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ON THE CONNECTION BETWEEN NONSTANDARD ANALYSIS AND CONSTRUCTIVE ANALYSIS

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Abstract
Constructive Analysis and Nonstandard Analysis are often characterized as completely antipodal approaches to analysis. We discuss the possibility of capturing the central notion of Constructive Analysis (i.e. algorithm, finite procedure or explicit construction) by a simple concept inside Nonstandard Analysis. To this end, we introduce Ω-*invariance* and argue that it partially satisfies our goal. Our results provide a dual approach to Erik Palmgren's development of Nonstandard Analysis inside constructive mathematics.

1. *Introduction: Two questions*

When comparing Nonstandard Analysis and Constructive Analysis, it is hard not to get blinded by the differences between the two. Indeed, the usual construction of the hyperreal field ${}^*\mathbb{R}$ involves an ultrafilter on $\mathbb{N}$, the existence of which is justified by appealing to the full axiom of choice (Kanovei and Reeken, 2004). The latter[1] is well-known to imply the principle of excluded middle (Diaconescu, 1975), the original sin of classical logic according to constructivist (and intuintionist) canon. Thus, the very basis of Nonstandard Analysis is seemingly rejected by the constructivist.

Furthermore, Nonstandard Analysis also seems problematic at a more conceptual level, from the constructivist point of view. Indeed, Errett Bishop, the founder of Constructive Analysis (Bishop, 1967), famously derided Nonstandard Analysis for its lack of 'computational content'.

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[1] The *countable* axiom of choice *is* constructively acceptable (Bishop, 1967; Ishihara, 2006).

> A more recent attempt at mathematics by formal finesse is non-standard analysis. I gather that it has met with some degree of success, whether at the expense of giving significantly less meaningful proofs I do not know. My interest in non-standard analysis is that attempts are being made to introduce it into calculus courses. It is difficult to believe that debasement of meaning could be carried so far. (Bishop, 1975, p. 513)

Ironically, Bishop was asked to review Keisler's introduction to Nonstandard Analysis (Keisler, 1976). The final sentence of Bishop's review sums up his views on Nonstandard Analysis quite well.

> Now we have a calculus text that can be used to confirm their experience of mathematics as an esoteric and meaningless exercise in technique. (Bishop, 1977, p. 208)

It should be noted that Bishop's views are not necessarily shared by other constructivists. For instance, Arend Heyting spoke highly of Abraham Robinson's Nonstandard Analysis (Heyting, 1973).

Despite this proverbial 'rocky start', there have been reconciliatory attempts between the communities of Nonstandard and Constructive Analysis. In particular, a conference entitled *Reuniting the antipodes* was organized in Venice in 1999 to bring together the two communities (Schuster et al., 2001). However, in (Van Oosten, 2006), the review of (Crosilla and Schuster, 2005), Van Oosten notes that little 'reunification' had taken place. Nonetheless, he also suggests a notable exception: in (Palmgren, 2001), Erik Palmgren develops some Nonstandard Analysis in a constructive system. Other results in this area include (Richman, 1981, p. 208), (Moerdijk and Palmgren, 1997) and (Palmgren, 1996a; 1996b; 1997; 2000). It should be noted that in the constructive approach to Nonstandard Analysis, objects may have 'strange' (i.e. non-classical) behaviour. A good example is the presence of nonzero nilpotent infinitesimals in certain constructive logical systems.

In this paper, we take the dual approach to the above: we investigate the possibility of formalizing basic notions from constructive mathematics *inside* classical Nonstandard Analysis. For instance, the notion of *algorithm* is central to constructive mathematics. Is there a definition in Nonstandard Analysis which captures this notion? Similarly, as the connectives in constructive mathematics are intuitionistic (Bridges, 1999, p. 96), do these have counterparts in Nonstandard Analysis?

For this paper, we limit ourselves to the following questions.

(1) Is there a (simple) notion in Nonstandard Analysis that captures Errett Bishop's notion of algorithm?

(2) How will we judge if the correspondence in the previous item is any good?

We first treat the second question in the next section. As noted in Remark 38 below, we do not attempt to capture the equally central constructive notion of 'proof' inside Nonstandard Analysis.

2. *The second question*

2.1. *The illusive notion of algorithm*

In this section, we formulate a partial answer to the second question in Section 1, i.e. we formulate a criterion that allows us to judge how good the correspondence is between Bishop's notion of algorithm and a potential (nonstandard) counterpart. Finding such a criterion is non-trivial, as Bishop nowhere exactly defines the notion of algorithm. We first discuss the various reasons for this omission.

First and foremost, by keeping the notion of algorithm vague, any result proved in Constructive Analysis is also a theorem of classical mathematics (called 'CLASS'), of intuitionistic mathematics (called 'INT') and Russian constructive mathematics (called 'RUSS'). In other words, by not committing to a particular definition of algorithm, Bishop's ensures a greater generality for his Constructive Analysis. The following quote by Douglas Bridges reflects this idea.

> Although Bishop has been criticised for being too vague in his concept of algorithm, by this very vagueness he left open the possibility of interpreting his work within a variety of formal systems. Not only is every theorem of BISH also a theorem of recursive constructive mathematics — which is, roughly, recursive function theory developed with intuitionistic logic — but it is also a theorem of Brouwer's intuitionistic mathematics, and, perhaps more significantly, of classical mathematics. (Bridges, 1999, p. 2)

A second reason for leaving the notion of algorithm vague may be found in (Bishop, 1985). Bishop argues that the 'naive' notion of algorithm is more basic and fundamental than e.g. the well-known notion of recursive function. Hence, we should forego the identification of algorithm and recursive function. The following quote by Bishop reflects this idea.

> [The recursive function theorists] admit only sequence of integers or rational numbers that are recursive (a concept we shall not define here: see (Kleene, 1952) for details). Their reasons are, that

> the concept is more precise than the naive concept of algorithm, that every naively defined algorithm has turned out to be recursive, and it seems unlikely we shall ever discover an algorithm that is not recursive. This requirement that every sequence of integers must be recursive is wrong on three fundamental grounds. First and most important, there is no doubt that the naive concept is basic, and the recursive concept derives whatever importance it has from some presumption that every algorithm will turn out to be recursive. (Bishop, 1985, p. 20)

Although Bishop has good reasons for leaving the notion of algorithm vague, the fact of the matter is that we do not have a direct definition of this fundamental entity. In this way, it seems difficult to judge whether any notion captures Bishop's notion of algorithm. Nonetheless, we *do* have access to an *indirect* definition of algorithm, discussed now.

In his writings, Bishop lists a large number of principles he deems unacceptable in his Constructive Analysis. We will refer to these principles as *non-algorithmic* or *non-constructive*. A well-known example is the *limited principle of omniscience*, which is an instance of the principle of excluded middle.

Principle 1: (LPO) *For every* φ *in* Δ_0, *we have* $(\exists n \in \mathbb{N})\varphi(n) \vee (\forall n \in \mathbb{N})\neg\varphi(n)$.

As intuitionistic logic is used in Constructive Analysis (Bridges, 1999, p. 96), LPO is interpreted as *there is a finite procedure which decides the truth of any existential statement*. As such a procedure would allow us to decide the truth of Goldbach's conjecture (and a slew of other famous open problems in mathematics), it seems highly unlikely that anyone will ever construct such a device. This is the reason behind the rejection of LPO (and therefore the law of excluded middle) in intuitionistic and constructive mathematics. Thus, by showing that a certain mathematical theorem implies a non-algorithmic principle, we can show that this theorem cannot be proved in Constructive Analysis. The reduction of a theorem to a non-algorithmic principle is called a 'Brouwerian counterexample'. We refer to (Mandelkern, 1989) for an overview of the latter.

