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SYMBOLS AND THEIR MEANING IN ANALYSIS

HARTLEY SLATER

Perhaps the most influential paradox about continua coming down from antiquity concerns The Arrow. The paradox of The Arrow considers a specific moment in time, and seems to show that there is no motion in that moment. For, Zeno argued, an arrow in flight occupies just the space it displaces. So, for the time it is in any place, it is not moving. Aristotle's answer to this was that neither motion nor rest could be assessed at a specific time, since each required measurement over some discrete, i.e. finite interval (Tiles 1989, 18):

> For, Aristotle argues (Physics VI, 8), from the fact that at any instant t during its flight the arrow does not move, it does not follow that the arrow is at rest at t. For, he says, motion and rest are terms which apply only in relation to periods of time. Since there can be no motion in an instant (all movement takes time) there can be no rest in an instant either. To qualify for being at rest, as opposed to being in motion, an object must occupy the same place for a period of time.

Nevertheless, some have still wanted to say that the arrow does have a velocity at each individual time, and just how we are to understand this notion of instantaneous velocity thus becomes a central question.

In this paper I defend Aristotle and Finitism on these matters, criticising in a new way the notion of instantaneous velocity so closely associated with the development of the calculus. Finitism, as it is here taken, is a cluster of views centring on the claim that there are no completed infinities, only potential ones. Amongst other things, as we shall see in the final section of this paper, it is precisely through defining irrational numbers in terms of the potential infinities involved in open intervals on the geometric line that we avoid the usual conclusions about irrational 'points' completing the rational ones on that line. In getting to this conclusion I look at some of the traditional philosophical questions about derivatives, from Berkeley's arguments against the emerging calculus of Newton, through the ideas in 'Smooth Infinitesimal Analysis' to another recent line of argument, which Robinson formulated, to make rigorous Leibniz' ideas about infinitesimals. That leads me to consider the nineteenth century thinkers, Dedekind, Cauchy and Weierstrass, who in stages formalised the notion of a limit, using the 'epsilon-delta' method, and established the belief in real numbers. The 'epsilon-delta' method is retained in a modified form, but twentieth century Finitists, such as Tiles, and Van Bendegem, in the tradition of Aristotle, have questioned whether real numbers provide an appropriate foundation for anything. Constructivists first tried improving on the mainline tradition in Analysis, but their doubts about real numbers were not thoroughgoing, since constructivists still countenance 'constructive reals'. The arguments given here allow not only constructive reals and classical reals, but also the two above varieties of infinitesimals their place, but point out in each case that that place is not where it is normally thought to be. The collective points enable us to settle on a finitary Analysis, and so a calculus using arbitrarily small, but still measurable units.

1

The philosophical problem at the base of the calculus arises in the calculation of the derivative of a function, say the function $y = x^2$, at the point (x_o, y_o). One supposes a small increment Dx is added to the value of x, and first calculates what the resultant value of y will be — here $(x_o + Dx)^2$. That means there has been an incremental growth in y of

$$(x_o + Dx)^2 - x_o^2,$$

i.e.

$$x_o^2 + 2x_oDx + Dx^2 - x_o^2,$$

which is

$$2x_oDx + Dx^2.$$

Calling this 'Dy' we can then estimate the gradient of the curve at (x_o, y_o), as Dy/Dx, which is

$$2x_o + Dx.$$

On one understanding of the matter, that gives us a final value of $2x_o$ for the gradient of the curve exactly at (x_o, y_o), by taking Dx to be zero. But with respect to the comparable calculation Berkeley found in Newton, Berkeley asked how 'Dx' can at one time be non-zero, to allow the division of

'$2x_o Dx + Dx^2$' by 'Dx', but then zero, to produce from '$2x_o + Dx$' the exact derivative '$2x_o$' (see, e.g. Jesseph 1993, 194):

> Hitherto I have supposed that x flows, that x hath a real Increment, that $[Dx]$ is something. And I have proceeded all along on that Supposition, without which I should not have been able to have made so much as one single Step. . . I now beg leave to make a new Supposition contrary to the first, i.e. I will suppose that there is no Increment of x, or that $[Dx]$ is nothing; which supposition destroys my first, and is inconsistent with it, and therefore with every thing that supposeth it.

