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MATHEMATICAL REASONING AND EXTERNAL SYMBOLIC SYSTEMS

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It is an almost trivial observation that the practice of mathematics typically involves a lot of 'scribbling and fiddling' with symbols, diagrams and special notations. Taking as a starting point the idea that both the written and the oral languages used by mathematicians are philosophically relevant aspects of their practices, the aim of this paper is to discuss in more detail the exact status of external symbolic systems, systems of writing in particular, for mathematical reasoning and mathematical practice. Are they merely convenient devices? Are they essentially heuristic components? Can mathematics be practiced without recourse to symbolic systems? In what sense, if any, can different forms of writing be said to be *constitutive* of doing mathematics?

The perspective adopted here is a combination of philosophical analysis with focus on empirical studies on numerical cognition (ranging from cognitive science to developmental psychology and anthropology), as well as on the history of notations in mathematics.[1] Indeed, the investigation takes into account three different levels: the synchronic level of a person 'doing math' at a given point in time; the diachronic, *developmental* level of how an individual learns mathematics; and the diachronic, *historical* level of the development of mathematics as a discipline through time. It will be argued that the use of external symbolic systems is constitutive of mathematical reasoning and mathematical practice in a fairly strong sense of 'constitutive', but not in the sense that manipulating notations is the *only* route to mathematical insight. Indeed, two case studies will illustrate this qualification: a man with acquired savant syndrome and a blind mathematician.

[1] This seems to me to be a fruitful way to adopt a practice-based philosophical perspective. As I have argued in (Dutilh Novaes 2012a), practice-based philosophy of any science must be thoroughly empirically-informed.

1. *The relations between mathematics and its languages*

In (Macbeth 2013), D. Macbeth discusses three different positions one could take on the role of notations and other graphic elements such as diagrams — what she describes as "writing, broadly conceived" — for mathematical reasoning.[2] (As she remarks, these are not only theoretically possible positions; each of them has actually been defended by real people.) Let us examine each of them in turn.

a. Written mathematical symbols are constitutive of mathematical reasoning

Macbeth attributes this position to Kant[3] and, as a more recent source, to Rotman (2000). But the most outspoken (recent) defenders of the idea that external devices such as inscriptions of any kind — writing, diagrams, notations etc. — are constitutive of the cognitive processes in which they are involved are the proponents of the *extended mind thesis* (such as (Clark 2008) and (Menary 2010a)). The extended mind thesis has also been applied specifically to mathematical notations in (De Cruz & De Smedt 2013), where a convincing case is made for the claim that mathematical notations are *constitutive*, in a strong sense of the term, of mathematical reasoning and mathematical practices. Quoting from the abstract:

> This paper draws on the extended mind thesis to suggest that mathematical symbols enable us to delegate some mathematical operations to the external environment. In this view, mathematical symbols are not only used to express mathematical concepts — they are

[2] Naturally, 'mathematical reasoning' and 'mathematical practices' are blanket terms covering a wide range of related but distinct phenomena. I use these terms merely for the sake of brevity; this should not be interpreted as an endorsement of an overly homogeneous view of mathematics as a discipline, or of the idea that there is a core essence to mathematical reasoning — there may well be one, but arguing for such a view is not the aim of the present analysis.

[3] "Kant famously held that writing broadly conceived as the inscribing of marks is a constitutive feature of mathematical practice." (Macbeth 2013, p. 25).

> constitutive of the mathematical concepts themselves. Mathematical symbols are epistemic actions, because they enable us to represent concepts that are literally unthinkable with our bare brains.[4] (De Cruz & De Smedt 2013, 3)

The two other positions described by Macbeth agree on their rejection of the idea of writing broadly construed (i.e. as manipulation of inscriptions) as constitutive of mathematical reasoning. But they disagree on whether mathematical reasoning is inherently *linguistic* or not. According to one of them, mathematical reasoning is in fact inherently tied to particular public languages, namely the vernacular languages of 'everyday life', but not to the specifically designed mathematical notations. In contrast, according to the third position, mathematical reasoning is not in any way language-dependent, or in any case not dependent on any *public* language such as speech or writing. Let me elaborate a bit on each of them.

b. Mathematical reasoning is conducted in vernacular languages; notations are merely convenient short-hands

This position, as described by Macbeth, views mathematical reasoning as constitutively independent of special systems of notation, but as inherently tied to vernacular languages. The medium in which mathematics is conducted is a specialized dialect of English, or whatever other vernacular language(s) the mathematician is proficient in. Macbeth offers the following quote from a textbook to illustrate the position: "The symbols are simply a convenience: It is easier to write 'x^2' than 'the square of x', and '$x \in$ A' is more compact than 'x is an element of the set A'. In each case the meaning is the same."[5]

