

A “DOWNWARDS LÖWENHEIM-SKOLEM-TARSKI THEOREM”
FOR SPECIFIC UNIFORM STRUCTURES

ROLAND HINNION

Abstract

This article explores the possibilities of getting an elementary sub-structure of some specific types of uniform first-order structures, so that the uniformity that is induced is still of the same type.

1. *Introduction*

The most outstanding example of a “size-reduction” result is the famous “Downwards Löwenheim-Skolem-Tarski Theorem” (abbreviation “LST”; [1], theorem 3.1.6). We explore here such possibilities for so-called “Malitz-structures” ([2], [3]). These are specific first-order structures on which universe there is a uniformity that presents (inter alia) the important technical advantage of having a basis purely made of equivalence relations. In such a context one has to take in account two kinds of “sizes”, namely the cardinal of the universe of the first-order structure, and the “weight” of the uniformity (or, as we will see here, “refined” versions of that classical notion [4]). So far, it is not clear at all whether or not the techniques that work here for Malitz-structures could or not be adapted successfully to more general uniform structures (the basis of equivalences playing an explicit role in the proofs) : a task for future research. . .

2. *Malitz-structures*

For the reader familiar with general uniform spaces, as studied f.ex. in [4], a Malitz-structure is a first-order-structure M , which universe is a particular kind of uniform space, namely one where the uniformity admits a basis made of equivalence relations, with the extra conditions that the resulting topology should be totally separated but not discrete; further should the proper functions of M be uniformly continuous. Hereunder we give an alternative

definition, that does necessitate no particular familiarity with the theory of general uniform spaces.

These structures were studied in [2],[3], where the reader can find more details and proofs. We recall here only those basic notions and facts which play a role in our size-reduction considerations "à la LST".

A Malitz-structure is a couple (M, \mathcal{F}) , where M is a first-order structure with universe U_M ; relations R_M, R'_M, \dots ; functions F_M, F'_M, \dots ; and constants c_M, c'_M, \dots .

The corresponding first-order language \mathcal{L} is supposed to be at most countable. Finally, \mathcal{F} is a family of equivalences on U_M (called "Malitz-family on M "), satisfying the following 4 conditions :

Cond 1 : \mathcal{F} is directed for the order \supseteq ("reverse inclusion").

Cond 2 : $=_M \notin \mathcal{F}$ (where $=_M$ is the equality relation on U_M)

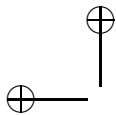
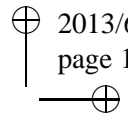
Cond 3 : $\forall a, b \in U_M (a \neq b \Rightarrow \exists \sim \in \mathcal{F} \neg a \sim b)$

Cond 4 : for any F_M (proper function of M) : $\forall \sim \in \mathcal{F} \exists \sim' \in \mathcal{F} \forall \vec{x}, \vec{y}$
(in U_M) : $\vec{x} \sim' \vec{y} \Rightarrow F_M(\vec{x}) \sim F_M(\vec{y})$.

Of course does " \vec{x} " stand for an n -tuple " x_1, x_2, \dots, x_n ", and " $\vec{a} \sim' \vec{b}$ " for $\forall i "a_i \sim' b_i"$.

Remarks :

1. The family \mathcal{F} is simply a basis for a uniformity, one can easily check that the induced topology is totally separated (by Cond 3), but not discrete (by Cond 2), Cond 4 exactly expresses that the proper functions of M are uniformly continuous. Notice that U_M and \mathcal{F} are necessarily infinite sets.
2. The "weight" of a uniformity (notation : w) is classically defined as the minimum of the cardinals of the uniformity bases cofinal in \mathcal{F} . For a Malitz-structure (M, \mathcal{F}) it will exactly be the minimum of the cardinals $|\mathcal{F}'|$, for \mathcal{F}' a cofinal subset of \mathcal{F} (i.e. $\forall \sim \in \mathcal{F} \exists \sim' \in \mathcal{F}' \sim \supseteq \sim'$).
Notice that such an \mathcal{F}' is always again a Malitz-family on M , inducing the same uniformity.
3. A technical advantage of Malitz-families is that the simple notion of "uniform-Cauchy- \mathcal{F} -net" allows to define the Cauchy-completion (in the usual sense [4]) : the net $(x_\sim)_{\sim \in \mathcal{F}}$ being "uniform-Cauchy" iff (definition) $\forall \sim, \sim' \in \mathcal{F} (\sim \supseteq \sim' \Rightarrow x_\sim \sim x_{\sim'})$.
The classical Cauchy-completion of U_M is then the set (adequately quotiented, of course) of all such uniform-Cauchy- \mathcal{F} -nets.