John Bell rightly seconds Berkeley's objection, making what is effectively Dx non-zero while Dx^2 is zero. In the above case that makes Dy equal to simply $2x_o Dx$ and allows the division by Dx to yield $2x_o$ as the final value for the derivative, supposedly without any question. For a start, Bell has this to say about the foundations of his 'smooth infinitesimal analysis' (Bell 1998, 6):

> If we now call two points a, b on the real line *distinguishable* or *distinct* when they are not identical, i.e. *not* $a = b$ — which as usual we shall write $a \neq b$ — and indistinguishable in the contrary case, i.e. if *not* $a \neq b$, then. . . indistinguishability of points will not in general imply their identity. As a result, the 'infinitesimal neighbourhood of 0' comprising all points indistinguishable from 0 — which we will denote by I — will. . . be nonpunctiform in the sense that it does not reduce to $\{0\}$, that is it is not the case that 0 is the sole member of I.

It is items in the 'infinitesimal neighbourhood of 0' that are supposed to replace 'Dx'. But if it was simply *indistinguishability* that was involved, then one might think that some modal operator could be used to express it. That would be quite in tune, also, with the Intuitionistic Logic that Bell uses, and the well-known modal interpretation of this logic that Gödel, for instance, produced (Gödel 1969). Something might be distinguishable from 0, i.e.

$$L(x \neq 0),$$

and something might be indistinguishable from 0, i.e.

$$\neg L(x \neq 0),$$

and then the indistinguishability of two points would not imply their identity, i.e.

$$\neg L(x \neq 0),$$

would not imply

$$x = 0.$$

With respect to the calculation of the derivative above that would mean we can simply say that $\neg L(Dx \neq 0)$, which means that we can consistently assume $Dx \neq 0$, with Dx an ordinary small quantity, and so obtain that $Dy/Dx = 2x_o + Dx$ without any problem. And that in turn, given $\neg L(Dx \neq 0)$, means that Dy/Dx is simply *indistinguishable* from $2x_o$, i.e. $\neg L(Dy/Dx \neq 2x_o)$. Moreover, if higher powers of Dx are involved in other, more complex cases there is then no requirement to identify them with zero. For if Dx is a small quantity then Dx^2, for instance, is smaller still, and so $\neg L(Dx \neq 0)$ entails $\neg L(Dx^2 \neq 0)$, etc. for the operator in question.

It is at this point, though, that one realises there is a considerable problem with Bell's use of the term 'indistinguishability'. For there might also be *multiples* of Dx and Dx^2 etc. remaining in Dy/Dx, and it is not so easy to see how one could justify $\neg L(Dx \neq 0)$ entailing $\neg L(nDx \neq 0)$ for arbitrary n. Enlargement clearly can lead to discriminations previously unavailable. So it starts to dawn that Bell's story about indistinguishability and his use of Intuitionistic Logic is really hiding the true rationale for his Smooth Infinitesimal Analysis. For it is by making his infinitesimals 'nilsquare' — which is quite another matter — that he gets round this problem with multiples of small quantities. The numerator of a quotient involved in calculating the gradient of a curve will in general have terms in all powers of Dx. Subsequent division by Dx is going to give the desired gradient, but with other additional terms involving powers of Dx. If these additional terms were all zero then one could claim to have derived the required gradient at the point, but that means making zero all terms in the previous numerator involving the *second* power of Dx. So making Dx^2 zero is an extra, additional stipulation beyond anything about 'indistinguishability'. It does not have a philosophical justification in the terms of the indistinguishability and Intuitionistic Logic that Bell spends so much time explaining. And clearly there are plenty of philosophical questions about 'nilsquares' if one looks not at the practical efficiency of Bell's procedure but at its possible interpretation. This is not to bring into doubt that the procedure is consistent, and so, from quite general logical considerations will have a model; indeed it must have a model. The question is: *does it have the right model*? Is there a non-zero *length* whose square is zero, i.e. which is a 'nilsquare'? For remember, in connection with the derivation of the gradient of the curve for the function $y = x^2$ there is a well-known diagram. Do Bell's infinitesimals fit into this picture? They do not. One can draw a straight line through two adjacent points on the curve, and one can draw the supposed tangent that touches the curve at just one point. But in neither case are there non-zero lengths whose squares are zero. In the one case there is a non-zero length, Dx, in the other case there is a length Dx whose square is zero — but only because the length itself is zero. The reason is simply that the diagram shows 'Dx' as a *length*, and non-zero lengths are not things whose squares could be zero quantities.