It is intuitively clear that, by considering a large number of non-algorithmic principles and theorems, we obtain an *indirect* qualification of the notion of algorithm: algorithms are those procedures that are strictly weaker than all non-algorithmic techniques. Thus, if a given notion $\mathbb{X}$ captures Bishop's primitive of algorithm, then $\mathbb{X}$ should give rise to the same class of non-algorithmic principles. For instance, LPO should also be non-algorithmic compared to $\mathbb{X}$ in the same way as it is for Bishop's primitive of algorithm. The same should hold for all non-algorithmic principles (in the sense of Bishop) and we arrive at the following (preliminary) criterion.

> For a formal notion $\mathbb{X}$ to capture Bishop's primitive of algorithm, all non-algorithmic principles should be interpreted as principles not derivable using $\mathbb{X}$.

Note that this definition is not circular, as 'non-algorithmic' is defined as the finite list of principles rejected in BISH. In order to work with this criterion, it is clear that we need a good overview of a large number of non-algorithmic principles and theorems, and their connections. Such is provided by the discipline *Constructive Reverse Mathematics*, introduced in Section 2.2. Inspired by these results, we will formulate a more detailed criterion.

2.2. *Introducing Constructive Reverse Mathematics*

In this section, we sketch an overview of the discipline *Constructive Reverse Mathematics* (CRM). This survey of CRM will allow us to refine the criterion formulated in the previous section. In order to describe CRM, we first need to briefly consider Errett Bishop's *Constructive Analysis*.

Inspired by L.E.J. Brouwer's famous foundational program of intuitionism (van Heijenoort, 1967), Bishop initiated the redevelopment of classical mathematics with an emphasis on *algorithmic* and *computational* results. In his famous monograph *Foundations of Constructive Analysis* (Bishop, 1967), he lays the groundwork for this enterprise. In honour of Bishop, the informal system of Constructive Analysis is now called 'BISH'. In time, it became clear to the practitioners of Constructive Analysis that intuitionistic logic provides a suitable logical basis for BISH:

> Now, our experience shows that when we *do* constructive mathematics, we are actually doing mathematics with intuitionistic logic. The desire for algorithmic interpretability forces us to use intuitionistic logic, and that restriction of our logic seems to result, inevitably, in arguments that are entirely algorithmic in character. (Douglas Bridges, (Bridges, 1999, p. 97); See also (Bridges and Vîţă, 2006, p. 7).)

In (Richman, 1990), Fred Richman has expressed a similar opinion. Hence, the meaning of the logical connectives in BISH differs from the 'usual' one in classical mathematics. The following interpretation of the logical connectives is found in (Bridges, 1999, p. 96) and (Bridges and Vîţă, 2006, p. 8).

Definition 2: (Connectives in BISH)

(1) The disjunction $P \vee Q$: we have an algorithm that outputs either P or Q, together with a proof of the chosen disjunct.
(2) The conjunction $P \wedge Q$: we have both a proof of P and a proof of Q.
(3) The implication $P \to Q$: by means of an algorithm we can convert any proof of P into a proof of Q.
(4) The negation $\neg P$: assuming P, we can derive a contradiction (such as $0 = 1$); equivalently, we can prove $P \to (0 = 1)$.
(5) The formula $(\exists x)P(x)$: we have (i) an algorithm that computes a certain object x, and (ii) an algorithm that, using the information supplied by the application of algorithm (i), demonstrates that $P(x)$ holds.
(6) The formula $(\forall x \in A)P(x)$: we have an algorithm that, applied to an object x and a proof that $x \in A$, demonstrates that $P(x)$ holds.

Evidently, the notion of *algorithm* is central to Constructive Analysis. We refer the reader to (Bishop, 1967), (Bishop and Bridges, 1985) and (Bridges and Vîţă, 2006) for a more detailed introduction to the latter.

We now introduce Constructive Reverse Mathematics (CRM) and list some of its results. We follow Hajime Ishihara's survey paper (Ishihara, 2006). In effect, CRM is a spin-off from Harvey Friedman's well-known foundational program *Reverse Mathematics*. In the latter, the aim is to find the *minimal* axioms that prove a certain theorem of *ordinary*[2] mathematics. In many cases, the theorem is also *equivalent* to the minimal axioms, where this equivalence is proved in the weak 'base theory' RCA_0. Stephen Simpson's monograph *Subsystems of Second-order Arithmetic* is an excellent introduction to Reverse Mathematics (Simpson, 2009). In CRM, the base theory is (inspired by) BISH and the aim is to find the minimal axioms that prove a certain *non-constructive* theorem. As in Friedman-Simpson Reverse Mathematics, we also observe many equivalences between theorems and the associated minimal axioms in CRM.

We now provide an overview of important CRM results, based on Hajime Ishihara's survey paper (Ishihara, 2006). These results suggest that the non-constructive principles exhibit a lot of *logical structure*. Indeed, although all these principles are rejected in BISH, some have a higher non-constructive content than others. Thus, CRM provides (or aims to provide) an exact classification of the non-constructive content of various well-known principles and theorems. As we will observe, this classification exhibits a lot of logical structure.

[2] See (Simpson, 2009, p. 2) for a description of 'ordinary mathematics'.

First of all, recall the definition of the arithmetical hierarchy.

Definition 3: *For $k \geq 0$, we have the following.*

(1) *A formula is* bounded, *if every occurrence of quantifiers is of the form $(\exists n \leq t(\vec{x}))$ and $(\forall m \leq s(\vec{y}))$, where s and t are terms.*
(2) *A formula is Δ_0 (or Σ_0, or Π_0) if it is bounded and has no occurrences of infinite numbers or the predicate 'is infinite'.*
(3) *A formula is Π_{k+1} if it has the form $(\forall n \in \mathbb{N})\varphi(n)$ with $\varphi \in \Sigma_k$.*
(4) *A formula is Σ_{k+1} if if has the form $(\exists n \in \mathbb{N})\varphi(n)$ with $\varphi \in \Pi_k$.*

Next, we consider the following theorem regarding LPO.

Theorem 4: *In* BISH*, the following are equivalent.*

(1) LPO: $P \vee \neg P \quad (P \in \Sigma_1)$.
(2) LPR: $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$.
(3) MCT: *(The monotone convergence theorem) Every monotone bounded sequence of real numbers converges to a limit.*
(4) CIT: *(The Cantor intersection theorem).*

By Definition 2, all connectives are intuitionistic and hence, the meaning of the items in the previous theorem differs a lot from that in the classical framework. Indeed, item (2) is read, in BISH, as *there is a finite procedure to decide between $x > 0$ and its negation*. As '$x > 0$' is an existential statement in BISH (See Definition 19 below or (Bishop, 1967, Definition 3)), LPR seems to be a non-trivial principle. We will discuss LPR and MCT in more detail in Sections 3.2 and 3.1.

Next, we list equivalences of LLPO, the *lesser limited principle of omniscience*

Principle 5: (LLPO) *For every P, Q in Σ_1, we have $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$.*

Note that LLPO is an instance of De Morgan's law, and is rejected in BISH. Indeed, LLPO states that *if a proof of $P \wedge Q$ leads to contradiction, then we can decide whether P leads to contradiction or Q leads to contradiction*, and the existence of such a decision procedure is highly doubtful.

Theorem 6: *In* BISH*, the following are equivalent.*

(1) LLPO.
(2) LLPR: $(\forall x \in \mathbb{R})[\neg(x > 0) \vee \neg(x < 0)]$.
(3) NIL: $(\forall x, y \in \mathbb{R})(xy = 0 \rightarrow x = 0 \vee y = 0)$.
(4) CLO: *For all $x, y \in \mathbb{R}$ with $\neg(x < y)$, $\{x, y\}$ is a closed subset of $\mathbb{R}$.*
(5) IVT: *a version of the intermediate value theorem.*

(6) WEI: *a version of the Weierstraß extremum theorem.*

We will investigate the principle LLPR in greater detail in Section 3.4.

As the last of the omniscience principles, we consider WLPO, the *weaker limited principle of omniscience*.

Principle 7: (WLPO) *For every P in Σ_1, we have $\neg P \vee \neg\neg P$.*

Note that in BISH, the principle of 'double negation elimination' is not available: the formula Q does imply $\neg\neg Q$, but not the other way around. Hence, we observe that WLPO is weaker than LPO. We have the following theorem.