[4] Philosophers working within the extended mind framework diverge on whether they seek to emphasize the *similarities* or the *differences* between cognitive processes involving or not involving external devices. Those who emphasize the similarities typically endorse the so-called 'parity principle' and concentrate on the metaphysical question of the boundaries of the mind. By contrast, those who emphasize the differences rely on the so-called 'complementarity principle', and seek to investigate the transformative power of engaging with external devices for human cognition (see the editor's introduction in (Menary 2010a)). De Cruz and De Smedt clearly belong to the second group, and so do I (Dutilh Novaes 2012b).

[5] (Schumacher 2001, 1). Arguably, this position would also require a more thorough elaboration of possible significant differences between vernacular speech and vernacular writing, as clearly the assumption being made is that vernacular writing is on a par with, or in any case much closer to, vernacular speech than to written mathematical notations. However, there are reasons to think that this is a misguided approach to writing — see (Krämer 2003), (Harris 1995) and (Menary 2007). Moreover, the very idea that there is a clear-cut distinction between 'natural' and 'artificial' languages is deeply problematic; insofar as writing is

c. Mathematical reasoning is not language-dependent

On this view, to 'do' mathematics would be a purely private, inner process, which can then be *expressed* and *communicated* a posteriori in some public medium such as systems of notations or spoken languages. (Macbeth's example of a proponent of this position is Fields medalist William Thurston.) Rotman's convenient term to describe this position is 'documentist': "This is the view that language, whether written or spoken, is invariably after the fact in mathematics, serving only to document or report results obtained independently." (Macbeth 2013, p. 27) In other words, the claim is that mathematical reasoning is metaphysically independent of, and temporally prior to, its expression in any form of public language; it requires no external medium at all to come about.

Thus, positions b. and c. both reject the notion that mathematical (written) notations are constitutive of mathematical practice. Positions a. and b. have in common the idea that mathematical reasoning requires some sort of external, linguistic medium to come about (as opposed to thoroughly internalist position c.), yet disagreeing on the exact nature of this medium.

The constitutivity question can thus be formulated with different degrees of generality. Is *writing* (mathematical notations in particular) constitutive of mathematical reasoning and mathematical practice? Alternatively, are *external media* in general (including inscriptions, utterances and, as we shall see shortly, other kinds of external representations) constitutive of mathematical reasoning and mathematical practice? Position c. answers 'no' to both questions; position b. answers 'no' to the first question but 'yes' to the second; position a. answers 'yes' to both (naturally, a positive answer to the first question entails a positive answer to the second question).

There is of course a weak sense in which mathematical practices require external media, namely in the sense that mathematics requires *public justification* of chains of reasoning. A mathematical demonstration is a piece of discourse presupposing a putative audience; it puts forward a chain of reasoning for public scrutiny. But here we can turn to the good old distinction between context of discovery and context of justification to argue that mathematical practices and mathematical reasoning, broadly construed, should not be reduced to contexts of justification; beyond justification, we may also want to understand the contexts of mathematical *discovery*. The question is thus whether external media are required for mathematical *discovery*, given that external media are trivially constitutive of mathematical justification.

However, there is another crucial dimension relevant for the notion of 'being constitutive of doing mathematics' (i.e. including both justification and

clearly a cultural technology, one could argue that it falls on the same side of the natural vs. artificial divide as mathematical notations.

discovery), namely a *temporal* dimension. We must consider three distinct levels: the synchronic level of the mathematician 'doing mathematics' at a given point in time; the diachronic, 'ontogenetic' (developmental) level of how mathematical skills are acquired by a given individual; and the diachronic, 'phylogenetic' (historical) level of how mathematical knowledge has emerged and developed throughout the history of mathematics. Distinguishing these three levels is essential if we are to make sense of the idea of a particular external medium (be it vernacular speech, special forms of writing such as mathematical notations or yet other media) being constitutive of mathematics. In the next section I discuss each of these three levels, drawing in particular on studies on numerical cognition, anthropology, developmental psychology, and the history of mathematics.[6]

2. *Numerical cognition and external symbolic systems*

2.1. *The synchronic level*

Superficial phenomenological observation strongly suggests that mathematical notations play a crucial role in mathematical practices. Indeed, it is a well-known fact that, in practice, anyone doing mathematics (professional mathematicians, those in the process of learning mathematics, the lay person doing a simple calculation, etc.) almost invariably makes extensive use of writing devices (pencil and paper, chalk and blackboard, etc.).[7]

Again, nobody would deny that making use of writing greatly facilitates mathematical reasoning, if nothing else because it offloads internal working memory. For example, it is much easier to undertake a long calculation if one can rely on pen and paper to keep track of its steps externally rather than internally. But this may also simply be (at least in part) a contingent result of how people happen to be trained to do mathematics, not necessarily something inherent to the cognitive activity of doing mathematics in general. The idea of 'being constitutive' seems to entail something stronger, namely that it should be *impossible* to engage in a given practice in any other way.