Bell certainly shows 'how elementary calculus and some of its principal applications can be developed within smooth infinitesimal analysis in a simple algebraic manner, using calculations with nilsquare infinitesimals in place of the classical limit concept' (Bell 1998, 15). But his algebraic procedures with his 'nilsquare infinitesimals' are not dealing with appropriate entities. We shall see this even more clearly with regard to Robinson's infinitesimals.

2

Robinson's Non-standard Analysis has very close similarities with Bell's Smooth Infinitesimal Analysis. For Robinson's style of analysis brings in infinitesimals simply by tackling the problem of multiplication and enlargement in another way. And that does not connect up with the right model, either. It is not now that the relevant infinitesimals have squares that are zero, but instead that they are non-Archimedean, being in magnitude less than any positive real, or rational number. These infinitesimals arise through extending the field of (standard) real numbers to the field of 'hyperreals' which (Robinson 1969, 154–5):

> contains non-trivial infinitely small (infinitesimal) numbers, i.e. [non-zero] numbers a such that $| a |< r$ for all standard positive r. (0 is counted as infinitesimal, trivially) ... If a is any finite [hyper]real number then there exists a uniquely determined standard real number r, called the standard part of a such that $r - a$ is infinitesimal or, as we shall also say, such that r is infinitely close to a...

The derivative is then defined as the standard part of 'Dy/Dx', and in finding its value we can avoid letting 'Dx' ever be zero (Keisler 1976a, 28, see also Hoskins 1990, 104):

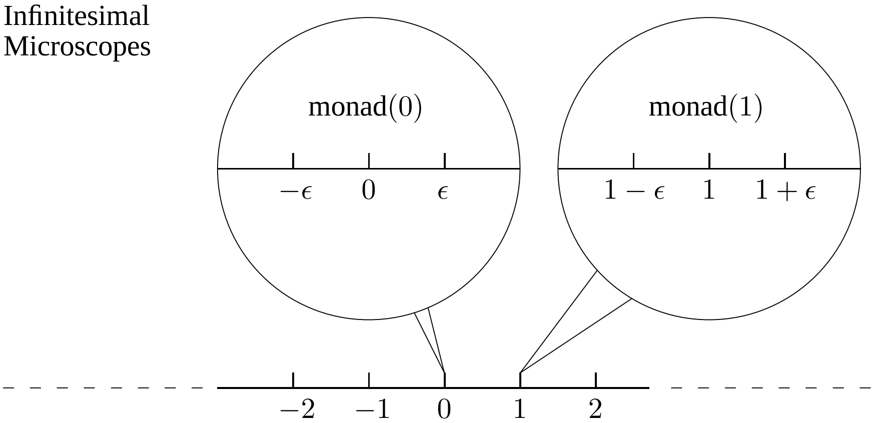
> Consider a real point (x_o, y_o) on the curve $y = x^2$. Let Dx be either a positive or a negative infinitesimal (but not zero), and let Dy be

> the corresponding change in y. Then the slope at (x_o, y_o) is defined in the following way:
>
> [slope at $(x_o, y_o)] = $ [the real number infinitely close to Dy/Dx].
>
> We compute...
>
> $$Dy/Dx = ((x_o + Dx)^2 - x_o^2)/Dx = 2x_o + Dx.$$
>
> This is a hyperreal number, not a real number. Since Dx is infinitesimal, the hyperreal number $2x_o + Dx$ is infinitely close to the real number $2x_o$. We conclude that
>
> [slope at $(xo, yo)] = 2x_o$.
>
> The process can be illustrated by the picture in Figure 1.4.5, with the infinitesimal changes Dx and Dy shown under a microscope.