Theorem 8: *In* BISH*, the following are equivalent.*

(1) WLPO.
(2) WPR: $(\forall x \in \mathbb{R})[\neg(x > 0) \vee \neg\neg(x > 0)]$.
(3) DISC: *A discontinuous function from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ exists.*

Finally, we consider several versions of Markov's principle, named after the Russian mathematician Andrey Markov (Jr.). The status of Markov's principle is ambiguous in constructive mathematics. Although it is accepted in the Russian constructivist school, it is rejected in Bishop's Constructive Analysis and in intuitionistic mathematics. An interesting discussion of this topic may be found in (Bridges and Vîţă, 2006, p. 10–11).

First of all, we consider the usual version of Markov's principle, a version of double-negation elimination.

Principle 9: (MP) *For every P in Σ_1, we have $\neg\neg P \to P$.*

We have the following theorem.

Theorem 10: *In* BISH*, the following are equivalent.*

(1) MP.
(2) MPR: $(\forall x \in \mathbb{R})[\neg\neg(x > 0) \to x > 0]$.
(3) EXT: *(The strong extensionality theorem).*

Next, we consider a weaker principle: the *disjunctive version of Markov's principle*. The latter is also a (complicated) instance of De Morgan's law.

Principle 11: (MP$^{\vee}$) *For every P, Q in Σ_1, we have $\neg(\neg P \wedge \neg Q) \to \neg\neg P \vee \neg\neg Q$.*

Theorem 12: *In* BISH*, the following are equivalent.*

(1) $\mathrm{MP}^{\vee}$.
(2) $\mathrm{MPR}^{\vee}$: $(\forall x \in \mathbb{R})[\neg\neg(x \neq 0) \to \neg\neg(x > 0) \vee \neg\neg(x < 0)]$.
(3) $\mathrm{CLO}^{\vee}$: *For all* $x, y \in \mathbb{R}$ *with* $\neg\neg(x < y)$, $\{x, y\}$ *is a closed subset of* $\mathbb{R}$.

Note that '$x \neq y$' is short for the existential statement $|x - y| > 0$ and is *stronger* than the negative statement $\neg(x = y)$.

Finally, we consider a weaker principle: the *weak version of Markov's principle*.

Principle 13: (WMP) *For every decidable P, if for every decidable Q,*

$$\neg\neg[(\exists n)Q(n)] \vee \neg\neg[(\exists n)(P(n) \wedge \neg Q(n))],$$

this implies $(\exists n)P(n)$.

We have the following theorem.

Theorem 14: *In* BISH*, the following are equivalent.*

(1) WMP.
(2) WMPR: $(\forall x \in \mathbb{R})\big(\big[(\forall y \in \mathbb{R})(\neg\neg(0 < y) \vee \neg\neg(y < x))\big] \to x > 0\big)$.

The following theorem summarizes the relations between the above principles.

Theorem 15: *The following hold in* BISH*.*

(1) $\mathrm{LPO} \leftrightarrow \mathrm{WLPO} + \mathrm{MP}$.
(2) $\mathrm{WLPO} \to \mathrm{LLPO}$.
(3) $\mathrm{MP} \leftrightarrow \mathrm{MP}^{\vee} + \mathrm{WMP}$.
(4) $\mathrm{LLPO} \to \mathrm{MP}^{\vee}$.

Note that we only have selected a number of equivalences and theorems from Ishihara's survey paper (Ishihara, 2006). For instance, we have not considered the famous *fan theorem*. Nonetheless, even with this partial overview, we may conclude that the non-algorithmic principles *exhibit a lot of logical structure*: we observe 'degrees' of non-constructiveness among the non-constructive principles, rather than just one set of 'equally non-constructive' principles.

2.3. *An answer to the second question*

In the previous paragraph, we have observed that the non-algorithmic principles in Constructive Analysis exhibit a lot of structure. This observation allows us to refine the criterion by which we judge whether a certain notion captures Bishop's primitive of algorithm. Our preliminary criterion from Section 2 was the following.

> For a formal notion $\mathbb{X}$ to capture Bishop's primitive of algorithm, all non-algorithmic principles should be interpreted as principles not derivable using $\mathbb{X}$.

Our final criterion is as follows.

> For a formal notion $\mathbb{X}$ to capture Bishop's primitive of algorithm, all non-algorithmic principles should be interpreted as principles not derivable using $\mathbb{X}$. Moreover, the interpretations of the non-algorithmic principles satisfy the same implications and equivalences as their originals in Constructive Reverse Mathematics.

By the previous criterion, a certain formal notion $\mathbb{X}$ captures Bishop's notion of algorithm if we can use it to 'reverse engineer' the results of Constructive Reverse Mathematics. For the rest of the paper, we attempt to find such a notion $\mathbb{X}$ in Nonstandard Analysis. This notion will give rise to a certain interpretation of Constructive Analysis in Nonstandard Analysis. In Remark 29, we discuss the exact nature of this interpretation. In two words, the main goal of the rest of this paper is as follows: We define a notion called Ω-invariance inside Nonstandard Analysis, which is intended to capture Bishop's notion of algorithm. Rather than providing a 'literal' translation from BISH to Nonstandard Analysis, we show that Ω-invariance gives rise to the same kind of Reverse Mathematics results inside Nonstandard Analysis.

3. *The first question*

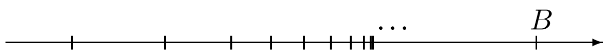
In this section, we explore the possibility of capturing Bishop's notion of algorithm by a simple notion from Nonstandard Analysis. For expository reasons, our presentation remains at the informal level. The reader only needs to be acquainted with the very basic notions of Nonstandard Analysis.

For the rest of this paper, we take $\mathbb{N} = \{0, 1, 2, \dots\}$ to denote the set of natural numbers, which is extended to ${}^*\mathbb{N} = \{0, 1, 2, \dots, \omega', \omega' + 1, \dots\}$, the set of *hypernatural* numbers, with $\omega' \notin \mathbb{N}$. The set $\Omega = {}^*\mathbb{N} \setminus \mathbb{N}$ consists of the *infinite* numbers, whereas the natural numbers are called *finite*. We tacitly assume that the domain of functions $f : \mathbb{N} \to \mathbb{N}$ can be extended to ${}^*\mathbb{N}$. We reason in an unspecified[3] system of Nonstandard Analysis which does not involve the transfer principle Σ_1-TR defined below, or stronger principles.

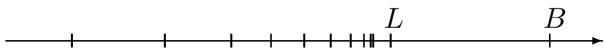
3.1. *The monotone convergence theorem*

We consider the following Brouwerian counterexample by Bishop concerning the monotone convergence theorem for sequences of reals.

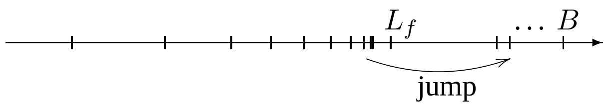
Example 16: (From (Bishop, 1985, p. 6)) We represent the terms of the sequence [in the monotone convergence theorem] by vertical marks marching to the right, but remaining to the left of the bound B.



The classical intuition is that the sequence gets cramped, because there are infinitely many terms, but only a finite amount of space to the left of B. Thus, it has to pile up somewhere. That somewhere is its limit L.



The constructivist grants that some sequences behave in precisely this way. I call those sequences stupid. Let me tell you what a smart sequence would do. It will pretend to be stupid, piling up at a limit, (in reality a false limit) L_f. Then when you have been convinced it really is piling up at L_f, it will take a jump and land somewhere to the right!



With this informal example, Bishop intends to cast doubt on the possibility that a *finite* procedure can compute the limit of a bounded increasing sequence. In other words, the example illustrates that it is impossible that we can prove MCT in BISH.

[3] The reader may check that a version of Nonstandard Analysis based on $I\Delta_0 + \exp$ suffices for our purposes. In general, a nonstandard version of RCA_0 (Simpson, 2009) seems to suffice.