[6] While it goes without saying that mathematics goes well beyond numerical cognition, thus far the empirical studies have almost exclusively focused on this particular area. Hence, at this point the analysis is less wide-ranging than what might be hoped for, but this is related to the somewhat limited availability of empirical data concerning other aspects of mathematics. At any rate, given limitations of space, it also appears to be a methodologically sound approach to focus on what is arguably the most basic level of mathematical cognition.

[7] "A … mathematician generally works by sitting around scribbling on paper: According to one legend, the maid of a famous mathematician, when asked what her employer did all day, reported that he wrote on pieces of paper, crumpled them up, and threw them into the wastebasket." (Jackson 2002, 1246)

Indeed, the claim that writing is constitutive of mathematical reasoning entails that it not only *records* independent processes; writing is in fact viewed as an integral part (embodiment) of these very cognitive processes — that is, at the very least *when* they are accompanied by writing. The opening passage in Clark's *Supersizing the Mind* (2008) recounts an anecdote in this spirit, involving the legendary physicist Richard Feynman:

> Consider this famous exchange between the Nobel Prize-winning physicist Richard Feynman and the historian Charles Weiner. Weiner, encountering with a historian's glee a batch of Feynman's original notes and sketches, remarked that the materials represented "a record of [Feynman's] day-to-day work." But instead of simply acknowledging this historic value, Feynman reacted with unexpected sharpness:
>
> "I actually did the work on the paper," he said.
> "Well," Weiner said, "the work was done in your head, but the record of it is still here."
> "No, it's not a record, not really. It's working. You have to work on paper and this is the paper. Okay?" (from Gleick 1993, 409)
>
> Feynman's suggestion is, at the very least, that the loop into the external medium was integral to his intellectual activity (the "working") itself. But I would like to go further and suggest that Feynman was actually thinking on the paper. (Clark 2008, xxv)

The 'thinking on the paper' claim must be spelled out. Of course, it is perfectly possible to engage in practices that are properly described as 'doing mathematics' without manipulating notations at a given time (e.g. mental calculation, to be discussed shortly). But the claim seems to be that, *when* a particular person is doing mathematics *accompanied by such manipulations*, then the manipulations themselves are constitutive of that particular process. This would be a weaker (but still non-trivial) sense in which the idea that writing is constitutive of doing mathematics can be cashed out: constitutive, but only *when present*.

Ultimately, whether mathematical reasoning requires the manipulation of external representational vehicles is essentially an *empirical* question. The different ways to formulate the issue and the various implications of each position may be discussed on a philosophical, abstract level, but at the end of the day we must look into how people actually 'do mathematics' to address these questions. I here argue that, based on empirical data drawn from different fields, a strong case can be made for the claim that external media are constitutive of mathematical knowledge and mathematical reasoning in the stronger sense that, even when a given person is apparently not manipulating symbols, such as a mental calculator, she in fact typically relies extensively on internalized versions of external devices (at least in most cases).

Let us thus start with the several reported cases of mental calculators, i.e. people who can perform incredibly long, complicated calculations without resorting to paper and pencil or anything of the kind.[8] Naturally, the general idea of 'mathematical reasoning' goes much beyond making numerical calculations, but mental calculators are a good example to illustrate the claims I am about to make.

The main question is: in what sense are mental calculators calculating *mentally*? By and large, they seem to be resorting to internalized external representations such as: words and symbols for numbers, paper-and-pencil algorithms which are simulated internally, mnemonic devices, tables of multiplication learned by heart by oral repetition etc. Importantly, there are different styles of mental calculation. For example, mental calculations performed by people trained in the Hindu-Arabic numeral system typically rely on strategies emerging from features of the decimal system. It is known for example that the fastest and least error-prone mental calculations are those consisting in adding a given number smaller than 10 to a multiple of 10 (10, 20, 30 etc.); this is arguably because the reasoner mentally 'replaces' the 0 on the right-side of one of the numerals with the other numeral.[9] So it seems that, while not using 'paper and pencil' at that particular moment, the operation being implemented relies significantly on the mode of presentation of the Hindu-Arabic numeral system as a place-value system, and thus on an internalization of external symbols.