But it is not the impossibility of a large enough microscope that makes Keisler's last remark quite inappropriate. It is the *impossibility* of what he tried to picture, on pages 27–29 of this book, namely, amongst other things, a line with minus epsilon, then zero, and then epsilon spaced out equally on it. For, of course, these lengths are Archimedean, and after enough extension would be longer than any measure that was previously specified. And it is not an accident that Keisler promoted this kind of incorrect visual picture. His 'Elementary Calculus' is a student text, and so it might be thought that his error is easily excused as just a heuristic, pedagogic device to help students get the hang of the technical processes. But no: the pictures are repeated in Keisler's more theoretical work 'Foundations of Infinitesimal Calculus' (Keisler 1976b, 3). So Keisler, astonishingly, really believed that spaced out items are what he was talking about, even though it is by definition impossible! One can see what is leading him into his confusions and contradiction: he clearly realises that *some* spatial interpretation of his symbolism is needed for it to be relevant to the traditional problem, yet *none* can be available.

So Keisler's last aside brings in a quite fatal philosophical objection to these procedures. As we saw in the quotation from Robinson, his positive infinitesimals when they are non-zero are still supposed to be less than any positive real number — and that means, amongst other things, that they cannot have a representation in graphical terms. *There cannot be any separation at all* between his minus epsilon and zero, or between zero and epsilon, otherwise it will break with their definition. So how is one to construe relations between these things? How can, for instance, $2x_o + Dx$ be *greater than* $2x_o$,

Infinitesimal
Microscopes



if there is no spatial separation between them? No answer of the required kind is forthcoming. It may be questioned, therefore, whether this theory of infinitesimals is in any way better, philosophically, than Newton’s theory of ‘fluxions’. If it takes Dx to be zero, the calculus has to meet Berkeley’s concerns. But if the calculus takes Dx to be a non-zero infinitesimal, in the manner of Robinson (or Bell), then it is no better, since while a consistent mathematical procedure for calculating derivatives is certainly then available (and so certainly some model for the infinitesimals), it is no longer talking about lengths and quantities, and so has no *graphical* interpretation.

Moreover, there is a clear mathematical reason why Robinson’s procedures are akin to Newton’s. For the derivative is defined by Robinson as the standard part of ‘Dy/Dx’, but standard parts of quotients are only obtainable if the standard part of the denominator is not zero (Keisler 1976a, 40). Thus the rule is that

If $st(b) \neq 0$ then $st(a/b) = st(a)/st(b)$.

In the cases in question, however, the standard part of ‘Dx’ is zero, so there is a repeat of Berkeley’s point against Newton. For ‘Dy/Dx’ now has to be put into a form where the denominator does not have a zero standard part before the calculation of the derivative can proceed. A worked practical example showing how this has to happen is to be found in Keisler 1976a, 42–3. For what is

$$st(c^2 + 2c - 24/c^2 - 16),$$

when $st(c) = 4$, and $c \neq 4$? The denominator has a zero standard part:

$$st(c^2 - 16) = st(c^2) - 16 = 4^2 - 16 = 0.$$

So the standard part of the quotient is not obtainable from this form of the quotient. On the other hand the numerator and denominator here have a common factor '$c - 4$', and if that is cancelled out then what becomes the form of the quotient has a standard part by the above rule:

$$st(c+6/c+4) = (st(c)+6)/(st(c)+4) = (4+6)/(4+4) = 10/8.$$

But that makes the supposed identity between

$$2x_oDx + Dx^2/Dx,$$

and

$$2x_o + Dx,$$

incomplete, since the standard part only of the latter form is obtainable directly under the rule. The idea is supposed to be that the two forms are forms of the same entity, and so the standard part worked out from the latter form will also be the standard part in connection with the former form. But if one must first derive the standard part of the latter in order to derive the standard part of the former, then the facts involved are notational facts about the given forms of the quotient. Indeed where is the specific *quotient* the two forms are supposed to be forms of? There is 'no entity without identity' and in this case the identity is incomplete.

