledge.[17]

ACKNOWLEDGEMENTS

For discussions over the years I thank John Corcoran, Albert Lewis, Michael Scanlan and José Miguel Sagüillo.

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[17] [Grattan-Guinness 2012b]. A much more elaborate philosophical and historical review [Grattan-Guinness 2012a] of the matters addressed in both articles builds upon a recent comparison of mathematics and logics in terms of different kinds of generality that theories within them exhibit [Grattan-Guinness 2011b].

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