Logique & Analyse **222** (2013), 149–156

A "DOWNWARDS LÖWENHEIM-SKOLEM-TARSKI THEOREM" FOR SPECIFIC UNIFORM STRUCTURES

ROLAND HINNION

Abstract
This article explores the possibilities of getting an elementary substructure of some specific types of uniform first-order structures, so that the uniformity that is induced is still of the same type.

1. *Introduction*

The most outstanding example of a "size-reduction" result is the famous "Downwards Löwenheim-Skolem-Tarski Theorem" (abbreviation "LST"; [1], theorem 3.1.6). We explore here such possibilities for so-called "Malitz-structures" ([2], [3]). These are specific first-order structures on which universe there is a uniformity that presents (inter alia) the important technical advantage of having a basis purely made of equivalence relations. In such a context one has to take in account two kinds of "sizes", namely the cardinal of the universe of the first-order structure, and the "weight" of the uniformity (or, as we will see here, "refined" versions of that classical notion [4]). So far, it is not clear at all whether or not the techniques that work here for Malitz-structures could or not be adapted successfully to more general uniform structures (the basis of equivalences playing an explicit role in the proofs) : a task for future research…

2. *Malitz-structures*

For the reader familiar with general uniform spaces, as studied f.ex. in [4], a Malitz-structure is a first-order-structure M, which universe is a particular kind of uniform space, namely one where the uniformity admits a basis made of equivalence relations,with the extra conditions that the resulting topology should be totally separated but not discrete; further should the proper functions of M be uniformly continuous. Hereunder we give an alternative definition, that does necessitate no particular familiarity with the theory of general uniform spaces.

These structures were studied in [2],[3], where the reader can find more details and proofs. We recall here only those basic notions and facts which play a role in our size-reduction considerations "à la LST".

A Malitz-structure is a couple $(M, \mathcal{F})$, where M is a first-order structure with universe U_M; relations R_M, $R'_M, \ldots$; functions $F_M, F'_M, \ldots$; and constants $c_M, c'_M, \ldots$
The corresponding first-order language $\mathcal{L}$ is supposed to be at most countable. Finally, $\mathcal{F}$ is a family of equivalences on U_M (called "Malitz-family on M"), satisfying the following 4 conditions :

Cond 1 : $\mathcal{F}$ is directed for the order $\supseteq$ ("reverse inclusion").

Cond 2 : $=_M \notin \mathcal{F}$ (where $=_M$ is the equality relation on U_M)

Cond 3 : $\forall a, b \in U_M$ $(a \neq b \Rightarrow \exists \sim \in \mathcal{F} \quad \neg a \sim b)$

Cond 4 : for any F_M (proper function of M) : $\forall \sim \in \mathcal{F}\ \exists \sim' \in \mathcal{F}\ \forall \vec{x}, \vec{y}$ (in U_M) : $\vec{x} \sim' \vec{y} \Rightarrow F_M(\vec{x}) \sim F_M(\vec{y})$.

Of course does "$\vec{x}$" stand for an n-tuple "$x_1, x_2, \ldots, x_n$", and "$\vec{a} \sim' \vec{b}$" for $\forall i$ "$a_i \sim b_i$".

Remarks :

1. The family $\mathcal{F}$ is simply a basis for a uniformity, one can easily check that the induced topology is totally separated (by Cond 3), but not discrete (by Cond 2), Cond 4 exactly expresses that the proper functions of M are uniformly continuous. Notice that U_M and $\mathcal{F}$ are necessarily infinite sets.

2. The "weight" of a uniformity (notation : w) is classically defined as the minimum of the cardinals of the uniformity bases cofinal in $\mathcal{F}$. For a Malitz-structure $(M, \mathcal{F})$ it will exactly be the minimum of the cardinals $|\mathcal{F}'|$, for $\mathcal{F}'$ a cofinal subset of $\mathcal{F}$ (i.e. $\forall \sim \in \mathcal{F}\ \exists \sim' \in \mathcal{F}' \sim \supseteq \sim'$).
Notice that such an $\mathcal{F}'$ is always again a Malitz-family on M, inducing the same uniformity.

3. A technical advantage of Malitz-families is that the simple notion of "uniform-Cauchy-$\mathcal{F}$-net" allows to define the Cauchy-completion (in the usual sense [4]) : the net $(x_\sim)_{\sim \in \mathcal{F}}$ being "uniform-Cauchy" iff (definition) $\forall \sim, \sim' \in \mathcal{F}\ (\sim \supseteq \sim' \Rightarrow x_\sim \sim x_{\sim'})$.
The classical Cauchy-completion of U_M is then the set (adequately quotiented, of course) of all such uniform-Cauchy-$\mathcal{F}$-nets.

