Logique & Analyse **221** (2013), 67–84

STRUCTURES, LANGUAGES AND MODELS: A UNIFYING APPROACH

JONAS R. BECKER ARENHART AND FERNANDO T.F. MORAES

Abstract

In this paper we present a unified approach to three topics: structures, formal languages and models. We begin by presenting a general theory of set theoretical structures. Formal languages and models are both structures inside this framework. We also present the link between languages and models given through a set theoretical predicate in the style of Suppes. The association of languages with structures satisfying certain conditions (given by set theoretical predicates) allow us to present an interesting application of the resulting framework: one is now able to characterize rigorously some classes of models. The classes of models so characterized play the role of scientific theories according to a version of the semantic approach to scientific theories. This is a first step in making explicit some of the underlying assumptions of the semantic approach to theories. In the end we give the example of how particle mechanics may be viewed as a theory according to that approach inside our framework.

1. *Introduction*

In this paper we present with great generality three topics of common interest for logicians and philosophers of science: structures, formal languages and scientific theories. Our approach follows the work of N.C.A. da Costa, who presents in a very general setting a theory of structures encompassing all usual mathematical theories. Of particular interest for us is the development of free algebras inside this theory of structures, for they can be used to give a rigorous account of formal languages. Both of these topics are well-known by logicians, but in general, a rigorous unifying treatment is still lacking. In particular, we deal with the problem of specifying a language to be used to talk about the elements of a structure. With these two notions in hand, scientific theories can then be viewed as classes of structures satisfying axioms formulated in an adequate language specified for them, which amounts to da Costa’s formalization of Suppes’ ideas on these topics. One of our aims is to unify these approaches appearing in da Costa’s work which were not completely linked by him yet.

We believe that the main interest of this kind of work relies precisely in its potential applicability to conceptual clarification in the philosophy of science[1]. According to a widespread philosophical conception of scientific theories, known as the ‘semantic view’ of scientific theories, a theory is defined as a kind of model or as a class of models. One of the troubles for this position is that the precise meaning of the word ’model’ in such discussions is not always as clear as it would be desirable in such contexts. However, some have claimed that for a rigorous treatment of the subject, all the relevant senses in which that word is used can be reduced to the set theoretical meaning (see [7], and [8]), the one which we shall deal with here. What is generally lacking in such discussions is a rigorous development of these models and the languages used to talk about them, mainly in case we want to axiomatize them or to discuss notions such as definability of elements of the domain or relations and operations on them. The approach of Suppes as formalized in da Costa and Chuaqui ([2]) is an option for making concepts clearer in these discussions and gives a rigorous meaning to the idea that a theory is a class of models specified by a set theoretical predicate. We show how this predicate can be written according to the approach to structures presented here.

Our strategy in this paper will be the following one: working inside first-order ZFC set theory, we first develop a theory of structures following da Costa and Rodrigues (see [3]). Using that theory as a basis, two distinct branches can then be developed: i) the algebraic conception of formal languages, and ii) the semantic conception of theories. Both branches, as we have said, are developed inside the theory of structures, that is, languages will be a kind of structure, and the models which are used to characterize theories are also structures. What is common to these two branches, as will be seen, is their underlying basis, the general theory of structures. Finally, we make the link between these last two points through Suppes’ predicate, as developed by da Costa and Chuaqui ([2]). In a Suppes’ predicate, as this notion is understood by da Costa and Chuaqui, we employ the formal languages previously developed to write the axioms that the structures constituting a certain theory must obey. In this way, for example, we need the language of field theory to express the axioms that fields must obey, and a field is a kind of structure that obeys those axioms, something expressed by the Suppes’ predicate for fields. So, as one can see, a concern with formal

[1] It is not our aim in this work to deal with all the details of those applications. Here we begin by putting the whole formal machinery to work. Other aspects of the discussion as based in this framework are forthcoming.

languages is not totally absent in the *semantic* view either (*cf.* the discussion in section 4).

In the following we give a rigorous treatment of those topics, without being exhaustive. Our main aim here is to give a unifying treatment of those themes, which somehow have already appeared in a fragmentary manner in different places and even serving different purposes, and show their coherent unity. Once the whole formal apparatus and its general motivation is set, it may be easier to simply depart from it and go ahead to provide the relevant philosophical discussions with adequate rigour.

2. *Structures*

We work in the first-order ZFC set theory, and although we keep our presentation informal, our development of the topics is rigorous. In this section, we follow the approach presented in [3], but, differently from the exposition there, we allow that the domain of the structures be composed of more than one set, and allow other simplifications which will be explained later.

Our first definition is of the set τ of types. These types are not to be confused with the ones of type theory. This set will be important also for the development of the languages in the next section so when we speak of the set of types, we always mean the set τ given by the following definition:

Definition 2.1: *The set τ of types is the least set satisfying the following conditions:*

1. *The symbols $0, 1, \ldots, n-1$ belong to τ;*
2. *If $a_0, a_1, \ldots, a_{n-1} \in \tau$, then $\langle a_0, a_1, \ldots, a_{n-1}\rangle \in \tau$, $1 \leq n < \omega$, where $\langle a_0, a_1, \ldots, a_{n-1}\rangle$ is the finite sequence of n terms, composed by $a_0, a_1, \ldots, a_{n-1}$.*

We now define the order of the elements of τ.