4. Notice that not any infinite first-order structure (for a countable \mathcal{L}) admits a Malitz-family : see [3], proposition 5.2.1.

Important parameters :

- $\delta_{\mathcal{F}}$ is the "characteristic" (or "additivity") of the directed set (\mathcal{F}, \supseteq) , i.e. the strict supremum of the cardinals of the upperly bounded subsets of (\mathcal{F}, \supseteq) .
- $k_{\mathcal{F}}$ is the "index" of \mathcal{F} , i.e. the strict supremum of the cardinals $|U_M/\sim|$, for $\sim \in \mathcal{F}$.

The role of these parameters has been studied in detail in [2], [3].

Some facts :

- $\delta_{\mathcal{F}} \leq |U_M|$
- $\delta_{\mathcal{F}} \leq k_{\mathcal{F}} \leq |U_M|^+$

(where α^+ is the successor cardinal of α).

3. The construction : "rough version"

Suppose that N is a (first-order) substructure of M (with U_N the universe of N). Then \mathcal{F} induces a family \mathcal{F}_N of equivalences on U_N , by simple restriction :

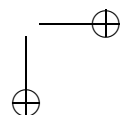
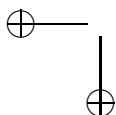
$$\mathcal{F}_N := \{ \sim \cap U_M^2 \mid \sim \in \mathcal{F} \}.$$

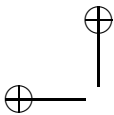
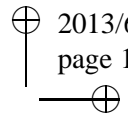
This family \mathcal{F}_N satisfies (easy to check) automatically Cond 1, 3, 4 (section 2), so that only Cond 2 should still be satisfied if we want to get a Malitz-family on N .

As we know that $=_M \notin \mathcal{F}$ (by Cond 2 for \mathcal{F}), we can choose for each $\sim \in \mathcal{F}$, one couple (a, b) such that $a \neq b$ & $a \sim b$; that couple "witnesses" that \sim is not $=_M$.

Then collecting the components of all the couples so obtained, we get a "witness-set" A , with the obvious size-bound :

$$|A| \leq |\mathcal{F}|.$$





And, if $A \subseteq U_N$ for such a set A , \mathcal{F}_N will obviously satisfy Cond 2 (section 2), so be a Malitz-family on N .

This suggests a first, rather "rough", approach in our search for a "Malitzean version" of *LST* :

Start with $X \subseteq U_M$, such that $|X| \leq |\mathcal{F}|$. Extend X to $X' := X \cup A$, where A is a witness-set as precedingly described. Then get, via the classical *LST*, and elementary substructure N of M , such that $|N| = |X'|$. Clearly $|N| \leq |\mathcal{F}|$ and \mathcal{F}_N is a Malitz-family for N .

This is without doubt a (first) result of size-reduction, but can be improved (Section 4) so that the "refined" versions (Sections 4,5) are really better, as is shown explicitly by our example in Section 6.

4. Refining the notion of "weight"

Fundamentally is the "rough" version in Section 3 linked to the weight of the uniformity, as the bound $|\mathcal{F}|$ can always be "optimized" by replacing \mathcal{F} by a cofinal subset \mathcal{F}' , of minimal size, so that $|\mathcal{F}'| = w$ (the weight of the uniformity).

But one can consider a subtler notion of "weight", that we will call the "Malitz-weight" (notation : w^*).

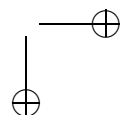
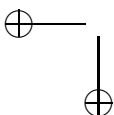
So, call "Malitz-witness-set" a set $A \subseteq U_M$, such that : $\forall \sim \in \mathcal{F} \quad \exists a, b \in A (a \neq b \ \& \ a \sim b)$.

Define then : $w^* :=$ the minimum of the $|A|$, for A a Malitz-witness-set.

Clearly : $w^* \leq w$.

And the straightforward adaptation of the "rough" construction in Section 3 (taking w^* as bound, instead of $|\mathcal{F}|$; and for A a Malitz-witness-set such that $|A| = w^*$) proves now clearly our

Main Theorem : If $X \subseteq U_M$ and $|X| \leq w^*$, then there exists an elementary substructure N of M , such that $|U_N| = w^*$ and (N, \mathcal{F}_N) is a Malitz-structure.