4. Notice that not any infinite firs-order structure (for a countable $\mathcal{L}$) admits a Malitz-family : see [3], proposition 5.2.1.

Important parameters :

- $\delta_{\mathcal{F}}$ is the "characteristic" (or "additivity") of the directed set $(\mathcal{F}, \supseteq)$, i.e. the strict supremum of the cardinals of the upperly bounded subsets of $(\mathcal{F}, \supseteq)$.
- $\mathrm{k}_{\mathcal{F}}$ is the "index" of $\mathcal{F}$, i.e. the strict supremum of the cardinals $|U_M/_{\sim}|$, for $\sim \in \mathcal{F}$.

The role of these parameters has been studied in detail in [2], [3].

Some facts :

- $\delta_{\mathcal{F}} \leq |U_M|$
- $\delta_{\mathcal{F}} \leq k_{\mathcal{F}} \leq |U_M|^+$

(where α^+ is the successor cardinal of α).

3. *The construction : "rough version"*

Suppose that N is a (first-order) substructure of M (with U_N the universe of N). Then $\mathcal{F}$ induces a family $\mathcal{F}_N$ of equivalences on U_N, by simple restriction :

$$\mathcal{F}_N := \left\{ \sim \cap U_M^2 \;\middle|\; \sim \in \mathcal{F} \right\}.$$

This family $\mathcal{F}_N$ satisfies (easy to check) automatically Cond 1, 3, 4 (section 2), so that only Cond 2 should still be satisfied if we want to get a Malitz-family on N.

As we know that $=_M \notin \mathcal{F}$ (by Cond 2 for $\mathcal{F}$), we can choose for each $\sim \in \mathcal{F}$, one couple (a, b) such that $a \neq b$ & $a \sim b$; that couple "witnesses" that $\sim$ is not $=_M$.
Then collecting the components of all the couples so obtained, we get a "witness-set" A, with the obvious size-bound :

$$|A| \leq |\mathcal{F}|.$$

And, if $A \subseteq U_N$ for such a set A, $\mathcal{F}_N$ will obviously satisfy Cond 2 (section 2), so be a Malitz-family on N.

This suggests a first, rather “rough”, approach in our search for a “Malitzean version” of LST :
Start with $X \subseteq U_M$, such that $|X| \leq |\mathcal{F}|$. Extend X to $X' := X \cup A$, where A is a witness-set as precedingly described. Then get, via the classical LST, and elementary substructure N of M, such that $|N| = |X'|$. Clearly $|N| \leq |\mathcal{F}|$ and $\mathcal{F}_N$ is a Malitz-family for N.

This is without doubt a (first) result of size-reduction, but can be improved (Section 4) so that the “refined” versions (Sections 4,5) are really better, as is shown explicitly by our example in Section 6.

4. *Refining the notion of “weight”*

Fundamentally is the “rough” version in Section 3 linked to the weight of the uniformity, as the bound $|\mathcal{F}|$ can always be “optimized” by replacing $\mathcal{F}$ by a cofinal subset $\mathcal{F}'$, of minimal size, so that $|\mathcal{F}'| = w$ (the weight of the uniformity).

But one can consider a subtler notion of “weight”, that we will call the “Malitz-weight” (notation : $w^\star$).

So, call “Malitz-witness-set” a set $A \subseteq U_M$, such that : $\forall \sim \in \mathcal{F} \quad \exists a, b \in A$ ($a \neq b$ & $a \sim b$).

Define then : $w^\star :=$ the minimum of the $|A|$, for A a Malitz-witness-set.

Clearly : $w^\star \leq w$.

And the straightforward adaptation of the “rough” construction in Section 3 (taking $w^\star$ as bound, instead of $|\mathcal{F}|$; and for A a Malitz-witness-set such that $|A| = w^\star$) proves now clearly our

Main Theorem : If $X \subseteq U_M$ and $|X| \leq w^\star$, then there exists an elementary substructure N of M, such that $|U_N| = w^\star$ and $(N, \mathcal{F}_N)$ is a Malitz-structure.