3

Now, the association between the derivative and motion, initiated by Newton's use of the term 'fluxion', was largely confined to England, while on the continent Leibniz' conception of infinitesimals had more hold. But both lost ground to Cauchy's and Weierstrass' definitions of the derivative in terms of limits, which is perhaps still the most common approach today. In fact Newton would seem to have had the idea of such limits, although he did not formalise it (Tiles 1989, p76). There are, in all, three theoretical options about how to understand calculations of a derivative like the paradigm one above — other than taking Dx to be (contradictorily) 0. Firstly one can avoid going to the limit, in the manner of Aristotle, and do 'analysis without actual infinity' (Mycielski 1981, see also Lavine 1994). The Dx is then effectively taken to be a non-negligible quantity of arbitrarily small size — which has the consequence that there is no strict derivative at a point, merely over an interval. Alternatively one can take Dx to be an infinitesimal in the way Bell or Robinson made formally rigorous. Or, thirdly, one can speak instead, with Cauchy and Weierstrass, of the derivative as merely the limit of '$2x_o + Dx$', and the like, as Dx tends to zero. Robinson, in fact, proves that derivatives arrived at in his non-standard way are equivalent to derivatives determined by such limits (Robinson 1969, 155). So if Robinson's, and Bell's procedures are philosophically unacceptable, what about Cauchy and Weierstrass'?

If one proceeds in some constructivist manner (e.g. Bishop 1967, Martin-Löf 1970, Troelstra 1977) one is doing no better than Cauchy and Weierstrass, with respect to the points that follow, so I will not consider that approach separately. What I shall concentrate on therefore is just the assessment of the Cauchy-Weierstrass definition of the derivative.

In fact there is next to nothing wrong with the Cauchy-Weierstrass definition of the derivative. It is a little remarked feature of that definition that it is easily made compatible with Finitism, and it is only other aspects of Analysis done the Cauchy-Weierstrass way, notably the theory of real numbers, that are really incompatible with this philosophical position. Separating out those other aspects allows a reconciliation to be made between Aristotle, Berkeley, Cauchy and Weierstrass at least on the foundations of the calculus. But gathering all these thinkers together also enables us to see more clearly just what is not to the point with the Cauchy-Weierstrass theory of real numbers.

The point about compatibility in the limited area of the derivative is very quickly established, since if one merely says that the limit of $2x_o + Dx$ is $2x_o$ as Dx tends to zero, then one can hold off saying that Dx ever is zero, and so whether the limit is reached, i.e. whether there is instantaneous velocity, or a derivative exactly at a point. So the Cauchy-Weierstrass epsilon-delta method for determining the derivative can be saved, so long as it is re-interpreted this way. What confuses the issue, of course, is that *in other areas of Analysis* one of the great achievements of the Cauchy-Weierstrass approach is said to be that it established that certain limits *are* reached. That the appropriate limits are reached is just the feature that supposedly shows that the real numbers are complete, for instance. But the trouble with the Cauchy-Weierstrass' account of real numbers is that Finitists say that such numbers do not actually exist, which I am taking to mean that from a Finitist perspective 'real numbers' are ideal elements in Hilbert's sense, i.e. formalistic parts of the symbolism that cannot be given any appropriate interpretation. As we shall see, the proper support for this Finitist position in this area lies again in adjusting the notion of limits present in the Cauchy-Weierstrass theory.