Remarks :

1. Here we have equality : $|U_N| = w^*$, because $X' = X \cup A$ and $|A| = w^*$.
2. That this version is really better than the "rough" one (of Section 3) is shown in Section 6.
3. Notice that w^* is "cofinally invariant", i.e. is the same for \mathcal{F} and any \mathcal{F}' cofinal in \mathcal{F} , so only depends on the uniformity.
4. What can be said about the substructures N of M constructed in this Main Theorem and its variants (hereunder) ? From the point of view of Model Theory, we have elementary substructures N of M , with some control on the size on N and on the uniformity on N (N being again a Malitz-structure). From the topological point of view we generally only get what is induced by the uniformity, for a Malitz-structure (as commented in Section 2, Remark 1) : a totally separated but not discrete topological space U_N . However, modulo particular further hypotheses, does one get some extra properties; example : the "compactness" properties for the Cauchy-completion of U_N discussed in section 5.5.

5. Strengthening the links between \mathcal{F} and \mathcal{F}_N

The construction (Section 3) gives us the following immediate links :

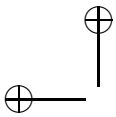
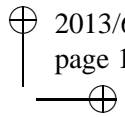
- the canonical map $f : \mathcal{F} \rightarrow \mathcal{F}_N : \sim \mapsto \sim \cap U_N^2$ is a surjective morphism of partial orders (for the directed order \supseteq).
- $\delta_{\mathcal{F}} \leq \delta_{\mathcal{F}_N}$ and $k_{\mathcal{F}_N} \leq k_{\mathcal{F}}$;
so that $\delta_{\mathcal{F}} \leq \delta_{\mathcal{F}_N} \leq k_{\mathcal{F}_N} \leq k_{\mathcal{F}}$
(as $\delta \leq k$ holds for any Malitz-family; see "last fact" in Section 2).

We give now some "improved" versions of our Main Theorem (Section 4), where the links between \mathcal{F} and \mathcal{F}_N are strengthened.

5.1 : A set $B (\subseteq U_M)$ will be called "Index-witness-set" iff (definition) the strict supremum of the cardinals $|B/\sim \cap B^2|$, for $\sim \in \mathcal{F}$, is exactly $k_{\mathcal{F}}$.

Optimizing the size of such sets B , we get the notion of "Index weight" :

$w_{\text{Index}} :=$ the minimum of the cardinals $|B|$, for B an Index-witness-set.



Notice (easy to check) that this notion is cofinally invariant (as is w^*), so only depends on the uniformity, not on the particular \mathcal{F} used.

It is clear that $B \subseteq U_N$, for B some Index-witness-set, forces : $k_{\mathcal{F}_N} = k_{\mathcal{F}}$.

All this gives us an immediate variant of our Main Theorem :

Variant 1 : If $X \subseteq U_M$ and $|X| \leq \text{maximum of } \{w^*, w_{\text{Index}}\}$, then there exists an elementary substructure N of M , such that $|U_N| \leq \text{maximum of } \{w^*, w_{\text{Index}}\}$, \mathcal{F} is a Malitz-family on N and $k_{\mathcal{F}_N} = k_{\mathcal{F}}$.

Proof : adapt the construction of Section 3, with this time $X' := X \cup A \cup B$, where A is a Malitz-witness-set, B is an Index-witness-set, and $|A| = w^*$, $|B| = \text{maximum of } \{w^*, w_{\text{Index}}\}$.

5.2 : A set $C(\subseteq U_M)$ will be called "Collapse-avoiding" iff (definition)

$$\forall \sim, \sim' \mathcal{F}[\sim \neq \sim' \Rightarrow \exists a, b \in C (a \sim b \Leftrightarrow \neg a \sim' b)]$$

Obviously will the canonical map $f : \mathcal{F} \rightarrow \mathcal{F}_N$ be bijective whenever $C \subseteq U_N$, for some Collapse-avoiding C .

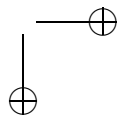
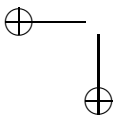
Optimizing the size of such sets C , we get the notion of "Collapse-weight of \mathcal{F} " :

$w_{\text{Coll}}^{\mathcal{F}} := \text{the minimum cardinal } |C|, \text{ for } C \text{ Collapse-avoiding subset of } U_M$

Remarks :

1. A bijective f is not necessarily an isomorphism (as f^{-1} can fail to be a morphism); we discuss in Section 5.4 circumstances where f is an isomorphism.
2. The "Collapse-weight of \mathcal{F} " is not a "cofinally invariant" notion : it depends on \mathcal{F} .

Variant 2 : If $X \subseteq U_M$, and $|X| \leq \text{maximum of } \{w^*, w_{\text{Coll}}^{\mathcal{F}}\}$, then there exists an elementary substructure N of M , such that $|U_N| \leq \text{maximum of } \{w^*, w_{\text{Coll}}^{\mathcal{F}}\}$, \mathcal{F}_N is a Malitz-family on N and the canonical map $f : \mathcal{F} \rightarrow \mathcal{F}_N$ is bijective.