Definition 2.2: *If $a \in \tau$, the order of a, denoted ord(a), is defined by:*

1. *ord(k) = 0, for k = 0, 1, . . ., n-1;*
2. $ord(\langle a_0, a_1, \ldots, a_{n-1}\rangle) = max\{ord(a_0), ord(a_1), \ldots, ord(a_{n-1})\}$ $+1$.

These definitions will be useful in the next sections, when we talk about the order of a language. The following definitions help us to construct relations and properties based on D_n, a non-empty family of sets, which will soon be called the domain of the structure. We hope that the context will make it clear when we write D_n whether we are speaking about a family of sets that constitute the domain or about the n-th member of that family, in case there is one. In the next definition, the usual set-theoretical operations of power set and cartesian product are being used, and are denoted respectively by "$\times$" and "$\mathcal{P}$".

Definition 2.3: *Given a family D_n of non-empty sets (that is, $D_0, D_1, \ldots, D_{n-1}$) we define a function t, called scale based on D_n, having τ as its domain, as follows:*

1. *t(k) = D_k, for $k = 0, 1, \ldots, n-1$;*
2. *If $a_0, a_1, \ldots, a_{n-1} \in \tau$, then $t(\langle a_0, a_1, \ldots, a_{n-1}\rangle) = \mathcal{P}(t(a_0) \times t(a_1) \times \ldots \times t(a_{n-1}))$.*

Thus, metaphorically speaking, the function t gives us a kind of layered universe based on the family D_n where, as the orders of types increase, the objects attributed to them by t increase in complexity too. Given $a \in \tau$, the elements of $t(a)$ are said to be of type a. To make this point clear, we give some examples. The elements of type 1, say, (that is, elements of D_1, since $t(1) = D_1$) are called individuals. The elements of $t(\langle\langle 1\rangle\rangle)$, are properties of properties of individuals, as one can check following the definition above. For another example, let us consider relations between objects of type 0 and 1. The elements of $t(\langle 0, 0, 1\rangle)$ are ternary relations in which the first two relata are of type 0 and the third is of type 1, and the elements of $t(\langle\langle 0, 1\rangle, 1\rangle)$ are relations between a relation and an object.

We denote the set $\bigcup(range\ t(D_n))$ by $\varepsilon(D_n)$.

Definition 2.4: *The cardinal K_{D_n} associated to $\varepsilon(D_n)$ is defined as*

$$K_{D_n} = sup\{|\bigcup_{k=0}^{n-1} D_k|, |\mathcal{P}(\bigcup_{k=0}^{n-1} D_k)|, |\mathcal{P}^2(\bigcup_{k=0}^{n-1} D_k)|, \ldots\}.$$

Here, $|\bigcup_{k=0}^{n-1} D_k|$ denotes the cardinal of the set $\bigcup_{k=0}^{n-1} D_k$.

Definition 2.5: *A structure e based on the family D_n is an ordered pair of the form*

$$e = \langle D_n, R_\iota\rangle.$$

Here, R_ι is a sequence of elements of $\varepsilon(D_n)$, and we suppose that the domain of this sequence is strictly less than K_{D_n}, where D_n, as we said before, is the family of sets on the domain of the function t. We say that K_{D_n} is the cardinal associated with *e*, and that $\varepsilon(D_n)$ is the scale associated with *e*.

As we said before, each element of $\varepsilon(D_n)$ has a certain type, for it belongs to $t(a)$ for some $a \in \tau$. Now, the order of a relation is defined as the order of its type. The order of *e*, denoted ord(*e*), is the order of the greatest of the types of the relations of the family R_ι, if there is one, and if there is no such relation, we put ord(*e*)=ω.

In the beginning of this section we remarked that our presentation makes some simplifications on [3]. Here we depart from da Costa’s original work in that we allow individuals and operations to occur in the structure, whereas da Costa reduced operations to relations and identified individuals with their unit sets. The main point of this change is to simplify the exposition in our paper, and from a mathematical point of view the difference is purely a matter of convention[2]. So, in the definition of structures, the objects in the family R_ι may be not only relations, but operations as well, that is, relations satisfying the well-known functional condition, or even distinguished elements from the domain, which we take to be $0-ary$ operations. In these cases we employ as usual the common notation for functions and objects.

3. *Formal Languages*

We now present an approach to formal languages which will make use of the notion of structure just developed. Here, languages will be a special kind of structure, a free-algebra. Our aim is to develop the language of simple type theory, but before doing that, first we present some topics on Universal Algebra.

3.1. *Topics on Universal Algebra*

In this section, to make our paper self-contained, we present the topics on Universal Algebra required for our purposes. We follow closely the exposition in Barnes and Mack ([1]), but our exposition is totally developed in the theory of structures presented above.