Besides inscriptions and utterances, numbers can be represented externally in a variety of ways. Indeed, humans have a long history of developing calculating devices/objects such as counting rods and abacuses, and each of them presupposes that quantities be represented so that they can be 'operated on' for calculation. So abacuses and counting rods can also be seen as symbolic systems, and as it turns out, they can also be 'internalized'.

In Japan, a popular technique for mental calculation relies on internal representations of abacuses, and is known as *anzan*.[10] A large number of children attend abacus clubs and first learn to perform calculations with a physical abacus (the Japanese version is called *soroban*), so that they become familiar with the processes. Once they are proficient calculators with the actual abacus (which usually involves years of training), they can begin

[8] See (Bellos 2010, 143–148) for an account of the variety of ways in which the practice of mental calculation has been and still is engaged in.

[9] These are known as *decade effects*. See (De Cruz et al. 2010, 93–94).

[10] See (Bellos 2010, 68–75) for details.

the training of mentally simulating the processes by visualizing an 'internal' abacus. The results are astonishing, including the ability to perform mental calculations while having one's attention distracted by other elements such as playing word games.

Thus, it would seem that, at least in most cases, mental calculations, even when performed by calculating prodigies, are essentially *internal manipulations* of previously mastered external symbolic systems. Additional support for this hypothesis comes from the study presented in (Tang et al. 2006), summarized in (De Cruz et al. 2010, 82):

> An intriguing fMRI study by Tang et al. (2006) provides indirect support for the role of symbolic representation in numerical cognition. In this study, both native English speakers and native Chinese speakers solved arithmetical operations. Although the IPS [intraparietal sulci] were active in both groups, they exhibited marked differences in other brain areas. Whereas the English speakers had a stronger activation in perisylvian, language-related areas such as Broca and Wernicke's areas, the Chinese speakers showed an enhanced response in premotor areas, involved in the planning of motor actions. The authors offered a possible reason for this: whereas English speakers learn arithmetical facts in verbal memory (e.g., when they learn multiplication tables), Chinese speakers rely on the abacus in their schooling. These differences in schooling might still be reflected in arithmetical practice, with English speakers mentally relying on language-based strategies, and Chinese speakers on motor-based strategies.

The exact neuronal details need not concern us here, but these results clearly suggest that mental calculations by Chinese participants engage motor processing; perhaps even unbeknownst to them (and without the special training given to the Japanese calculating prodigies mentioned above), they are to some extent reenacting calculating procedures performed externally with the abacus. English speakers, by contrast, having learned to calculate mostly relying on language-based strategies, apparently continue to rely on these strategies even when performing mental calculations.

2.2. *The diachronic, ontogenetic (developmental) level*

Hence, it seems that the cognitive processes of 'doing mathematics' on the synchronic level vary per individual, and in particular in function of the way in which she learned mathematics, i.e. what I have described as the diachronic, ontogenic, developmental level. This is for now most conspicuous for number cognition, but it may well carry over to 'higher levels' of mathematical practice — for example, when a professional mathematician mentally formulates a mathematical theorem (but again, it is a hypothesis to be investigated empirically).

Thus, the ontogeny of mathematical skills in a given individual must be brought into the picture. Naturally, the topic has been extensively studied, and it would be pointless to try to summarize the main results here. For our purposes, a key issue is the extent to which humans possess innate mathematical concepts, and the extent to which mathematical concepts and mathematical reasoning must be learned (from experience, exposure, formal education etc.) (De Cruz & De Smedt 2010). Specifically on numerical cognition, one currently widely accepted account is the one proposed by S. Dehaene (1997); according to his model, humans possess an innate grasp of very small exact quantities (up to 3 or 4), and an innate inexact, approximate grasp of larger quantities. Still on this model, basic exact arithmetic emerges from a combination of these two innate capacities, but it crucially depends on the existence of external symbolic representations for numbers (exact quantities) and on extensive training to arise. In other words, it requires the very practice of counting, i.e. associating 'names' to exact quantities, which, although highly pervasive, is not a universal feature of human languages (more on this shortly). One implication of this model is thus that, in the absence of external symbolic representations of exact quantities, even very basic arithmetical abilities should not emerge; seemingly, a person who was never explicitly taught to count will not be able to learn basic arithmetic operations at a later stage.