Proof : adapt the construction of Section 3, with this time $X' := X \cup A \cup C$, where A is a Malitz-witness set, C is a Malitz-witness set, C is a Collapse-avoiding set, and $|A| = w^*$, $|C| = w_{\text{Coll}}^{\mathcal{F}}$.

5.3 : The reader can easily imagine now the Variant 3, combining the ideas of the Variants 1 and 2; the initial bound for X will then be the maximum of $\{w^*, w_{\text{Coll}}^{\mathcal{F}}, w_{\text{Index}}\}$; the final N obtained has the same bound, and : $k_{\mathcal{F}_N} = k_{\mathcal{F}}$, f is bijective.

5.4 : It is clear, just by basic considerations about morphisms for partial orders that : if (\mathcal{F}, \supseteq) is totally ordered, and the canonical map $f : \mathcal{F} \rightarrow \mathcal{F}_N$ is bijective, then f is an isomorphism.

Consequence : in that case $\delta_{\mathcal{F}_N} = \delta_{\mathcal{F}}$.

5.5 : The adequate "compactness notion" for a Malitz-structure (M, \mathcal{F}) is the one of " $\delta_{\mathcal{F}}$ -cover-compactness", i.e. the fact that any open covering of U_M admits a $\delta_{\mathcal{F}}$ -subcovering (see [2], [3]).

Further is a Malitz-family called "compactifying" when the Cauchy-completion of the concerned structure is $\delta_{\mathcal{F}}$ -cover-compact; and it is known that in that case one has : $\delta_{\mathcal{F}} = k_{\mathcal{F}}$. Notice that any \mathcal{F} with $k_{\mathcal{F}} = \aleph_0$ is necessarily "compactifying".

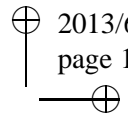
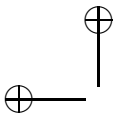
From all this we can deduce that :

- if \mathcal{F} is compactifying, then $\delta_{\mathcal{F}} = \delta_{\mathcal{F}_N} = k_{\mathcal{F}_N} = k_{\mathcal{F}}$ (for N constructed as in our Main Theorem);
- if $k_{\mathcal{F}} = \aleph_0$, then $\delta_{\mathcal{F}} = \delta_{\mathcal{F}_N} = k_{\mathcal{F}_N} = \aleph_0$.

6. Example

Consider the first-order structure $M : \mathcal{R}, \leq, F_M$ (with the usual order-relation and the function $F_M(x) = x + 1$). Our Malitz-family is the collection \mathcal{F} of all the equivalence-relations on \mathcal{R} , of type $\sim_{(r_0, r_1, r_2, \dots, r_k)}$, for all the possible choices for k (natural number) and strictly increasing sequences of real numbers : $r_0 < r_1 < r_2 < \dots < r_k$; such an equivalence being defined by its list of equivalence-classes :

$$] - \infty, r_0[, [r_0, r_1[, [r_1, r_2[, \dots, [r_k, +\infty[.$$



One checks easily that \mathcal{F} is indeed a Malitz-family for M , with $\delta_{\mathcal{F}} = k_{\mathcal{F}} = \aleph_0$.

But the weight of the corresponding uniformity is obviously not \aleph_0 ! So that the "rough" version of the construction in Section 3 allows to obtain an N with upper bound : $|U_N| \leq w$, not guaranteeing at all that N is countable !!

In contrast to that, any of the variants of our Main Theorem guarantees that (starting with X = the empty set, of course) we get indeed a countable N , as thanks to the density of \mathcal{Q} in \mathcal{R} (\mathcal{Q} being the set of rational numbers) it is easy to see that :

$$w^* = w_{\text{Index}} = w_{\text{Coll}}^{\mathcal{F}} = \aleph_0 \quad .$$

This shows that the "refined" notions of weight really improve the results provided by the type of construction in Section 3.

7. Conclusion

The "Downwards LST" can be adapted to Malitz-structures, but at the price of size-constraints unknown to the classical LST. In particular is there no general guarantee for the existence of a countable elementary substructure, that is still a Malitz-structure for the induced uniformity.

ULB
CP211, Bd du Triomphe
Brussels 1050
Belgium
E-mail: rhinnion@ulb.ac.be

REFERENCES

- [1] Chang, C.C. and Keisler, H.J., *Model Theory* (1973), North-Holland.
- [2] Hinnion, R., "A general Cauchy-completion process for arbitrary first-order structures", (2007), *Logique & Analyse* 197, 5–41.
- [3] Hinnion, R., "Directed sets and Malitz-Cauchy completions", (1997), *Mathematical Logic Quarterly* 43, 465–484.
- [4] Kelley, J.L., *General Topology*, (1953), Van Nostrand, New York.