Definition 3.1: *The similarity type of a structure e* $= \langle D_n, R_\iota \rangle$ *is a family* $s_{\lambda<\iota}$ *of types, such that for each* $\lambda < \iota$, s_λ *is the type of* R_λ.

[2] In fact, to allow individuals as we are doing or to identify them with their unit sets are both commitments to individuals anyway. If one is wishing to avoid individuals for structuralist reasons, then maybe ZFC is not the best framework to work with, since individuals are always given in the domain of the structure. Interesting as it is, we shall not concern ourselves with structuralism in this particular paper.

We say that two structures have the same similarity type when the family s of types is equal for both, that is, their relations have the same type and the family in the domain is composed of the same number of elements. Since a family is always ordered, the same type of relations occur always in the same order when the structures have the same similarity type.

Definition 3.2: *A s-algebra $\mathfrak{A}$ is a structure $\langle A, R_\iota \rangle$ such that $ord(R_n) \leq 1$ for each n and $\mathfrak{A}$ has similarity type s.*

The restriction to order 1 or less serves to make each s-algebra into an algebra in the usual sense, that is, a set with a family of operations defined over this set and distinguished elements taken from the domain.

Definition 3.3: *Let $\mathfrak{A}$, $\mathfrak{B}$ be s-algebras with $\mathfrak{A} = \langle A, R_\iota \rangle$ and $\mathfrak{B} = \langle B, R'_\iota \rangle$. A homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ is a function $\varphi : A \to B$ such that, for all $R_{\lambda<\iota}$ of the s-algebra $\mathfrak{A}$ whose type is $\langle 0_1, \ldots, 0_k \rangle$,*

$$\varphi(R_{\lambda<\iota}(a_1, \ldots, a_k)) = R'_{\lambda<\iota}(\varphi(a_1), \ldots, \varphi(a_k)).$$

Definition 3.4: *Let $\mathfrak{A}$, $\mathfrak{B}$ be s-algebras and $\varphi : A \to B$ a homomorphism. If φ is a bijection we say that it is an isomorphism between the algebras.*

Definition 3.5: *Let X be any set, let $\mathfrak{F}$ be a s-algebra with domain F and let $\sigma : X \to F$ be a function. We say that $\langle F, \sigma \rangle$ (also, for simplicity, denoted by F) is a free s-algebra on the set X of free generators if, for every s-algebra $\mathfrak{A}$ and function $\alpha : X \to A$, there exists a unique homomorphism $\varphi : F \to A$ such that $\varphi\sigma = \alpha$.*

Theorem 3.1: *For any set X and any similarity type s, there exists a free s-algebra on X. This free s-algebra on X is unique up to isomorphism.*

Proof. a) *Uniqueness*: We show first that if $\langle F, \sigma \rangle$ is free on X, and if $\varphi : F \to F$ is a homomorphism such that $\varphi\sigma = \sigma$, then $\sigma = 1_F$, the identity map on F. To show this , we take $A = F$ and $\tau = \sigma$ in the defining conditions. Then $1_F : F \to F$ has the required property for φ, and hence by its uniqueness is the only such map.

Now let $\langle F, \sigma \rangle$ and $\langle G, \varsigma \rangle$ be free s-algebras on X. Since $\langle F, \sigma \rangle$ is free, there exists a homomorphism $\varphi : F \to G$ such that $\varphi\sigma = \varsigma$. Since $\langle G, \varsigma \rangle$ is free, there exists a homomorphism $\phi : G \to F$ such that $\phi\varsigma = \sigma$. Hence $\phi\varphi\sigma = \phi\varsigma = \sigma$, and by the result above, $\phi\varphi = 1_F$. Similarly, $\varphi\phi = 1_F$. Thus φ and ϕ are mutually inverse isomorphisms, and so uniqueness is proved.

b) *Existence*: An algebra F will be constructed as a union of sets F_n $(n \in \omega)$, which are defined inductively as follows:

1) $F_0 = X$

2) F_r is defined for $0 \leq r < n$. Then define $F_n = \{(s_t, a_1, \ldots, a_k) | s_t$ is a term of s, $s_t = \langle 0_1, \ldots, 0_k \rangle, a_i \in F_{r_i}, \sum_{i=1}^{k} r_i = n-1\}$

3) Put

$$F = \bigcup_{n \in \omega} F_n$$

The set F is now given. In order to make it into a s-algebra, we must specify the action of the operations corresponding to the types of s.

4) If $s_t = \langle 0_1, \ldots, 0_k \rangle$, put $\overline{s_t}(a_1, \ldots, a_k) = (s_t, a_1, \ldots, a_k)$.

This makes F into a s-algebra. Now, this point deserves some comments. We are constructing an s-algebra, and for that we must specify the operations of that algebra. As an abuse of language, we allow that the operation being constructed be named almost exactly as the type of the element of sequence s which determines the similarity type of the algebra being constructed, as can be seen from item 4 above, where on the left side of the equality $\overline{s_t}$ denotes the operation being defined, and on the right, without overline, s_t denotes the type of the sequence s. The reason for that identification comes from the fact that we still do not have a structure with operations, for the operations are being built from the elements of the sets F_n. To complete the construction, we must give the map $\sigma : X \to F$.