This hypothesis is further corroborated by anthropological studies of cultures whose languages have very few words for numbers. The Pirahã and the Mundukuru, two tribes in the Amazon, do not have words for numbers beyond very low ones (the Mudukuru have words to name quantities up to five, but beyond three they are not consistently used).[11] Even when, later in life, speakers of these languages learn another language which does have words for numbers, they never seem to be able to learn basic exact arithmetic. Their ability to estimate quantities and to calculate with approximations, however, is very similar to that of people fully embedded in 'numerical cultures'.[12]

But spoken words are not the only external devices used for counting. Some cultures have counting systems based on body parts, so that each exact amount would correspond to a given body part. The Oksapmin in Papua New Guine, for example, point at the corresponding body part and utter its name to indicate a certain quantity. (Naturally, such counting systems have a

[11] See (De Cruz et al. 2010, section 2.4).
[12] See (Pica et al. 2004).

fairly low upper bound; in the case of the Oksapmin, it goes up to 27.) Crucially, when members of such cultures are probed for their abilities in exact arithmetic (both as a result of formal training and of spontaneous learning), their performance is much more aligned with that of e.g. schooled speakers of English than with the performance of members of cultures which simply do not systematically engage in the practice of keeping track of exact quantities with external symbols at all (such as the Pirahã and the Murukundu).[13] These results suggest that, more than language-dependent, exact numerical cognition is *external-symbol-dependent*; it presupposes the very concept of exact quantities, which may only emerge by means of explicit association to external symbols and the practice of counting beyond very small amounts.

Developmental studies with young children seem to confirm this hypothesis. According to a popular (though not unanimously accepted) view, infants and young children have a logarithmic, non-linear representation of quantities in that the 'distance' between 1 and 2, for example, is larger than the 'distance' between 8 and 9. This was observed in experiments where they were asked to represent quantities in a number line. With schooling, however, their conception of quantities gradually converges towards the linear conception characteristic of the series of the natural numbers.[14] Moreover, in similar experiments with members of a culture with few words for numbers (the Mundukuru), the same logarithmic representation was observed.[15]

In fact (and as parents of young children know all too well), for most children, learning to count, i.e. to associate exact quantities to particular external symbols (usually spoken words), requires rather intensive training to be mastered. Counting is a practice which only emerges upon explicit instruction. The development of the concept of the linear, well-ordered series of natural numbers then typically continues in formal educational settings (pre-school etc.), which is a pre-condition for the child to learn arithmetic operations.

What do these considerations entail for the question of whether spoken language, and external symbols more generally, is (are) constitutive for mathematical reasoning? It would seem that even at the most basic level of numerical cognition, external representations are indeed required for the development of the concept of exact quantities beyond very small amounts (arguably, up to three). What both the anthropological and the developmental data suggest is that humans in fact require the existence of external symbols

[13] See (Saxe 1982) and (Saxe 1985).
[14] See (Siegler & Booth 2004).
[15] See (Dehaene et al. 2008).

and the appropriate training in order to develop the very concept of exact quantities.[16]

Going back to the three positions presented above, it seems clear that, even if at specific occasions (i.e. the synchronic level), 'doing math' does not require the act of manipulating external symbols (as the defender of position c., the 'documentist', would have it), from a diachronic, developmental point of view, external symbols appear to be a necessary condition for the emergence of mathematical concepts and mathematical reasoning. Moreover, I have argued that many of the processes which appear to take place exclusively 'in the head' are in fact *internal simulations of external processes*; in such cases, there is a clear sense in which external representations are constitutive of mathematical reasoning, even if they are simulated and manipulated mentally.

Thus, I claim that position c. is severely weakened by the empirical evidence discussed so far. But this body of evidence does not seem to offer sufficient data to 'break the tie' between positions a. and b. Indeed, I have stressed the crucial role of *spoken* words in particular for the emergence of exact numerical cognition; how could the proponent of a. then further argue against b.? Let us turn to the third level of analysis, the diachronic-phylogenetic-historical level, which will provide important additional elements for the analysis.

2.3. *The diachronic, phylogenetic (historical) level*

It is common knowledge that major advancements in the history of mathematics were almost invariably accompanied by significant changes in mathematical notation: the adoption of the Hindu-Arabic numeral system and the astonishing progress in Indian mathematics; the development of special notations in North Africa which were then brought to Europe by the abbaco schools tradition;[17] Viète's and Descartes' groundbreaking algebraic innovations;[18] Leibniz and the development of calculus;[19] among others. Here is an apt summary of the overall 'phylogeny' of mathematical notations:

> Over the course of history, mathematical formalisms have evolved so that they are cognitively helpful devices, and this evolution has entailed making apparently superficial, but practically crucial, form

[16] It may seem that this will end up being a chicken-egg situation from a historical point of view: how did external symbols for exact numbers first emerge? What came first, the concept or the representation? We will briefly discuss these matters below.