Grattan-Guinness explains how crucial it is to Weierstrass's approach that it is about 'real' numbers (Grattan-Guinness 1980, 141):

> As part of his programme Weierstrass introduced a definition of irrational numbers. One motivation for the definition was to make sure that infinitesimals could be dispensed with entirely, although some of the Weierstrassians affirmed forms of actual infinitesimals. More important was the need to avoid the question-begging proofs by his predecessors of theorems, such as the intermediate value theorem, on the existence of limits. An important lemma in the proof of such theorems was the 'Bolzano-Weierstrass theorem', which asserted that an infinite bounded set (of real numbers) contains at least one limit point.

The parenthesis here '(of real numbers)' is what is crucial, since the intermediate value theorem and the Bolzano-Weierstrass theorem do not hold for the rationals, for instance. Centrally the real numbers are 'complete' whereas the rationals are not. We shall see there is a better way of putting the latter matter later, but one common way of illustrating the incompleteness of the rationals is to point out that a succession of fractions which have as their limit $\sqrt{2}$ or 2π do not contain this limit point, since $\sqrt{2}$ and 2π are irrational. The rational numbers are 'dense', i.e. between any two there is a third, but they are said to be not 'continuous', which Dedekind found a definition for in terms of his 'cuts' (Grattan-Guinness 1980, 222). A 'cut' in a densely ordered system is a pair of classes of its elements which exhaust the system, and which are such that every element of one class precedes every element of the other. A system is said to be continuous if, for every cut, there is an element that is either the maximum of the lower class, or the minimum of the upper class. Any cut in the rational numbers not produced by a rational number is taken to be produced by an irrational number, in order to obtain the totality of the 'real' numbers.

But the problem with Dedekind's definition of the real numbers, which supports relevant aspects of Finitism in this area, is that the cut $\{x : x^2 < 2\}$, for instance, is neither greater than 1 nor less than 2, and so it cannot be a number in this region. Indeed, as a consequence, it is not a number at all, since it could only be in this region if it was in fact a number. For sure, all of the members of this cut are less than 2, but that does not mean that the set of those members (in the mathematical sense of 'set') is less than 2. That set is strictly incommensurable with any number like 1 and 2; it is only comparable, for instance, with other sets of numbers like $\{x : x > 1\}$ and $\{x : x < 2\}$. I have laid out in fine detail the parallel feature of the sets involved in the Cauchy-Weierstrass definition of real numbers, at the end of Slater 2006. What one primarily has to remember is that while the decimal expansion of a rational or irrational number naturally generates partial sums that form a Cauchy sequence of rationals, the number itself, i.e., *the sum* *of the decimal increments*, is not a Cauchy sequence of rationals. Nor is it the equivalence class of such a sequence, which is the definition of a real number from the Cauchy-Weierstrass point of view (c.f. Suppes 1972). So real numbers, in neither of the above the above senses, have a representation on a geometric line. Fascination with the mathematics of Dedekind cuts, and equivalence classes of Cauchy sequences of rationals no doubt is what has blocked sight of the fact they are *not the relevant items to attend to*. Just as Bell's and Robinson's infinitesimals were consistent and had models, but not the right model in terms of quantities, lengths and entities, so Dedekind's, Cauchy's and Weierstrass' 'real numbers' are certainly consistent, and have models, but not models consisting in rational and irrational numbers.

4

But still, it may be said, surely irrational numbers, with their non-terminating decimal expansions, must have a place on the geometric line alongside the rational points. The diagonal of a unit square, and the circumference of a unit circle may easily be compared, linearly, with the unit they are measured against. Are not these irrational numbers the limits of sequences of rationals, which not only complete the rationals, but also show thereby that the rationals are incomplete? It is here that the emendation flagged above, in connection with the statement about the incompleteness of the rationals, has to come in. For what corresponds to an irrational number like $\sqrt{2}$ or 2π is not a *closed* interval, as with the rationals, but an *open* interval. Only in this way is the endlessness of the decimal expansion of an irrational number properly paralleled. There are certainly other intervals on the line if there are rational ones, but those others starting from 0 (say) are not distinguished by ending at a different set of points. Instead they are distinguished by being open intervals, and so by not ending anywhere. They might end *by* a certain point but not *at* that point.