5) For each $x \in X$, put $\sigma(x) = x \in F_0$.

Finally, we need to prove that F is free on X, i.e., we must show that if A is any s-algebra and $\tau : X \to A$ any map of X into A, then there exists a unique homomorphism $\varphi : F \to A$ such that $\varphi\sigma = \tau$. We do this by constructing inductively the restriction φ_n of φ to F_n and by showing that φ_n is completely determined by τ and by φ_k for $k < n$.

We have $F_0 = X$. For $x \in X$ the homomorphism condition requires $\varphi\sigma(x) = \tau(x)$, and since $\sigma(x) = x \in F_0$, we must have $\varphi_0(x) = \tau(x)$. Thus $\varphi_0 : F_0 \to A$ is defined, and it is uniquely determined by the conditions to be satisfied by φ.

Suppose that φ_k is defined and uniquely determined for $k < n$. An element of $F_n (n > 0)$ is of the form $(s_t, a_1, \ldots, a_k)$ with $s_t = \langle 0_1, \ldots, 0_k \rangle$ $a_i \in F_{r_i}$, and

$$\sum_{i=1}^{k} r_i = n - 1.$$

Thus $\varphi_{r_i}(a_i)$ is already uniquely defined for $i = 1, \ldots, k$. Furthermore, since $(s_t, a_1, \ldots, a_n) = s_t(a_1, \ldots, a_k)$ and since the homomorphism property of φ requires that $\varphi(s_t, a_1, \ldots, a_k) = s_t(\varphi(a_1), \ldots, \varphi(a_k))$, we define $\varphi_n(s_t, a_1, \ldots, a_k) = s_t(\varphi_{r_i}(a_1), \ldots, \varphi_{r_i}(a_k))$.

This determines φ_n uniquely, and as each element of F belongs to exactly one subset F_n, in putting $\varphi(\alpha) = \varphi_n(\alpha)$ for $\alpha \in F_n$ $(n \geq 0)$, we see that φ is a homomorphism from F into A satisfying $\varphi\sigma(x) = \varphi_0 = \tau(x)$ for all $x \in X$ as required, and that φ is the only such homomorphism, and this completes the proof. □

Now, we pass to the development of languages properly, which will be considered as specific free s-algebras as presented above.

3.2. *The Language of Simple Type Theory*

In this section we shall present as a specific s-algebra the language of simple type theory.

For each type $a \in \tau$, we have:

1. a denumerable set V_a, called the set of variables of type a.
2. a set R_a of the constant symbols of type a, which may be eventually empty for some of the elements of τ.

Also, we define the set of all variables and constants of the language.

Definition 3.6: *For all $a \in \tau$:*

1. *$V = \bigcup V_a$, the set of variables;*
2. *$R = \bigcup R_a$, the set of constants.*

As we can see there will be in particular variables for each set of the domain, that is, for each set D_k, there will be a family of variables restricted to D_k, so that our language is a many-sorted one when $1 \leq k$, for only in these cases we will have more than one set in the domain.

For each element of τ, we define the terms of that type:

Definition 3.7: *If $a \in \tau$ the set T of terms is defined by:*

$$T_a = V_a \cup R_a$$

Since we are treating formal languages as algebras, the next step is to specify the set of generators of a free algebra accordingly. We take X as the following set: $X = \{T^a(t_0, \ldots, t_{n-1}) | T^a \in T_a, a = \langle a_0, \ldots, a_{n-1}\rangle \in \tau, t_k \in T_k\}$.

As one can see, in this definition $a \neq 0, 1, \ldots, n-1$. The intuitive idea is that individual terms should not by themselves form atomic formulas.

Let $s = \{\neg, \rightarrow, (\forall x) | x \in V_a, a \in \tau\}$, where $\neg = \langle 0\rangle$, $\rightarrow = \langle 0, 0\rangle$, and $(\forall x) = \langle 0\rangle$ for each such symbol. Here, the reader should keep in mind the identification made in the proof in the last section of the operations in the free s-algebra with the elements of the sequence s. The connectives and quantifiers of the language are not really types, but, as the construction made above indicates, they are constructed from the corresponding types of the sequence s and, according to the convention made above, the same symbol is used to name both. $P(V, R)$ is the free s-algebra on the set X above, with V and R as defined above. Once again, the reader should not confuse the symbols, which will represent relations defined on the algebra with their types, which tells us what kinds of objects it relates and the weight of the relation.

An element w of $P(V, R)$ is called a formula.

Definition 3.8: *Given $w \in P(V, R)$, the set $V(w)$ is the set of variables of the formula w, defined as*

$$V(w) = \bigcap \{U | U \subseteq V, w \in P(U, R)\}$$

The variables occurring in a formula may be bound or free. We define the set of free variables as follows:

Definition 3.9: *Let $w \in P(V, R)$. The set $var(w)$ of free variables of w is inductively defined:*

1. $var(R^a(t_0, \ldots, t_{n-1})) = \{t_i : t_i \in V_i\}$;
2. $var(X^a(t_0, \ldots, t_{n-1})) = \{t_i : t_i \in V_i\} \cup \{X^a\}$;
3. $var(\neg w) = var(w)$;
4. $var(w_1 \rightarrow w_2) = var(w_1) \cup var(w_2)$;
5. $var(\forall x w) = var(w) - \{x\}$.