To see that the monotone convergence theorem actually implies LPO in BISH, consider the following sequence,

$$z_n := \begin{cases} w_n & (\forall m \leq n)\psi(m) \\ w + \sum_{i=1}^{n} \frac{w+B}{2^i} & \text{otherwise} \end{cases}, \tag{1}$$

where ψ is Δ_0 and w_n is an increasing sequence below B, converging to $w < B$. By definition, z_n converges to w if and only if $(\forall n \in \mathbb{N})\psi(n)$. By MCT, we can decide if z_n converges to w or not. By the definition of z_n in (1), this allows us to decide if $(\exists n \in \mathbb{N})\neg\psi(n)$ or not, i.e. we have LPO. Moreover, the usual proof of MCT can be used to prove the implication LPO $\rightarrow$ MCT. Hence, MCT is equivalent to LPO.

In light of the equivalence between MCT and LPO, the following two remarks are important here.

First of all, in (Sanders, 2011), it is shown that a certain (complicated) version of MCT from Nonstandard Analysis is equivalent to the following principle, to be compared to LPO.

Principle 17: (Σ_1-TR) *For all* $\varphi \in \Delta_0$, *we have*

$$(\exists n \in \mathbb{N})\varphi(n) \vee (\forall n \in {}^*\mathbb{N})\neg\varphi(n). \tag{2}$$

The previous principle is the transfer principle of Nonstandard Analysis, limited to Σ_1-formulas. Note that Σ_1-TR is a kind of 'hyperexcluded' middle: it excludes the possibility that

$$(\forall n \in \mathbb{N})\psi(n) \wedge (\exists n \in {}^*\mathbb{N})\neg\psi(n), \tag{3}$$

for any $\psi \in \Delta_0$. Moreover, in (Moerdijk and Palmgren, 1997), it is shown that the full transfer principle implies the principle of excluded middle in intuitionistic logic. Hence, we suspect there to be some connection between Σ_1-TR and LPO.

Secondly, in the absence of Σ_1-TR, we cannot exclude that (3) holds for some $\psi \in \Delta_0$. In this case, the sequence z_n in (1) has exactly the behaviour depicted in Example 16. Indeed, z_n seems to converge to w for any finite $n \in \mathbb{N}$, but at some point, z_n jumps over w. Hence, there seems to be a connection between the *standard* version of MCT and Σ_1-TR.

In the following paragraph, we investigate these — admittedly vague — connections further by studying another famous principle equivalent to LPO. We finish this paragraph with a remark on Σ_1-TR.

Remark 18: We tacitly assumed that parameters $\vec{x}$ of natural numbers are allowed in φ in (2). Written out in full, the latter formula thus reads, for fixed $k \in \mathbb{N}$,

$$(\forall \vec{x} \in \mathbb{N}^k)[(\exists n \in \mathbb{N})\varphi(n, \vec{x}) \vee (\forall n \in {}^*\mathbb{N})\neg\varphi(n, \vec{x})].$$

For the rest of this paper, we will assume that such parameters are allowed everywhere. However, we usually omit parameters for aesthetic reasons.

3.2. *The constructive continuum*

In this paragraph, we study Brouwer's well-known theorem that the intuitionistic continuum cannot be split in two parts (van Heijenoort, 1967, p. 446). To this end, we need some definitions concerning real numbers in Constructive Analysis.

Definition 19:

(1) A real number x is a sequence $q_k : \mathbb{N} \to \mathbb{Q}$ such that

$$(\forall n, m \in \mathbb{N})\big(|q_m - q_n| < \tfrac{1}{m} + \tfrac{1}{n}\big). \tag{4}$$

(2) We write '$x > 0$' if $(\exists k \in \mathbb{N})(q_k > \frac{1}{k})$, and '$x < 0$' if $(\exists k \in \mathbb{N})((-x) > 0)$.
(3) We write '$x \geq 0$' if $(\forall k \in \mathbb{N})(q_k \geq -\frac{1}{k})$, and '$x \leq 0$' if $(\forall k \in \mathbb{N})((-x) \geq 0)$.
(4) We write '$x = 0$' if $x \leq 0 \wedge x \geq 0$.

Thus, in Constructive Analysis, a real number is a Cauchy sequence of rational numbers which converges quickly. The usual operations $+$ and $\times$ can be defined easily on the real numbers (Bishop, 1967, Definition 2).

Now consider the following principle.

Principle 20: (LPR) $(\forall x \in \mathbb{R})(x > 0 \vee \neg(x > 0))$.

With the above definition, it is clear that LPR has the same syntactical form as LPO: they both express the existence of a decision procedure for (certain) Σ_1-formulas and their negations. By (Ishihara, 2006, Theorem 1), LPO and LPR are indeed equivalent. Thus, LPR is rejected in Constructive Analysis, and, by Definition 2, there is indeed no way to (constructively) split the continuum in the two sets $(-\infty, x_0]$ and $[x_0, +\infty)$, for any $x_0 \in \mathbb{R}$.

We now study the connection between Σ_1-TR and LPR. The latter expresses that *we can decide, by means of a finite procedure, whether* $x > 0$ *holds or not*. Now, if Σ_1-TR is available, then the existential formula $x > 0$ is equivalent to the formula $(\exists k \leq \omega)(q_k > \frac{1}{k})$, for any choice of $\omega \in \Omega$. As the latter is a *bounded* formula, it is easy to verify whether it holds. Thus, we observe that if Σ_1-TR is available, then we can easily judge whether $x > 0$ holds or not (modulo a procedure to decide bounded formulas). Similarly, if we have $(\exists k \leq \omega)(q_k > \frac{1}{k})$ for *all* $\omega \in \Omega$, then we obtain, by underflow, $(\exists k \in \mathbb{N})(q_k > \frac{1}{k})$, i.e. $x > 0$.

In the previous paragraph, we observed that Σ_1-TR implies a version of LPR: given the former principle, we can decide if $x > 0$ or $\neg(x > 0)$ by considering $(\exists k \leq \omega)(q_k > \frac{1}{k})$. However, the most important observation to be made is that the choice of ω in the latter formula does not matter: by Σ_1-TR, the formula $(\exists k \leq \omega)(q_k > \frac{1}{k})$ is equivalent to $(\exists k \in \mathbb{N})(q_k > \frac{1}{k})$, for *any* choice of $\omega \in \Omega$.

Finally, we note that Σ_1-TR also yields a way to decide Σ_1-formulas. Indeed, in the same way as in the previous paragraphs, we have that $(\exists n \in \mathbb{N})\varphi(n)$ is equivalent to $(\exists n \leq \omega)\varphi(n)$, for any choice of $\omega \in \Omega$, if Σ_1-TR is available. Thus, Σ_1-TR provides a certain decision procedure for Σ_1-formulas, which is similar to the content of LPO.

In this paragraph, we obtained a more concrete connection between LPO and Σ_1-TR. Indeed, we observed that both give rise to a certain decision procedure for Σ_1-formulas. However, the most important observation was that, in the case of Σ_1-TR, the decision procedure does *involve* an infinite number ω, but that the procedure does not depend of the *choice* of $\omega \in \Omega$.

3.3. *Turing machines and independence*

In the previous paragraph, we hinted at a certain — still vague — notion of independence as the key to the connection between Σ_1-TR and LPO. To make this notion more precise, we now study a concrete example of computabilty: the *Turing machine*. We refer to (Soare, 1987) for an introduction to the latter.

By (Soare, 1987, Theorem 2.2, p. 64), the membership relation of a set A may be decided by a Turing machine, if and only if A is Δ_1, i.e. there are $\varphi_1, \varphi_2 \in \Delta_0$, s.t.

$$A = \{m \in \mathbb{N} : (\exists n_1 \in \mathbb{N})\varphi_1(n_1, m)\} = \{m \in \mathbb{N} : (\forall n_2 \in \mathbb{N})\varphi_2(n_2, m)\}. \tag{5}$$

We now show that Δ_1-sets satisfy the following very *concrete* independence condition.