[17] See (Høyrup 2010).

[18] See (Macbeth 2004), and (Heeffer 2010) for precursors of Viète's innovations.

[19] See (Knobloch 2010).

> changes. For example, the convention introduced by Descartes in which letters near the beginning of the alphabet are used to denote constants and those near the end to denote variables, frees us from the burden of remembering which are which and allows us to use our memory for other aspects of a mathematical situation. (Landy and Goldstone 2007, 2039)

Thus, formalisms and notational systems are cognitive technologies which must be *usable* by agents with specific sensorimotor and cognitive characteristics, but which must also be *useful* in that they allow these agents to perform cognitive tasks which would be otherwise more difficult or even impossible to perform.[20]

Still, it may well be that mathematical notations are not essential for these bouts of progress, but rather that they simply *register* newly made progress. In other words, it may well be that the development of new notational techniques is the *consequence* rather than the *cause* of progress in mathematics. To quote D. Macbeth again (2013, p. 29): "The systems of written signs that have been devised for mathematics were devised for mathematics that already existed; it would be impossible to design a notation for mathematics without knowing at least some of the mathematics that the notation was designed to capture."

But in practice, is this really always the case? There seem to be a number of examples in the history of mathematics where specific notations were adopted even before it became clear which concept(s), if any, they singled out. Indeed, mathematical notation is often (though not always) characterized by what can be described as a process of *de-semantification*, in S. Krämer's (2003) fitting terminology. If a given system of notations has a well-defined syntax, and in particular clearly formulated rules of transformation, it is possible to operate within the system even if some of its symbols have no fixed reference — in fact, even if one is not sure that they have a reference at all.

> [...] signs can be manipulated without interpretation. This realm separates the knowledge of how to solve a problem from the knowledge of why this solution functions. (Krämer 2003, 532)

[20] It is important that they should represent a significant improvement over cognitive processes unaided by such devices, otherwise the cost of learning how to operate with them would not justify their use; there must be a favorable trade-off between learning investment and benefits. For more on the historical development of mathematical and logical notations, see (Dutilh Novaes 2012b, chapter 3).

The concept of zero is a case in point, a mathematical concept which emerged in virtue of the characteristics of a notational convention, namely the place-value system. Within such numerical systems, some device expressing the fact that a given position is not occupied by any number is essential, roughly functioning as a place-holder. Place-holder devices were used for the first time by the ancient Babylonians over 4000 years ago, and later further developed within Indian mathematics. To express a number such as 2046, for example, some device is required to indicate the fact that the 'x00' position is not filled; otherwise, it would be impossible to disambiguate 2046 from 246.[21] Other ancient mathematical traditions (Greek, Roman, Jewish) did not use a place-value system, and did not have a place-holder symbol or the concept of zero. In effect, the historical emergence of the concept of zero in Indian mathematics is closely related to the notational device used to indicate a 'gap' in the Indian place-value numerical system; only later did it develop into a mathematical concept properly speaking.[22] Importantly, until the beginning of modern times in Europe, zero was not viewed as a number on a par with other numbers; instead, it was viewed as a 'gap', but this did not prevent mathematicians and users of mathematics to calculate with the symbol *as if* it was a number.[23]

Zero is not the only such case in the history of mathematics. With the advent of calculus and infinitesimal mathematics, the concepts of infinitely small and infinitely large numbers essentially emerged from the very formalism developed by Leibniz; they were not first introduced conceptually so as to be put to use afterwards. In effect (as reported by Krämer (2003, fn. 36)), commenting on whether infinitely small or infinitely large numbers are 'actual' numbers, Leibniz remarked: "On n'a point besoin de faire dépendre l'analyse mathématique des controverses métaphysiques." (There is no need to let mathematical analysis depend on metaphysical controversies.)

As discussed in (Macbeth 2013), in the second half of the 19th century there was a general rejection of a 'calculative' approach to mathematics (viewed as not sufficiently 'concept-oriented'). However, if one considers the totality of the history of mathematics, it is clear that there have been several instances of mathematical concepts actually emerging from formalisms

[21] See (Bellos 2010, chap. 3) for further details. According to (Kaplan 1999), already in the 6th century AD zero began to be treated as a number rather than as a notational device in Indian mathematics.

[22] See (Seife 2000) and (Kaplan 1999) for comprehensive histories of the concept of zero.