The language thus created is the language of type theory. One can now easily go ahead and present axioms and inference rules for it, but we shall not do such things here.

3.3. *The order of a language*

We can now consider fragments of the language of type theory. It is easy to see how one can obtain, for example, from the language of the theory of types as exposed here, what is usually known as first-order, second-order or n-th order languages in general. We begin approaching the topic more rigorously with the following definition:

Definition 3.10: *The order of a term is the order of its type.*

Now, restricting ourselves to terms of order 1 or less, and in particular to variables of order 1 or less, we can easily obtain second order languages. When we restrict ourselves this way, we consider in building the similarity type s only quantifiers over variables of the types available to us, which are the ones of order one or less, and so, we quantify only over individuals or properties and relations over individuals. In the same way, we can obtain n-th order languages by restricting ourselves to the sets of terms of order $n-1$ and less, and, in this way, allowing quantification over variables of order $n-1$ and less.

Definition 3.11: *The order of a language £ is defined as*

$$ord(\pounds) = max\{ord(x_a) : x_a \in V_a\} + 1.$$

So, the order of a language gives us information on the sets over which we quantify, or, more precisely, over the order of objects over which we quantify.

4. *The language of a structure*

In this section, we relate both proposals made above: the theory of structures and the language of types seen as a free algebra. An interesting discussion in the philosophy of mathematics and foundations is whether there is a most natural language to use associated with a given structure. According to our approach, associated with every structure *e* there will be a language which is the one in which we will talk about the elements of the structure, having as constant symbols exactly one symbol for each relation occurring in the structure, with both the symbol and the corresponding relation being of the same type. As we said before, given a structure *e*, to build one language for this structure we consider ord(*e*) and restrict ourselves to terms of this order or less. Also, the set X of generators of the free algebra will be restricted accordingly, that is, in building the language as a free algebra, the generators will be the atomic formulas built from the symbols available to us.

As a consequence of this discussion, the languages in which we can treat more adequately the elements of a structure are the languages $\pounds$ such that $ord(e) \leq ord(\pounds)$[3]. So, for example, in first-order structures, we must use at least first-order languages. To consider a simple example, let’s take group theory, which deals with groups (first-order structures). These structures, according to the approach adopted in this work, are most naturally treated by second-order languages, for we must talk about subgroups and quantify over subsets of the domain. That does not mean that it is impossible or not fruitful to use first-order languages; in fact, the work is commonly done in first-order language, as when we use set theory to develop group theory, or other mathematical theories such as well-ordered structures or Dedekind-complete fields, theories which are not first-order (see discussion in [5, pp. 89, 90]).

There is a live discussion on this topic in the philosophy and foundations of mathematics. Should we restrict ourselves to first-order languages or should we adopt higher-order languages? We shall not enter into the dispute here (for a defence of second-order logic in these contexts, see, for example, Shapiro in [6]). The general approach followed in this work suggests that a pluralist view should be pursued: it is fruitful to explore higher-order languages as well as first-order languages. Even though the latter have the preference of most philosophers, one must recognize that some kinds of structures employ more naturally dealt with higher-order languages. In fact, the reader may even wonder whether the problem of a “better” language for certain mathematical theories is a legitimate one (for agnosticism about this problem, see Hodges [4, pp. 71-73]).

Having a language with which we can talk about the elements of *e*, it is now simple to define for this language the notions of the structure modeling a sentence α of the language, that is, $e \models \alpha$, as well as other semantical notions, but we will not do that here.

5. *Suppes’ Predicate*

Given a structure and some language adequate for this structure, we now discuss how to formulate a Suppes’ predicate for that structure using that language (we follow da Costa and Chuaqui, see [2]). First of all, we recall the definition of the similarity type of a structure *e*, which is a family of types that determines the kinds of the relations present in the structure. According

[3] We are following here also some suggestions made by da Costa through personal communications in seminars.

to this definition, two structures have the same similarity type if the types of their relations form the same family and they have the same number of sets in their domain.

Now, given structures $e = \langle D_n, R_\iota \rangle$ and $g = \langle E_n, L_\iota \rangle$ of the same similarity type, we consider how to extend a given function $f : D_k \mapsto E_k$ for $k < n$ to a function mapping from $\varepsilon(D_n)$ to $\varepsilon(E_n)$.