Remarks :

1. Here we have equality : $|U_N| = w^\star$, because $X' = X \cup A$ and $|A| = w^\star$.
2. That this version is really better than the "rough" one (of Section 3) is shown in Section 6.
3. Notice that $w^\star$ is "cofinally invariant", i.e. is the same for $\mathcal{F}$ and any $\mathcal{F}'$ cofinal in $\mathcal{F}$, so only depends on the uniformity.
4. What can be said about the substructures N of M constructed in this Main Theorem and its variants (hereunder) ? From the point of view of Model Theory, we have elementary substructures N of M, with some control on the size on N and on the uniformity on N (N being again a Malitz-structure). From the topological point of view we generally only get what is induced by the uniformity, for a Malitz-structure (as commented in Section 2, Remark 1) : a totally separated but not discrete topological space U_N. However, modulo particular further hypotheses, does one get some extra properties; example : the "compactness" properties for the Cauchy-completion of U_N discussed in section 5.5.

5. *Strenghtening the links between $\mathcal{F}$ and $\mathcal{F}_N$*

The construction (Section 3) gives us the following immediate links :

- the canonical map $f : \mathcal{F} \to \mathcal{F}_N : \sim \mapsto \sim \cap U_N^2$ is a surjective morphism of partial orders (for the directed order $\supseteq$).
- $\delta_{\mathcal{F}} \leq \delta_{\mathcal{F}_N}$ and $k_{\mathcal{F}_N} \leq k_{\mathcal{F}}$;
  so that $\delta_{\mathcal{F}} \leq \delta_{\mathcal{F}_N} \leq k_{\mathcal{F}_N} \leq k_{\mathcal{F}}$
  (as $\delta \leq k$ holds for any Malitz-family; see "last fact" in Section 2).

We give now some "improved" versions of our Main Theorem (Section 4), where the links between $\mathcal{F}$ and $\mathcal{F}_N$ are strenghtened.

5.1 : A set B ($\subseteq U_M$) will be called "Index-witness-set" iff (definition) the strict supremum of the cardinals $|B/_{\sim\cap B^2}|$, for $\sim\in \mathcal{F}$, is exactly $k_{\mathcal{F}}$. Optimizing the size of such sets B, we get the notion of "Index weight" :

w_{Index} := the minimum of the cardinals $|B|$, for B an Index-witness-set.

Notice (easy to check) that this notion is cofinally invariant (as is $w^\star$), so only depends on the uniformity, not on the particular $\mathcal{F}$ used.

It is clear that $B \subseteq U_N$, for B some Index-witness-set, forces : $k_{\mathcal{F}_N} = k_{\mathcal{F}}$.

All this gives us an immediate variant of our Main Theorem :

Variant 1 : If $X \subseteq U_M$ and $|X| \leq$ maximum of $\{w^\star, w_{\text{Index}}\}$, then there exists an elementary substructure N of M, such that $|U_N| \leq$ maximum of $\{w^\star, w_{\text{Index}}\}$, $\mathcal{F}$ is a Malitz-family on N and $k_{\mathcal{F}_N} = k_{\mathcal{F}}$.

Proof : adapt the construction of Section 3, with this time $X' := X \cup A \cup B$, where A is a Malitz-witness-set, B is an Index-witness-set, and $|A| = w^\star$, $|B| =$ maximum of $\{w^\star, w_{\text{Index}}\}$.

5.2 : A set $C (\subseteq U_M)$ will be called "Collapse-avoiding" iff (definition)

$$\forall \sim,\ \sim' \mathcal{F}[\sim \neq \sim' \Rightarrow \exists a, b \in C\ (a \sim b \Leftrightarrow \neg a \sim' b)]$$

Obviously will the canonical map $f : \mathcal{F} \to \mathcal{F}_N$ be bijective whenever $C \subseteq U_N$, for some Collapse-avoiding C.
Optimizing the size of such sets C, we get the notion of "Collapse-weight of $\mathcal{F}$" :

$w^{\mathcal{F}}_{\text{Coll}} :=$ the minimum cardinal $|C|$, for C Collapse-avoiding subset of U_M

Remarks :

1. A bijective f is not necessarily and isomorphism (as f^{-1} can fail to be a morphism); we discuss in Section 5.4 circumstances where f is an isomorphism.
2. The "Collapse-weight of $\mathcal{F}$" is not a "cofinally invariant" notion : it depends on $\mathcal{F}$.

Variant 2 : If $X \subseteq U_M$, and $|X| \leq$ maximum of $\{w^\star, w^{\mathcal{F}}_{\text{Coll}}\}$, then there exists an elementary substructure N of M, such that $|U_N| \leq$ maximum of $\{w^\star, w^{\mathcal{F}}_{\text{Coll}}\}$, $\mathcal{F}_N$ is a Malitz-family on N and the canonical map $f : \mathcal{F} \to \mathcal{F}_N$ is bijective.