[23] "The rules of calculus apply exclusively to the syntactic shape of written signs, not to their meaning: thus one can calculate with the sign '0' long before it has been decided if its object of reference, the zero, is a number, in other words, before an interpretation for the numeral '0' — the cardinal number of empty sets — has been found that is mathematically consistent." (Krämer 2003, 532)

and notational conventions. We need not claim that all or even most mathematical concepts emerged from 'paper-and-pencil' manipulations of formalisms; it is sufficient to notice that some crucial developments in the history of mathematics seem to have been prompted by the creative, productive power of well-designed notations.

Indeed, an important difference between 'doing mathematics' in spoken language or 'doing mathematics' in written language seems to be that the written medium lends itself more easily for operations to be carried out *on* and *with* the medium itself. Given that written symbols have a physical permanence (on paper or other surfaces) that spoken symbols do not have, written symbols can be used not only to *represent* concepts (such as the concepts of exact quantities), but also to be *operated on* — what Menary (2010b) describes as 'cognitive manipulations'. As Krämer observes about the decimal place-value system,

> this system made it possible[24] not only to *depict* all natural numbers with ten written signs, but also to *calculate* with numbers. The decimal place-value system is both a *medium* for representing numbers and a *tool* for operating with numbers. (Krämer 2003, 531)

In sum, this brief excursion on the history of mathematics and mathematical notations suggests that, even if it might be possible (or at least conceivable) to do mathematics with no recourse to any writing medium whatsoever, mathematics *as we know it* only emerged in close connection with the development of specially designed systems of writing. They allow humans not only to represent previously formulated mathematical concepts, but arguably also to produce new ones, and perhaps more importantly, to operate on them 'hands-on' by means of manipulations of symbols.[25]

3. *The exceptions confirming the rule?*

I have argued that manipulating systems of writing is an important route to mathematical knowledge, a fact deeply related to the cognitive makeup of humans. There are, however, extraordinary cases which simply do not seem

[24] This claim is too strong; it would be more accurate to say that the place-value system *facilitated* the process of calculating with symbols rather than making it possible in an absolute sense. Calculations are also possible with other number systems. (I owe this point to Dirk Schlimm.)

[25] Notice that my emphasis on the importance of notations for mathematical practice does not commit me to a formalistic conception of mathematics (Weir 2011). The claim is that writing is usually essential for mathematical reasoning, but not that manipulation of symbols is all there is to mathematical practice.

to fit into this general picture, i.e. cases where 'something' beyond notations seems to be playing a decisive role for the production of mathematical insight. In this section, I discuss two such cases: Jason Padgett, a man with acquired savant syndrome who has the uncanny ability to see shapes and numbers as fractals; and Bernard Morin, the blind mathematician who in the 1960s provided the first model for the eversion of the sphere.

As noted above, the existence of mental calculators is a fact, but most of them cannot be said to be 'doing away' completely with external representations, given that they essentially seem to simulate these representations mentally/internally. However, it is likely that at least some prodigies, some savants in particular, might be attaining mathematical knowledge through alternative routes.[26] Some people seem to be able to 'see' mathematical facts through means other than their embodiment in external representations.

Jason Padgett has acquired savant syndrome, most likely as a result of having been beaten at a mugging incident in 2002 (Brogaard 2011). Since then, he has developed the unique ability to hand-draw fractal renditions of objects and images; he claims to perceive some images as fractals. Padgett is now being studied by cognitive scientists, who seek to understand the neurological basis of his mathematical abilities. In a videotaped interview, he says the following:[27]

> Many mathematicians *see numbers as digits*, and while they can do amazing things when it comes to understanding the very root of an equation, only when *you 'see' it and do the mathematics together* can you really understand where it comes from.

Padgett's claim that many (most?) mathematicians 'see numbers as digits' is indeed very much in the spirit of the views defended here. But his final observation is what is most revealing: he refers to a form of 'seeing' the root of an equation that is independent of seeing numbers as digits, and thus a form of mathematical insight which is presumably not inherently tied to external representations (crucially, the fractals he draws are renditions of what he 'sees' prior to making the drawings themselves). Notice however Padgett's suggestion that *both* this 'seeing' ability *and* 'doing the mathematics' (presumably, operating with symbols) are required to understand where the root of an equation 'comes from'. So Padgett seems to be saying that

[26] The wide majority of mathematical savants seem to rely on heuristic shortcuts, and at any rate it is worth noticing that the mathematical feats of savants are typically not of the creative kind, i.e. they do not represent genuine advancement in the discipline (Koziol et al. 2010).

[27] Link for the video: `http://www.q13fox.com/news/kcpq-scientists-still-calling-in-on-federal-way-savant-20110622,0,7610032.story`

external representations are necessary — although not sufficient — for this kind of mathematical insight, even beyond the obviously social aspect of communicating problems and results to peers.