Theorem 21: *For every* Δ_1*-set* $A \subset \mathbb{N}$*, there are* $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$*-functions* $p_A(n,m)$ *and* $q_A(n,m)$ *such that, for any fixed* $\omega \in \Omega$*, we have*

$$(\forall k \in \mathbb{N})[k \in A \leftrightarrow p_A(k,\omega) < q_A(k,\omega)]. \tag{6}$$

Proof. Assume A is Δ_1, i.e. we have (5) for some φ_1, φ_2 in Δ_0. Define $p_A(n,m)$ as the least $n_1 \le m$ such that $\varphi_1(n_1,n)$, if such exists and m otherwise. Let $q_A(n,m)$ be the least $n_2 \le m$ such that $\neg\varphi_2(n_2,n)$ if such exists and m otherwise. We now prove that p_A and q_A indeed satisfy (6).

First of all, fix $\omega \in \Omega$. For $k \in \mathbb{N}$, if $k \in A$, then $p_A(k,\omega)$ is finite and $q_A(k,\omega)$ is infinite, by (5). In particular, we have $p_A(k,\omega) < q_A(k,\omega)$. Now suppose there is some $k_0 \in \mathbb{N}$ such that $p_A(k_0,\omega) < q_A(k_0,\omega)$ and $k_0 \notin A$. By (5), we have $(\forall n_1 \in \mathbb{N})\neg\varphi_1(n_1,m_0)$ and, by definition, the number $p_A(k_0,\omega)$ must be infinite. Similarly, the number $q_A(k_0,\omega)$ must be finite. However, this implies $p_A(k_0,\omega) \ge q_A(k_0,\omega)$, which yields a contradiction. Thus, we have $k \in A \leftrightarrow p_A(k,\omega) < q_A(k,\omega)$, for all $k \in \mathbb{N}$. It is clear that we obtain the same result for a different choice of $\omega \in \Omega$. □

By the previous theorem, for every set $A \subset \mathbb{N}$ in Δ_1, there is a formula ψ in Δ_0 such that, for any fixed $\omega \in \Omega$, we have

$$(\forall k \in \mathbb{N})[k \in A \leftrightarrow \psi(k,\omega)].$$

In other words, the set A is fully described by a simple formula $\psi(n,\omega)$. Moreover, the description involves an infinite number ω, but is *independent* of the *choice* of $\omega \in \Omega$. Thus, we have found a *concrete* independence property which captures the Turing computable sets. Motivated by this results, we introduce the following definition.

Definition 22: *Let* $\psi(n,m)$ *be* Δ_0 *and fix* $\omega \in \Omega$*. Then* $\psi(n,\omega)$ *is* Ω-invariant *if*

$$(\forall n \in \mathbb{N})(\forall \omega' \in \Omega)(\psi(n,\omega) \leftrightarrow \psi(n,\omega')). \tag{7}$$

For $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$*, the function* $f(n,\omega)$ *is called* Ω*-invariant, if*

$$(\forall n \in \mathbb{N})(\forall \omega, \omega' \in \Omega)(f(n,\omega) = f(n,\omega')).$$

The following theorem shows that the (truth) value of an Ω-invariant object is already determined at some finite stage.

Theorem 23: *For every Ω-invariant formula $\psi(n,\omega)$, we have*

$$(\forall n \in \mathbb{N})(\exists m_0 \in \mathbb{N})(\forall m, m' \in {}^*\mathbb{N})[m, m' \geq m_0 \to \psi(n,m) \leftrightarrow \psi(n,m')]. \tag{8}$$

For every Ω-invariant function $f(n,\omega)$, we have

$$(\forall n \in \mathbb{N})(\exists m_0 \in \mathbb{N})(\forall m, m' \in {}^*\mathbb{N})[m, m' \geq m_0 \to f(n,m) = f(n,m')]. \tag{9}$$

In each case, the number m_0 can be computed by an Ω-invariant function.

Proof. This proof requires the techniques *underflow* and *overflow* from Nonstandard Analylsis, available even in very weak systems of Nonstandard Analysis (Im- pens and Sanders, 2008). For expository reasons, we do not go into details. □

In light of Theorems 21 and 23, it seems justified to claim that Ω-invariance (partially) captures the notion of 'algorithm' and 'finite procedure'. We will refer to Ω-invariant functions and formulas as $*$-computable functions, $*$-algorithms, $*$-finite procedures, $*$-decision procedures, etc., to avoid possible confusion with the original nomenclature.

3.4. *Reverse engineering Constructive Reverse Mathematics*

In this section, we use the notion of $*$-algorithm to obtain results similar to Theorem 4, inside Nonstandard Analysis. The theorems in these section are proved in an unspecified[4] system of Nonstandard Analysis which does not involve the transfer principle Σ_1-TR or stronger principles.

We first prove the following theorem.

Theorem 24: *Given* Σ_1-TR, *a $*$-algorithm can decide which disjunct holds in*

$$(\exists n \in \mathbb{N})\varphi(n,\vec{x}) \vee (\forall n \in {}^*\mathbb{N})\neg\varphi(n,\vec{x}), \qquad (\vec{x} \in \mathbb{N}^k, \varphi \in \Delta_0) \tag{10}$$

i.e. there is an Ω-invariant formula $\psi(\vec{x},\omega)$ such that

$$(\forall \vec{x} \in \mathbb{N}^k)(\psi(\vec{x},\omega) \to (\exists n \in \mathbb{N})\varphi(n,\vec{x}) \wedge \neg\psi(\vec{x},\omega) \to (\forall n \in {}^*\mathbb{N})\neg\varphi(n,\vec{x})). \tag{11}$$

[4] The reader may check that a version of Nonstandard Analysis based on $I\Delta_0 + \exp$ suffices for our purposes. In general, a nonstandard version of RCA_0 (Simpson, 2009) suffices.

Proof. By Σ_1-TR, the formula $(\exists n \in \mathbb{N})\varphi(n,\vec{x})$ is equivalent to $(\exists n \leq \omega)\varphi(n,\vec{x})$, for any fixed $\omega \in \Omega$ and $\vec{x} \in \mathbb{N}^k$. Also, if $(\forall n \leq \omega)\neg\varphi(n,\vec{x})$, then $(\forall n \in \mathbb{N})\neg\varphi(n,\vec{x})$, and we obtain $(\forall n \in {}^*\mathbb{N})\neg\varphi(n,\vec{x})$, by Σ_1-TR.

We define the function $f(\vec{x},m)$ as follows.

$$f(\vec{x},m) = \begin{cases} 1 & \text{If } (\exists n \leq m)\varphi(n,\vec{x}) \\ 0 & \text{If } (\forall n \leq m)\neg\varphi(n,\vec{x}) \end{cases}.$$

By the previous paragraph of the proof, we have

$$(\exists n \in \mathbb{N})\varphi(n,\vec{x}) \leftrightarrow f(\vec{x},\omega) = 1 \text{ and } (\forall n \in {}^*\mathbb{N})\neg\varphi(n,\vec{x}) \leftrightarrow f(\vec{x},\omega) = 0.$$

for any fixed $\omega \in \Omega$ and $\vec{x} \in \mathbb{N}^k$. In particular, this implies that $f(\vec{x},\omega)$ is Ω-invariant, and we have found a $*$-algorithm to decide (10), i.e. (11). $\square$

Recall that LPO is interpreted in Constructive Analysis as *there is an algorithm to decide whether* $(\exists n \in \mathbb{N})\varphi(n)$*, or its negation, holds*. Although the previous theorem is a step in the right direction, formula (10) is not quite the same as the disjunction in LPO. The following definitions (inside Nonstandard Analysis) will make (10) look more like LPO.