Definition 5.1: *Given the function f as described above, for each type $a \in \tau$ we define:*

1. *For the objects of type k, with $0 \leq k < n$ $f(D_k) = \{f(x) : x \in D_k)\}$;*
2. *For $a \in \tau$ such that $a = \langle a_0, \ldots, a_{n-1} \rangle$, and R the set of objects of type a, we have $f(R) = \mathcal{P}(f(t_{a_0}) \times f(t_{a_1}) \times \ldots \times f(t_{a_{n-1}}))$*

This function maps objects of the type a in $\varepsilon(D_n)$ to objects of type a in $\varepsilon(E_n)$. The interesting case occurs when the following definition is verified:

Definition 5.2: *Given structures e $= \langle D_n, R_\iota \rangle$ and g $= \langle E_n, L_\iota \rangle$ of the same similarity type $s_{\lambda<\iota}$, and f a bijection from D_k to E_k with $0 \leq k < n$, we say that the family $f' = f_{s_{\lambda<\iota}}$ is an isomorphism between e and g when $f_a(R^a) = L^a$, where R^a and L^a means that R and L have type a.*

Definition 5.3: *A sentence Φ of the language appropriate for the structure e is called transportable if for any structure g isomorphic to e*

$$e \models \Phi \Leftrightarrow g \models \Phi.$$

Definition 5.4: *A Suppes'predicate is a formula P(e) of set theory which says that e is a structure of similarity type s satisfying Γ, a set of transportable sentences Φ of the language adequate for e.*

When P(*e*), that is, when *e* satisfies P, we say that *e* is a P-structure. According to da Costa and Chuaqui ([2, p.104]), this definition captures the sense in which we can say that a theory is a class of models, precisely, the class of models that are P-structures for some adequate P.

We now consider some examples of Suppes' predicates for some theories. Our goal is to show how classical particle mechanics can be axiomatized in the approach exposed here.

Example 5.1: *A Suppes' Predicate for Group Theory*

Let G be a set; as defined above, we introduce the function t_G, or simply t, whose domain is the set τ, to create the scale $\varepsilon(G)$. Then, we choose a relation $\circ$ of type $\langle 0,0,0\rangle$, that is, $\circ \in \mathcal{P}(G\times G\times G)$, a relation $-$ of the type $\langle 0,0\rangle$, that is, $-\in \mathcal{P}(G\times G)$ and an element i of type 0, that is, $i\in G$. As one can check from the definitions, the order of each of the relations is 1 and the order of i is 0. Remembering that a n-ary function is a $n+1$-ary relation, we have that the usual composition operation becomes a ternary relation, the opposite relation becomes a binary relation.

The structure of groups is $\mathfrak{G} = \langle G,\circ,-,i\rangle$ and the order of this structure is the greatest order of its relations, so $ord(\mathfrak{G})$ is 1. The theory is not done yet, for we need to write the postulates and give the set-theoretical predicate.

As defined above, the language for $\mathfrak{G}$ is a second-order language. The set T of terms is formed by a set of variables $V=\bigcup V_a$, where $ord(a)\leq 1$ and the set $\{\circ^{\langle 0,0,0\rangle}, -^{\langle 0,0\rangle}, i^0\}$ of constant symbols. So, the set X of free generators is $X=\{T^a(t_0,\ldots,t_{n-1})|T^a\in T_a, a=\langle a_0,\ldots,a_{n-1}\rangle\in\tau, t_k\in T_k\}$, and by the theorem above, there is a free algebra upon the set of free generators X, and it is the language for the structure $\mathfrak{G}$.

With the language so developed we can write the usual axioms for group theory, let's call them A1, A2 and A3, respectively as follows:

1. $\forall x\forall y\forall z(((x\circ y)\circ z)=(x\circ(y\circ z)))$
2. $\forall x\exists y(x-y=i)$
3. $\forall x(x\circ i=x)$

Then, a Suppes predicate for group theory can be written as follows:

$$\mathfrak{G}(X) \Longleftrightarrow \exists G\exists\circ\exists-\exists i(X=\langle G,\circ,-,i\rangle\wedge A1\wedge A2\wedge A3)$$

Now, we finally proceed to present the Suppes' predicate for classical particle mechanics.

Example 5.2: *First, we need to present some mathematical structures: the first one is the field of real numbers $\mathfrak{R}$. There is just one base set, the set $\mathbb{R}$ of real numbers. The objects of this set are of type 0. The operations are $+$, $\cdot$, 0, 1, $<$, which are of types $\langle 0,0,0\rangle$, $\langle 0,0,0\rangle$, 0, 0 and $\langle 0,0\rangle$ respectively (one must not confuse the symbol 0 of types with the element 0 of the field). As usual, these are the operations of addition, multiplication, the identity element of addition, the identity element of multiplication and the relation of less than between real numbers. The language of the field of real numbers* *according to our approach is a second-order language and the axioms for the field are well known and we can easily see that they are transportable.*

The next structure is the vector space over the field of real numbers $\mathfrak{V}$*. In this case there are two base sets, the set* V *of vectors and the set* $\mathbb{R}$ *of real numbers. The objects of* V *are of type* 1*. Besides the field's operation we have here* $+$, $\cdot$, $\mathsf{0}$ *which are respectively of types* $\langle 1,1,1\rangle$, $\langle 0,1,1\rangle$, 1*. A Euclidean vector space is a vector space with addition of inner product and vector product, denoted by* (x,y) *and* $[\mathsf{x},\mathsf{y}]$ *of types* $\langle 1,1,0\rangle$ *and* $\langle 1,1,1\rangle$ *respectively. The order of vector space language and of the Euclidean vector space is* 2 *according to our approach, that is, a second-order language. The axioms for these structures are well known and clearly transportable.*