Proof : adapt the construction of Section 3, with this time $X' := X \cup A \cup C$, where A is a Malitz-witness set, C is a Malitz-witness set, C is a Collapse-avoiding set, and $|A| = w^\star$, $|C| = w^{\mathcal{F}}_{\text{Coll}}$.

5.3 : The reader can easily imagine now the Variant 3, combining the ideas of the Variants 1 and 2; the initial bound for X will then be the maximum of $\{w^\star, w^{\mathcal{F}}_{\text{Coll}}, w_{\text{Index}}\}$; the final N obtained has the same bound, and : $k_{\mathcal{F}_N} = k_{\mathcal{F}}$, f is bijective.

5.4 : It is clear, just by basic considerations about morphisms for partial orders that : if $(\mathcal{F}, \supseteq)$ is totally ordered, and the canonical map $f : \mathcal{F} \to \mathcal{F}_N$ is bijective, then f is an isomorphism.
Consequence : in that case $\delta_{\mathcal{F}_N} = \delta_{\mathcal{F}}$.

5.5 : The adequate "compactness notion" for a Malitz-structure $(M, \mathcal{F})$ is the one of "$\delta_{\mathcal{F}}$-cover-compactness", i.e. the fact that any open covering of U_M admits a $\delta_{\mathcal{F}}$-subcovering (see [2], [3]).

Further is a Malitz-family called "compactifying" when the Cauchy-completion of the concerned structure is $\delta_{\mathcal{F}}$-cover-compact; and it is known that in that case one has : $\delta_{\mathcal{F}} = k_{\mathcal{F}}$. Notice that any $\mathcal{F}$ with $k_{\mathcal{F}} = \aleph_0$ is necessarily "compactifying".

From all this we can deduce that :

- if $\mathcal{F}$ is compactifying, then $\delta_{\mathcal{F}} = \delta_{\mathcal{F}_N} = k_{\mathcal{F}_N} = k_{\mathcal{F}}$ (for N constructed as in our Main Theorem);
- if $k_{\mathcal{F}} = \aleph_0$, then $\delta_{\mathcal{F}} = \delta_{\mathcal{F}_N} = k_{\mathcal{F}_N} = \aleph_0$.

6. *Example*

Consider the first-order structure $M : \mathcal{R}, \leq, F_M$ (with the usual order-relation and the function $F_M(x) = x + 1$). Our Malitz-family is the collection $\mathcal{F}$ of all the equivalence-relations on $\mathcal{R}$, of type $\sim_{(r_0, r_1, r_2, \ldots, r_k)}$, for all the possible choices for k (natural number) and strictly increasing sequences of real numbers : $r_0 < r_1 < r_2 < \cdots < r_k$; such an equivalence being defined by its list of equivalence-classes :

$$]-\infty, r_0[, [r_0, r_1[, [r_1, r_2[, \ldots, [r_k, +\infty[.$$

One checks easily that $\mathcal{F}$ is indeed a Malitz-family for M, with $\delta_{\mathcal{F}} = k_{\mathcal{F}} = \aleph_0$.

But the weight of the corresponding uniformity is obviously not $\aleph_0$! So that the “rough” version of the construction in Section 3 allows to obtain an N with upper bound : $|U_N| \leq w$, not guaranteeing at all that N is countable !!

In contrast to that, any of the variants of our Main Theorem guarantees that(starting with X = the empty set, of course) we get indeed a countable N, as thanks to the density of $\mathcal{Q}$ in $\mathcal{R}$ ($\mathcal{Q}$ being the set of rational numbers) it is easy to see that :

$$w^\star = w_{\text{Index}} = w^{\mathcal{F}}_{\text{Coll}} = \aleph_0 \quad .$$

This shows that the “refined” notions of weight really improve the results provided by the type of construction in Section 3.

7. *Conclusion*

The “Downwards LST” can be adapted to Malitz-structures, but at the price of size-constraints unknown to the classical LST. In particular is there no general guarantee for the existence of a countable elementary substructure, that is still a Malitz-structure for the induced uniformity.

ULB
CP211, Bd du Triomphe
Brussels 1050
Belgium
E-mail: `rhinnion@ulb.ac.be`

REFERENCES

[1] Chang, C.C. and Keisler, H.J., *Model Theory* (1973), North-Holland.
[2] Hinnion, R., “A general Cauchy-completion process for arbitrary first-order structures”, (2007), *Logique & Analyse* 197, 5–41.
[3] Hinnion, R, “Directed sets and Malitz-Cauchy completions”, (1997), *Mathematical Logic Quarterly* 43, 465–484.
[4] Kelley, J.L., *General Topology*, (1953), Van Nostrand, New York.