Definition 25: [$*$-disjunction] The formula $A \curlyvee B$ is short for the statement *There is an* Ω*-invariant formula* $\psi(\vec{x},\omega)$ *such that*

$$(\forall \vec{x} \in \mathbb{N}^k)(\psi(\vec{x},\omega) \to A(\vec{x}) \wedge \neg\psi(\vec{x},\omega) \to B(\vec{x})). \tag{12}$$

Note that $A \curlyvee B$ indeed implies $A \vee B$. In addition, there is an Ω-invariant formula which tells us which disjunct of $A \vee B$ holds. Hence, $A \curlyvee B$ indeed expresses *There is a* $*$*-algorithm that decides which disjunct of* $A \vee B$ *holds*.

Definition 26: [$*$-negation] For φ in Δ_0, the formula $\dashv[(\exists n \in \mathbb{N})\varphi(n)]$ is defined as $(\forall n \in {}^*\mathbb{N})\neg\varphi(n)$ and the formula $\dashv[(\forall n \in \mathbb{N})\varphi(n)]$ is defined as $(\exists n \in {}^*\mathbb{N})\neg\varphi(n)$.

Note that blocks of existential (resp. universal) quantifiers can be combined into *one* existential (resp. universal) quantifier. Of course, it is possible to define $*$-negation in full generality, but this would lead us too far.

The newly introduced connectives will also be called '$*$-connectives'. The previous definitions yield the following principle, to be compared to LPO.

Principle 27: ($\mathfrak{LPO}$) *For every* $\varphi \in \Delta_0$, $(\exists n \in \mathbb{N})\varphi(n) \curlyvee \dashv[(\exists n \in \mathbb{N})\varphi(n)]$.

The following remark shows that the $*$-negation is not just an aesthetic device, but behaves like its intuitionistic counterpart.

Remark 28: In intuitionistic logic, $\neg[(\exists n \in \mathbb{N})\varphi(n)]$ implies the universal formula $(\forall n \in \mathbb{N})\neg\varphi(n)$, but $\neg[(\forall n \in \mathbb{N})\varphi(n)]$ is weaker than the existential formula $(\exists n \in \mathbb{N})\neg\varphi(n)$. By Definition 2, the intuitive justification of this asymmetry is that, even if it is impossible that $\varphi(n)$ holds for all $n \in \mathbb{N}$, this does not provide a method to compute a counterexample. Similarly, $\dashv[(\exists n \in \mathbb{N})\varphi(n)]$ implies $(\forall n \in \mathbb{N})\neg\varphi(n)$, but $\dashv[(\forall n \in \mathbb{N})\varphi(n)]$ is weaker than the existential formula $(\exists n \in \mathbb{N})\neg\varphi(n)$. The latter formula can be stated as *there is a $*$-algorithm that computes* $n_0 \in \mathbb{N}$ *such that* $\neg\varphi(n_0)$. Indeed, if $(\exists n \in \mathbb{N})\neg\varphi(n)$, then $(\mu n \leq \omega)\neg\varphi(n)$ computes the least such number, in an Ω-invariant way. Furthermore, if there is a proof of the universal formula $(\forall n \in \mathbb{N})\varphi(n)$ in Nonstandard Analysis, then — under certain conditions — there is also a proof of $(\forall n \in {}^*\mathbb{N})\varphi(n)$. Such results are called *conservation results*. See e.g. (Avigad and Helzner, 2002, Theorem 4.4) for a textbook example. This partially explains the definition of $*$-negation.

At this point, we believe we should discuss the exact nature of the interpretation we are establishing between Constructive and Nonstandard Analysis.

Remark 29: In this paper, we will *not* obtain a literal translation of intuitionistic logic (or Constructive Analysis) inside Nonstandard Analysis. Moreover, we do not provide some version of the well-known *realizability interpretation*. Our aim is similar, but different: we define new connectives ($*$-disjunction and $*$-negation) which are based on Ω-invariance in the same way the intuitionistic connectives are based on the primitive notion of algorithm. The definitions of these new connectives are *inspired* by their intuitionistic counterparts, but, a priori, that is the only connection.

After introducing these new objects, we will translate a number of well-known principles (like LPO and LLPO) from CRM to Nonstandard Analysis, using the $*$-connectives. For the most part, this 'translation' consists in replacing the intuitionistic connectives with their nonstandard counterpart, i.e. the translation is usually purely mechanical. We prove that these translated principles (called $\mathfrak{LPO}$ and $\mathfrak{LLPO}$) satisfy the same equivalences in Nonstandard Analysis as their counterparts in CRM do. Whenever a mechanical translation was not possible for a given principle W in CRM (e.g. for the principle DISC or Π_1-LEM), we have used the *meaning* of W in BISH to obtain a reasonable counterpart $\mathfrak{W}$ of W in Nonstandard Analysis. It is beyond the scope of this paper to discuss examples of the latter sort. However, in a sense, the translation is both syntactic and semantic in nature.

Hence, the main goal of this paper becomes clear: We define a notion called Ω-invariance inside Nonstandard Analysis, which is intended to capture Bishop's notion of algorithm. Rather than providing a 'literal' translation from BISH to Nonstandard Analysis, we show that Ω-invariance gives rise to the same kind of Reverse Mathematics results inside Nonstandard Analysis.

We have the following theorem.

Theorem 30: *The principle* $\mathfrak{LPO}$ *is equivalent to* Σ_1-TR.

Proof. The reverse implication follows immediately from Theorem 24 and Definitions 25 and 26. For the forward implication, let φ be as in Σ_1-TR. By $\mathfrak{LPO}$, one of the disjuncts of

$$(\exists n \in \mathbb{N})\varphi(n) \vee (\forall n \in {}^*\mathbb{N})\neg\varphi(n)$$

must hold. This immediately implies Σ_1-TR. $\square$

By Theorem 4, LPO is equivalent to LPR. The latter principle closely resembles the following one.

Principle 31: ($\mathfrak{LPR}$) $(\forall x \in \mathbb{R})[x > 0 \vee \neg(x > 0)]$.

By Definitions 25 and 26, $\mathfrak{LPR}$ is the statement *there is a* $*$*-decision procedure for* $x > 0$ *and* $\neg(x > 0)$. This interpretation is similar to that of LPR in Constructive Analysis. We now prove the equivalence between $\mathfrak{LPO}$ and $\mathfrak{LPR}$, to be compared to Theorem 4.

Theorem 32: *The principle* $\mathfrak{LPO}$ *is equivalent to* $\mathfrak{LPR}$.

Proof. For the forward implication, recall that $x > 0$ is defined as $(\exists k \in \mathbb{N})(q_k > \frac{1}{k})$ and that $\neg(x > 0)$ is defined as $(\forall k \in {}^*\mathbb{N})(q_k \leq \frac{1}{k})$. By Theorem 30, we may use Σ_1-TR. By the latter, if $(\forall k \in \mathbb{N})(q_k \leq \frac{1}{k})$ then also $\neg(x > 0)$ follows. Moreover, the number x is a real, i.e. we have (4). By Σ_1-TR, we obtain

$$(\forall n, m \in {}^*\mathbb{N})(|q_m - q_n| < \tfrac{1}{m} + \tfrac{1}{n}).$$

Thus, for infinite ω, ω', the difference between q_ω and $q_{\omega'}$ is only infinitely small. By this observation, in case $x = 0$ is false, it suffices to check if $q_\omega > 0$ or if $q_\omega < 0$ to know whether $x > 0$ or $x < 0$. Given Σ_1-TR, $x = 0$ is equivalent to $(\forall k \leq \omega)(|q_k| \leq \frac{1}{k})$, for any fixed $\omega \in \Omega$.

Finally, we define the $*$-algorithm which decides between $x > 0$ and $\neg(x > 0)$. Fix some $\omega \in \Omega$. First, check if $(\forall k \leq \omega)(|q_k| \leq \frac{1}{k})$. If this formula holds, then we have $x = 0$ and return '$\neg(x > 0)$'. Otherwise, check if $q_\omega > 0$. If this formula holds, return '$x > 0$'. Otherwise, return '$\neg(x > 0)$'.