It is the failure to appreciate this difference that has led to many of the mysteries exposed in Zeno's paradoxes of motion (c.f. Slater 2000). One must first remember, for instance, such facts as that $1 + 1/2 + 1/4 + \ldots$ is not equal to 2. For even though 2 is the limit of the partial sums (and these partial sums form a Cauchy sequence of rational numbers) this limit is never reached but only approached with increasing nearness. So how does Achilles catch the Tortoise? He gets to $1/2$ of the distance away, then to $1/4$ of the distance away, etc., and in doing so may seem to need to go on for ever. But what allows him to catch up is not the fact that the successive small increments might themselves take increasingly smaller units of time, as is sometimes said. For this 'infinite task' will still not get him to the meeting point: he will have just covered an open interval, and still not have got exactly to where the Tortoise is. As above, this task might end *by* a certain point, but not *at* that point. So how does Achilles catch up? What must be noticed to solve the paradox is that at any of the points along the way all Achilles needs to do to catch up the Tortoise is traverse the remaining, complementary closed interval to the limit point of the open interval (c.f. Chihara, 1965). It may be asked: how does he do that? But the answer to this is that he does it the same way as he traversed all the incremental, closed intervals before. It might be thought that this is just an *ad hominem* response, but it is still applicable to the immediate point, and we shall see later that there is indeed something more to be said about the basic assumption of the paradox that closed intervals can be traversed. For the problem is set up assuming that traversing closed intervals can be achieved in a finite way, and the paradox arises by the considering that one such finite achievement must be the result of an infinite task. But the supposed infinite accumulation of tasks on the way to the goal gets one nowhere, i.e. to no place from which one can take a further step to the goal, while only a finite part of that accumulation is needed as a basis for successfully reaching the goal, if a closed interval can be traversed, since at any point in the accumulation there remains only a further closed interval before the goal is reached.

Irrational measures, therefore, are not the limits of sequences of rationals in the sense that they are end points just beyond them; indeed it is exactly the failure of appropriate sequences to come to an end — because their decimal representations are non-terminating — that makes irrational measures not those further points. We are assuming, of course, that rational intervals can be located on a line, but while there are other intervals there, so the rationals are certainly 'incomplete', those other intervals are simply open intervals, which do not end (both ways) at points.

Still, what is wanted but has not yet been obtained, is a proper confirmation of the kind of point Aristotle originally made (Tiles 1989, 17):

> . . . the Aristotelian will first deny that points are parts of time; they are not limits of division and cannot be reached by division. No extended whole, no continuous magnitude can be made up out of what has no magnitude, for adding together two things of no magnitude cannot increase their size.

Asking for the sum of the number of points on a line, from this point of view, is therefore like asking for the number of angels that can sit upon a pin. For what has no extension — such as, *ex hypothesi*, a real point — has no part in the extensional, i.e. physical world. Thus we know from the above that *if* Achilles can reach the half-way point, then all he has to do is repeat this kind of achievement to catch up with the Tortoise. But how can he even reach the half-way point if he has to traverse all the open infinity of points in between, and that doesn't get him right to the spot? The full truth, of course, is that there are no such *points* in between, and not even a half-way point.

There is something further we have yet to see in connection with this, but at least it is clear that if the above is true then it resolves one remaining problem with Finitist views of the calculus. For if one considers only things that have an extension, then, in connection with derivatives, one merely gets, following Cauchy and Weierstrass, the determination of rates of change over ever-decreasing, finite intervals, providing no strict derivatives at a point. But not only are there no strict derivatives at a point, there are no *points*, either, in the required sense of the word. Certainly, when we estimate some quantity to within a certain degree of error, say a certain number of decimal places, the belief might be that reality itself has no such latitude. So maybe a Finitist should not talk about 'estimates', if this word implies there is some final, exact answer to be approximate about. But it need not have that implication, and instead need mean merely that, while the estimate is accurate to the given number of decimal places, still more accuracy may be obtained, in a process that is endless.