Next, we present the real affine space which is a vector space with an addition domain A*, whose elements are of type* 2*. The new operation is difference of points which, for* $p,q\in A$ *is denoted by* $q-p$*, whose type is* $\langle 2,2,1\rangle$*. The new axiom needed is the statement that difference of points is vector and the law of addition of points, which says that for* $p,q,r\in A$*,* $q-p+r-q=r-p$*. If the vector space is Euclidean, the affine space is a Euclidean space. In Euclidean space we can define the distance between points* $p,q\in A$ *in the following way:* $d(p,q)=|q-p|=\sqrt{(\mathsf{q}-\mathsf{p},\mathsf{p}-\mathsf{q})}$*.*

Now, we present a Galilean space-time system. We add to the four dimensional affine space presented earlier a new universe V_1 *which is a subset of* V*, and operation* t *from* A *into* $\mathbb{R}$ *of type* $\langle 2,0\rangle$ *and relations of type* $\langle 3,3,1\rangle$ *and* $\langle 3,3,3\rangle$ *denoted by* $(,)$ *and* $[,]$ *respectively.* t *represents the measure of time. The two axioms following must be satisfied:*

1. V_1 *is a three dimensional vector subspace of* V *and* $(,)$ *and* $[,]$ *are its scalar product and vector product, respectively;*
2. t *is a function from* A *to* $\mathbb{R}$ *such that for each* $P\in A$ *the set* $\{Q : \mathsf{t}(Q)=\mathsf{t}(P)\}$ *is a three dimensional euclidian space with vector space* V_1*. The affine space for* $\mathsf{t}(P)=r$ *denoted by* $A(r)$*.*

For a classical mechanical system we add new universe P*, the set of particles and the set* $\mathbb{N}$ *of natural numbers to index the external forces. So the family of universe can be given by the sequence* $\mathbb{R}$, V, A, V_1, P *and* $\mathbb{N}$*. The operations on these sets are those necessary to make* $\mathbb{R}$, V, A, V_1 *a Galilean space-time system plus the following new relations:*

1. *A function* a *of type* $\langle 0,2\rangle$ *which gives the origin at each time;*
2. *A function* s *of type* $\langle 4,0,2\rangle$ *for the position of a particle in each time. We write* $\mathsf{s}_p(t)$ *for this function;*
3. *A mass function m of type* $\langle 4,0\rangle$*;*

4. *A force function* f *of type* $\langle 4,4,0,1\rangle$ *which represents the internal forces;*
5. *A force function* g *of type* $\langle 4,0,5,1\rangle$ *which represents the external forces.*

For the specific axioms of mechanics we need notions of analysis such as derivatives and convergence of series. The field of real numbers must be completed with the corresponding operations of differentiation, integration and addition of series. Since differentiation, for example, takes functions of real numbers to functions of real numbers, the order of this operation will be 2. So, the language needed to talk about this is structure is at least third-order language.

The kinematical axioms are:

1. *The range of* t *is an interval* I *of real numbers;*
2. P *is a finite and non-empty set;*
3. a *is a function from* I *to* A *such that for each* $i \in I$ $\mathsf{a}(i) \in A(i)$*;*
4. s *is a function from* $\mathsf{P} \times I$ *into* A *such that for each* $p \in \mathsf{P}$ *and* $i \in I$ *we have that* $\mathsf{s}_p(i) \in A(t)$*;*
5. *m is a function from* P *into* $\mathbb{R}$*;*
6. f *is a function from* $\mathsf{P} \times \mathsf{P} \times I$ *into* V_1*;*
7. g *is a function from* $\mathsf{P} \times I \times \mathbb{N}$ *into* V_1*;*
8. *For every* $p \in \mathsf{P}$ *and* $i \in I$ *the vector function* $\mathsf{s}_p(i) - \mathsf{a}(i)$ *is twice differentiable at i.*

Dynamical Axioms

9. *For* $p \in \mathsf{P}$ $m(p)$ *is a positive real number;*
10. *For* $p, q \in \mathsf{P}$ *and* $i \in I$ $\mathsf{f}(p,q,i) = -(q,p,i)$*;*
11. *For* $p, q \in \mathsf{P}$ *and* $i \in I$ $[\mathsf{s}(p,i) - \mathsf{s}(q,i), \mathsf{f}(p,q,i) - \mathsf{f}(q,p,i)] = 0$*;*
12. *For* $p \in \mathsf{P}$ *and* $i \in I$ *the series* $\Sigma_n(\mathsf{g}(p,i,n)$ *is absolutely convergent;*
13. *For* $p \in \mathsf{P}$ *and* $i \in I$ $m(p)D^2(\mathsf{s}_p(i)) = \Sigma_{q\in \mathsf{P}}\mathsf{f}(p,q,i) + \Sigma_n(\mathsf{g}(p,i,n)$.

where D^2 *is the second derivative with respect to* i.