This $*$-algorithm is easily brought in the form (12). Indeed, the formula $\psi(\omega) \equiv [(\forall k \leq \omega)(|q_k| \leq \frac{1}{k}) \vee (q_\omega < 0 \wedge (\exists k \leq \omega)(|q_k| > \frac{1}{k}))]$ is Ω-invariant. Hence,

$$\neg\psi(\omega) \to x > 0 \wedge \psi(\omega) \to \neg(x > 0).$$

For the reverse implication, assume $\mathfrak{LPR}$ and let φ be as in Σ_1-TR and assume $\varphi(n)$ holds for all $n \in \mathbb{N}$. Suppose there is an $\omega \in \Omega$ such that $\neg\varphi(\omega)$ and let ω_0 be the least of these. We first define the function $g_\varphi(i)$ as follows

$$g_\varphi(i) = \begin{cases} 1 & (\exists n \leq i)\neg\varphi(n) \\ 0 & (\forall n \leq i)\varphi(n) \end{cases}.$$

Secondly, $h_\varphi(i)$ is the least $n \leq i$ such that $\neg\varphi(n)$, if such exists, and i otherwise. Finally, we define the real x as follows

$$q_k = \sum_{i=0}^{k} \frac{1}{2^{i-h_\varphi(i)}} g_\varphi(i).$$

Note that x satisfies (4), i.e. that x is indeed a real number.

As $q_k = 0$ for $k \in \mathbb{N}$, we cannot have $x > 0$. Similarly, as $q_m = \sum_{i=0}^{\omega_0 - m} \frac{1}{2^i}$ for $m \geq \omega_0$, we cannot have $\neg(x > 0)$. However, we have just showed that both $x > 0$ and $\neg(x > 0)$ are impossible. This contradicts $\mathfrak{LPR}$ and we conclude that there cannot be $\omega \in \Omega$ such that $\neg\varphi(\omega)$. Together with our assumption that $(\forall n \in \mathbb{N})\varphi(n)$, the principle $\mathfrak{LPR}$ thus implies $(\forall n \in {}^*\mathbb{N})\varphi(n)$. From this, Σ_1-TR follows easily and, by Theorem 30, $\mathfrak{LPO}$ is obtained. $\square$

The previous theorem is our first step towards 'reverse engineering' Constructive Reverse Mathematics. We now obtain a result similar to Theorem 32 for the principles LLPO and LLPR. By Definitions 25 and 26, the latter correspond to the following principles in Nonstandard Analysis.

Principle 33: ($\mathfrak{LLPO}$) *For every* P, Q *in* Σ_1, *we have* $\neg(P \wedge Q) \to \neg P \vee \neg Q$.

Principle 34: ($\mathfrak{LLPR}$) $(\forall x \in \mathbb{R})(\neg(x > 0) \vee \neg(x < 0))$.

Before we prove the equivalence between the previous principles, we need to consider the following remark regarding the constructive continuum.

Remark 35: In (Bridges, 1999, Axiom set R2), we find $\neg(x > y \wedge x < y)$ among the axioms for the constructive continuum. Furthermore, Bishop states that this formula, though negative in nature, is provable in his Constructive Analysis (Bishop, 1967, p. 21). Thus, it seems justified to tacitly assume that $\urcorner(x > 0 \wedge x < 0)$ holds for all real numbers. We need this property in the proof of Theorem 36.

We have the following theorem.

Theorem 36: *The principles $\mathfrak{LLPO}$ and $\mathfrak{LLPR}$ are equivalent.*

Proof. We first study the exact content of $\mathfrak{LLPO}$. Consider the following formula

$$\urcorner[(\exists n \in \mathbb{N})\varphi_1(n) \wedge (\exists m \in \mathbb{N})\varphi_2(m)], \tag{13}$$

with $\varphi_1, \varphi_2 \in \Delta_0$. It is clear that the antecedent of $\mathfrak{LLPO}$, i.e. $\urcorner(P \wedge Q)$, has exactly this form. By Definition 26, (13) is equivalent to

$$(\forall n \in {}^*\mathbb{N})\neg\varphi_1(n) \vee (\forall m \in {}^*\mathbb{N})\neg\varphi_2(m). \tag{14}$$

The consequent of $\mathfrak{LLPO}$, i.e. $\urcorner P \curlyvee \urcorner Q$, then has the form

$$(\forall n \in {}^*\mathbb{N})\neg\varphi_1(n) \curlyvee (\forall m \in {}^*\mathbb{N})\neg\varphi_2(m). \tag{15}$$

Thus, $\mathfrak{LLPO}$ is the statement that *if* (14) *holds, then there is a $\ast$-algorithm deciding which disjunct of this formula holds.* i.e. (15).

Now assume $\mathfrak{LLPO}$ and fix $x \in \mathbb{R}$. As discussed in Remark 35, we may assume the formula $\urcorner(x > 0 \wedge x < 0)$. By Definition 26, the latter formula is equivalent to

$$(\forall k \in {}^*\mathbb{N})\big[q_k \leq \tfrac{1}{k} \vee (\forall l \in {}^*\mathbb{N})(q_l \geq -\tfrac{1}{l})\big]. \tag{16}$$

We apply $\mathfrak{LLPO}$ to decide which of the disjuncts of this formula holds. As the first disjunct of (16) is $\urcorner(x > 0)$ and the second one is $\urcorner(x < 0)$, we obtain $\mathfrak{LLPR}$.

For the other direction, assume $\mathfrak{LLPR}$, let φ_1 and φ_2 be as in (13). We now define a $\ast$-algorithm to decide between $(\forall n \in {}^*\mathbb{N})\neg\varphi_1(n)$ and $(\forall m \in {}^*\mathbb{N})\neg\varphi_2(m)$. To this end, fix $\omega' \in \Omega$ and consider the following three cases.

First of all, if $(\forall n \leq \omega')\neg\varphi_1(n)$ is false, we must have $(\forall m \in {}^*\mathbb{N})\neg\varphi_2(m)$, by (14). Thus, we output '$\urcorner[(\exists m \in \mathbb{N})\varphi_2(m)]$'.

Secondly, if $(\forall m \leq \omega')\neg\varphi_2(m)$ is false, we must have $(\forall n \in {}^*\mathbb{N})\neg\varphi_1(n)$, and we output '$\neg[(\exists n \in \mathbb{N})\varphi_1(n)]$'.

For the third case, we may assume $(\forall n \leq \omega')\neg\varphi_1(n)$ and $(\forall m \leq \omega')\neg\varphi_2(m)$. We first define the function g_{φ_1,φ_2} as follows

$$g_{\varphi_1,\varphi_2}(i) = \begin{cases} -1 & \text{if } (\exists n \leq i)\varphi_1(n), \\ 1 & \text{if } (\exists m \leq i)\varphi_2(m), \\ 0 & \text{if } (\forall m \leq i)\neg\varphi_2(m) \text{ and } (\forall n \leq i)\neg\varphi_1(n) \end{cases}.$$

Note that the previous function is well-defined, as the case $(\exists n \in {}^*\mathbb{N})\varphi_1(n) \wedge (\exists m \in {}^*\mathbb{N})\varphi(m)$ cannot occur, by (14). We also define the function $h_{\varphi_1,\varphi_2}(i)$ as the least $n' \leq i$ such that $\varphi_1(n') \vee \varphi_2(n')$, if such exists, and i otherwise. Finally, we define the real number x as

$$q_k = \sum_{i=0}^{k} \frac{1}{2^{i-h_{\varphi_1\varphi_2}(i)}} g_{\varphi_1,\varphi_2}(i).$$

Note that x is indeed a real number by (4), and the assumptions made in this case. By $\mathfrak{LLPR}$, we can decide between $\neg(x > 0)$ and $\neg(x < 0)$. If the first formula holds, we have $(\forall k \in {}^*\mathbb{N})(q_k \leq \frac{1}{k})$. This is only possible if $(\forall m \in {}^*\mathbb{N})\neg\varphi_2(m)$ holds, and we output this formula. Similarly, if $\neg(x < 0)$ is holds, we have $(\forall k \in {}^*\mathbb{N})(q_k \geq -\frac{1}{k})$, which is only possible if $(\forall n \in {}^*\mathbb{N})\neg\varphi_1(n)$ holds, and we output this formula.