But there is a last point that has to be added to this before we can get the matter entirely clear. The point that has not been mentioned so far is that there are only *approximations* even to the rationals on a geometric line. The rationals can only be understood *exactly* in terms of the ratios of whole numbers of discrete elements, and so not in connection with a 'continuum'. Thus if there are 12 people in a room and 4 of them are female then there are half as many women as men, etc. That is a fact about the rational number $1/2$, and it is an exact fact, but it could only arise with discrete items such as those illustrated. By contrast there are properly no exact rational intervals on a line, i.e. points that are a rational measure apart. A Finitist account of the calculus, and Analysis more generally, is the only account that respects this fact.

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Dept. of Philosophy
The University of Western Australia
Nedlands, Western Australia 6009
E-mail: hartley.slater@uwa.edu.au

REFERENCES

Bell, J.L. 1998, *A Primer of Infinitesimal Analysis*, C.U.P. Cambridge.
Bishop, E. 1967, *Foundations of Constructive Analysis*, Mc-Graw-Hill, New York.
Chihara, C. 1965, 'On the Possibility of Completing an Infinite Process', *Philosophical Review* 74.1, 74–87.
Gödel, K. 1969, 'An Interpretation of the Intuitionistic Sentential Logic', in J. Hintikka (ed.) *The Philosophy of Mathematics*, O.U.P. Oxford.
Grattan-Guinness, I. 1980, *From the Calculus to Set Theory 1630–1910*, Duckworth, London.
Grattan-Guinness, I. 2000, *The Search for Mathematical Roots 1870–1940*, Princeton U.P., Princeton.
Hoskins, R.F. 1990, *Standard and Non-Standard Analysis*, Ellis Norwood, Chichester.
Jesseph, D.M. 1993, *Berkeley's Philosophy of Mathematics*, University of Chicago Press, Chicago.
Keisler, H.J. 1976a, *Elementary Calculus*, Prindle, Weber and Schmidt, Boston.
Keisler, H.J. 1976b, *Foundations of Infinitesimal Calculus*, Prindle, Weber & Schmidt, Boston.
Kneale, W. and Kneale, M. 1962, *The Development of Logic*, O.U.P., Oxford.
Lavine, S. 1994, *Understanding the Infinite*, Harvard University Press, Cambridge MA.
Martin-Löf, P. 1970, *Notes on Constructive Mathematics*, Almqvist and Wiksell, Stockholm.
Mycielski, J. 1981, 'Analysis without Actual Infinity', *Journal of Symbolic Logic*.
Priest, G.G. 1979a, 'The Logic of Paradox', *Journal of Philosophical Logic*.
Robinson, A. 1969, 'The Metaphysics of the Calculus', in J. Hintikka (ed.) *The Philosophy of Mathematics*, O.U.P., Oxford.
Slater, B.H. 2000, 'Finite Achievements', *Logique et Analyse* 171–2, 341–344.
Slater, B.H. 2006, 'Grammar and Sets', *Australasian Journal of Philosophy* 84, 59–73.
Suppes, P. 1972, *Axiomatic Set Theory*, Blackwell, Oxford.
Tiles, M. 1989, *The Philosophy of Set Theory*, Blackwell, Oxford.
Troelstra, A.S. 1977, *Choice Sequences: A Chapter in Intuitionistic Mathematics*, Clarendon, Oxford.
Van Bendegem, J.P. 2010, 'Finitism in Geometry'. *The Stanford Encyclopedia of Philosophy*, Edward N. Zalta (ed.), url = http://plato.stanford.edu /entries/geometry-finitism/.

Van Bendegem, J.P. 2011, 'The Possibility of Discrete Time', in Craig Callender (ed.), *The Oxford Handbook of Time*, O.U.P., Oxford, 145–162.