These formulas are transportable, in the sense defined previously. The motivations for these formulas can be found in the works of Suppes cited in the bibliography.

6. *Final Remarks*

As a first remark, one should note that everything developed here could in principle be done inside category theory. Our use of ZFC set theory reflects primarily the fact that this kind of framework is the most usual in mathematical practice. Obviously, the specific apparatus in which one works may have important consequences for the development of theories according to the semantic view. As an example, if we are working inside a countable model of first-order ZFC (which exists by Löwenheim-Skolem theorem if ZFC is consistent), then the set of real numbers, for one example, is denumerable. How does that impinge on physical theories that employ real numbers in an essential way? For another example, consider paraconsistent set theories. Inside those frameworks one may allow for contradictions in the development of theories; could that bear fruitful consequences for our understanding of current scientific theories? Those are questions which, to our mind, are more easily noticeable and dealt with through the use of a rigorous framework such as the one presented here, even though we shall not pursue those particular points now[4].

Now that we reached our goal, it’s time to make one final point clear. The presentation of particle mechanics above may have conveyed to the reader the impression that the method proposed here is quite complex and may not repay our efforts. Suppes’ predicates, as they appear in the work of Suppes and his collaborators may seem preferable, it may be argued, since they are easier to work with, they are more akin to the informal style of the working mathematician. Given this situation, what can we say in favor of the approach to scientific theories developed in this work, and in particular to the theory of structures proposed here?

Our claim is that, despite our being conscious that the method proposed here involves some additional complications when compared to Suppes’ original approach, it has the great advantage of precision and the extra effort spent in working through its’ complications repays when we consider the great promises of important results that may be achieved. To cite but one, we mention da Costa and Rodrigues Generalized Galois theory, developed in [3]. Difficult problems of definability of concepts may be studied according to their approach and the useful techniques developed by them apply to the framework presented here. When one is interested in precise results and conceptual clarification, one needs a rigorous framework to begin with, and we think the one proposed here is apt for many applications. For most practical applications in general philosophical discussions we can relax the rigour and proceed more or less informally, but when it comes the time for some kinds of technical applications, such as demanded by some specific

[4] We thank an anonymous referee for pressing on this point.

philosophical labor in foundational studies, it is better to shift to a precise framework like the one proposed here.

As one example of a philosophical investigation in which the technical apparatus proposed here may help us, we could mention once again the uncovering of some underlying assumptions of the semantic view on scientific theories, such as the use of formal languages to determine the class of models that a particular theory is. Philosophers of science tend to believe that this particular approach to determine a theory as a class of models is independent of a formal language; as we have seen, if 'model' is taken in the rigorous set theoretical sense, then such a remark may not be true. This particular topic, as others previously mentioned, is of the greatest relevance for the philosopher of science, and its investigation may follow on the basis of the present one. Another relevant application not noticed by many concerns the logical foundations of theories, and relates to the distinct kinds of models obtained by the use of distinct kinds of languages. For instance, to keep talking about real numbers, if we change a little bit the language employed in the axiomatization of mechanics and allow that the field of real numbers be axiomatized by a first-order language only, then non-standard models of the reals enter in the theory too, in particular enumerable models. As far as we know, little attention has been given to the problems such facts may raise, mainly in the relationship of the field of real numbers and the development of mechanics, and we believe that the apparatus presented here may be helpful in such investigations.

Jonas R. Becker Arenhart:
Department of Philosophy
Fronteira Sul Federal University – Chapecó, Brazil
E-mail: jonas.becker2@gmail.com
Fernando T.F. Moraes:
Logic and Foundations of Science Research Group, Brazil
E-mail: fermatadeu@uol.com.br

REFERENCES

[1] Barnes, D.W., and Mack, J.M., 1975, *An Algebraic Introduction to Mathematical Logic*, Springer-Verlag, New York.
[2] da Costa, N.C.A. and Chuaqui, R., 1988, "On Suppes' Set Theoretical Predicates", *Erkenntnis*, 29: 95–112.
[3] da Costa, N.C.A. and Rodrigues, A.A.M., 2007, "Definability and Invariance", *Studia Logica*, 86: 1–30.

[4] Hodges, W., 2001, "Elementary Predicate Logic", in Gabbay, Dov. M., and Guenthner, F., (eds.), *Handbook of Philosophical Logic*, vol. 1, 2nd. edition, 2001, Springer.
[5] Kunen, K., 2009, *The Foundations of Mathematics*, College Publications, London.
[6] Shapiro, S., 1991, *Foundations without foundationalism, a case for second-order logic*, Clarendon Press, Oxford.
[7] Suppes, P., 1960, "A Comparison of the Meaning and Uses of Models in Mathematics and the Natural Sciences", *Synthese*, 12, pp. 287–301.
[8] Suppes, P., 1967, *Set-theoretical Structures in Science*, Notes Un. Stanford.