Hence, $\mathfrak{LLPR}$ provides a $*$-algorithm to decide which of the disjuncts of (14) holds. From this, $\mathfrak{LLPO}$ easily follows and we are done. $\square$

Finally, we prove one partial result from Theorem 15.

Theorem 37: *The principle $\mathfrak{LPO}$ implies $\mathfrak{LLPO}$.*

Proof. Assume $\mathfrak{LPO}$. By Theorem 30, we may use Σ_1-TR. Let φ_1 and φ_2 be as in (14). We can easily verify which disjunct in the latter holds, by checking if $(\forall n \leq \omega)\neg\varphi_1(n)$ and $(\forall m \leq \omega)\neg\varphi_2(m)$, for any fixed $\omega \in \Omega$. Indeed, by Σ_1-TR, these bounded formulas are equivalent to $(\forall n \in {}^*\mathbb{N})\neg\varphi_1(n)$ and $(\forall m \in {}^*\mathbb{N})\neg\varphi_2(m)$, respectively. Thus, $\mathfrak{LLPO}$ follows. $\square$

3.5. *An answer to the first question*

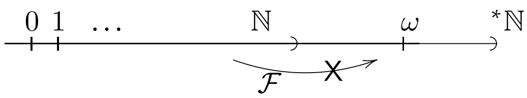
In this paragraph, we formulate a partial answer to the first question from Section 1.

Although suggestive, Theorems 32 and 36 do not even scratch the surface of Constructive Reverse Mathematics (CRM). Indeed, as is clear from Section 2.2, there is a large (and growing) number of principles and equivalent theorems that constitute CRM. Furthermore, we did not treat formulas of the form $\neg\neg\varphi$, nor did we attempt to interpret intuitionistic implication[5]. Thus, the results in this paper are more suggestive than definitive in nature. They do suggest an interesting avenue of research, however.

Hence, the answer to the first question is a careful 'maybe': we need to treat a large number principles from CRM inside Nonstandard Analysis before we can accurately judge if the notion of Ω-invariance gives rise to the 'same' kind of equivalences as we find in CRM. In (Sanders, 2012b), this investigation is undertaken 'in full' and a large number of equivalences was obtained in a similar fashion to the above. However, one encounters a significant problem with the current 'naive' approach and new ideas are needed to overcome this hurdle. In particular, an interpretation for the constructive notion of 'proof' inside Nonstandard Analysis is required, as discussed in Remark 38 below.

As usual, any answer leads to many questions. The first question that comes to mind in light of our above results is: *Why is there a connection between Nonstandard and Constructive Analysis?* We now briefly speculate on this topic.

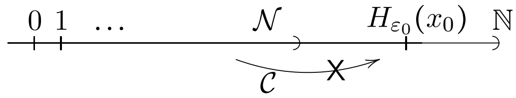
In Nonstandard Analysis, the notion of *infinite number* is central. The finite numbers are exactly the natural numbers. The new numbers in ${}^*\mathbb{N} \setminus \mathbb{N}$ are the infinite numbers and no 'finite' operation $\mathcal{F}$ can take a finite number to an infinite number. The 'finite' operations include all the usual $\mathbb{N} \to \mathbb{N}$-functions.



We now propose a similar (but vague) interpretation for the natural numbers in Constructive Analysis, when assuming an *external* point of view. In BISH,

[5] Note that it is possible to define '$\neg$' in such a way that the counterpart of Markov's principle, i.e. $\neg\neg P \to P$ ($P \in \Sigma_1$), is not derivable in (our base theory of) Nonstandard Analysis. Hence, our results are not just an instance of the fact that recursive mathematics RUSS is a model for BISH, as might be wrongly suggested by Theorem 21. We thank Martin Davis for the discussion regarding this question.

the notion of *algorithm* is central. A number only exists after an algorithm has been given to compute it, i.e. when it has been constructed. The (vague) set $\mathcal{N}$ of numbers that have been constructed is always expanding. However, there are some numbers we can never hope to construct. For instance, it is generally agreed that Constructive Analysis can be formalized in HA^{ω}, Heyting arithmetic augmented with all finite types. However, the function[6] $H_{\varepsilon_0}(x)$ cannot be defined in the latter system. Hence, for most elements $x_0 \in \mathcal{N}$, we can never construct the number $H_{\varepsilon_0}(x)$. Thus, we obtain the following picture of $\mathbb{N}$: the numbers $\mathcal{N}$ are the 'constructible' (or 'constructed') numbers, whereas the numbers in $\mathbb{N} \setminus \mathcal{N}$ are *non-constructible*. Moreover, no 'constructive' operation $\mathcal{C}$ can ever take a number in $\mathcal{N}$ outside of this set.



We emphasize that the above comparison is vague and informal. We do believe it to serve an explanatory purpose.

In the following final remark, we discuss the proverbial 'elephant in the room' regarding Definition 2.

Remark 38: In this paper, we have concentrated on finding a concept from Nonstandard Analysis which captures Bishop's notion of (constructive) algorithm. However, as is clear from Definition 2, the notion *proof* plays an important role in Constructive Analysis. Nonetheless, we have not attempted to provide an interpretation for this equally central notion.

In (Sanders, 2012a), a first attempt is made to establish an interpretation of the constructive notion of *proof* in Nonstandard Analysis. As it turns out, in the same way Constructive Analysis is limited to formulas that come with proofs, the limitation to formulas A which satisfy (a version of) the Transfer Principle '$A \leftrightarrow {}^*A$' from Nonstandard Analysis, provides a suitable interpretation of 'proof'. Note that by Definition 26 and Remark 28, a kind of Transfer is already built into $*$-negation. We refer to (Sanders, 2012a) for details. A full interpretation of Constructive Analysis inside Nonstandard Analysis in this spirit is forthcoming in (Sanders, 2012c). These results endow Constructive Analysis with a certain 'robustness', as discussed in (Sanders, 2013).

[6] The function $H_{\alpha}(x)$ is called the *Hardy function* of level $\alpha \in$ ORD. See (Buss, 1998).

4. *Conclusion and Future research*

4.1. *Conclusion*

In this paper, we made an attempt at bringing Nonstandard and Constructive Analysis closer together, i.e. at *reuniting the antipodes*. This was accomplished by attempting to isolate *algorithm*, the central notion of Constructive Analysis, inside Nonstandard Analysis. We used the following two questions from the introduction as guiding principles.

(1) Is there a (simple) notion in Nonstandard Analysis that captures Errett Bishop’s notion of algorithm?
(2) How will we judge if the correspondence in the previous item is any good?

By reviewing the main results in the discipline Constructive Reverse Mathematics (a foundational program based on Constructive Analysis), we arrived at a criterion by which we might *indirectly* capture Bishop’s primitive notion of algorithm. In short, a formal notion captures Bishop’s primitive of algorithm if it gives rise to the same equivalences as found in Constructive Reverse Mathematics. Indeed, if the latter is the case, then the same principles are non-algorithmic in both cases, i.e. with regard to the formal notion and with regard to Bishop’s algorithm. Hence, the formal notion must (approximately) capture Bishop’s primitive of algorithm.

In answer to the first question, we defined ‘Ω-invariance’, a candidate nonstandard counterpart of the notion of algorithm. We then applied our criterion to Ω-invariance. In particular, we showed that several famous non-algorithmic principles (e.g. LPO and LLPO) behave in the same way in Nonstandard Analysis based on Ω-invariance. To this end, we defined counterparts in Nonstandard Analysis of the intuitionistic connectives $\vee$ and $\neg$. In conclusion, we suggested that more equivalences need to be proved in Nonstandard Analysis before we can answer the first question positively. We also provided an explanation why there is *überhaupt* a connection between Nonstandard Analysis and Constructive Analysis.

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