Importantly, Padgett had virtually no mathematical training prior to the incident, and when he started drawing the fractals, he did not master the mathematical concepts to be able to discuss his drawings in a mathematically-informed way (Brogaard 2011). Since then, he has taken some college-level mathematics courses, but the fact remains that the cognitive processes leading to his mathematical insights do not seem to involve the manipulation of external symbols in any fundamental way.

Now, if in typical cases manipulating portions of writing (broadly construed) is a fundamental aspect of mathematical practice, one important implication seems to be that mathematicians rely substantially on *vision* to do mathematics (as well as in motor abilities to operate writing devices). Does this mean that it is *impossible* to do mathematics unless a given individual is sighted? Naturally, this suggestion is immediately refuted by the sheer existence of a number of very gifted blind mathematicians. Obviously, they ‘do mathematics’ while not relying on the hands-on manipulation of written symbols (not in the same way, at any rate; many of them do rely on Braille or similar systems). But rather than disproving the claim that mathematical practice is fundamentally tied to forms of writing, at least some blind mathematicians seem in fact to *confirm* it in that the ways in which they produce mathematical knowledge are often significantly different from those of sighted mathematicians. To use Jason Padgett’s terminology, we might say that, unlike most mathematicians, blind mathematicians arguably do not predominantly see ‘numbers as digits’, and at times this seems to provide them with privileged insight with respect to some specific problems.

The American Mathematical Society published a fascinating notice on blind mathematicians a few years ago (Jackson 2002), where Euler, Saunderson, Pontryagin and Bernard Morin, among others, are mentioned. The notice focuses not only on their mathematical accomplishments, but also on the methods and strategies they developed to practice mathematics while being blind.[28] Although the goal of the notice is not to offer a detailed and scientifically-grounded account of the deeper cognitive aspects of being a blind mathematician, some observations already suggest that there might

[28] Let me now offer in full a passage already quoted above, which presents the general goal of the AMS notice: “A sighted mathematician generally works by sitting around scribbling on paper. [...] So how do blind mathematicians work? They cannot rely on back-of-the-envelope calculations, half-baked thoughts scribbled on restaurant napkins, or hand-waving argument in which “this” attaches “there” and “that” intersects “here”. Still, in many ways, blind mathematicians work in much the same way as sighted mathematicians do.” (Jackson 2002, 1246)

be significant differences with respect to sighted mathematicians (against a background of predominant commonalities).

To illustrate these possible differences, let us focus on Bernard Morin, the mathematician who in 1967 first exhibited a homotopy that carries out the eversion of a sphere, by creating clay models of it. In 1959, S. Smale had proved that the sphere can be everted, or turned inside out (in fact, he proved a more complicated statement which is equivalent to the eversion claim). However, actually constructing a sphere eversion following the arguments in Smale's paper is in no way straightforward; originally, the proof was above all a theoretical construction, a proof of possibility rather than a display of the process itself. (Indeed, we might say that the proof relied essentially on the 'manipulation of mathematical symbols'.) Morin was the first person who could actually describe and implement the eversion process, and he claims himself that being blind contributed to the feat. His observations on mathematical cognition, while of course remaining essentially anecdotal, are very insightful:

> Far from detracting from his extraordinary visualization ability, Morin's blindness may have enhanced it. Disabilities like blindness, he noted, reinforce one's gifts and one's deficits, so "there are more dramatic contrasts in disabled people," he said. Morin believes there are two kinds of mathematical imagination. One kind, which he calls "time-like", deals with information by proceeding through a series of steps. This is the kind of imagination that allows one to carry out long computations. "I was never good at computing," Morin remarked, and his blindness deepened this deficit. What he excels at is the other kind of imagination, which he calls "space-like" and which allows one to comprehend information all at once. (Jackson 2002, 1248)

Indeed, what Morin describes as the 'time-like' mathematical way of reasoning is presumably intimately tied to performing computations and proceeding step-wise, and as we have seen, manipulating external symbolic systems greatly enhances calculating power. It is thus not so surprising that a blind mathematician would be hampered on this level.[29] But what does the 'space-like' kind of imagination consist in for Morin in the particular case of the eversion of the sphere?

[29] But this is obviously not to be taken as a general rule. Larry Wos is a blind mathematician-logician with remarkable results in automated theorem-proving. He works on Braille terminals for programming.

> One thing that is difficult about visualizing geometric objects is that one tends to see only the outside of the objects, not the inside, which might be very complicated. By thinking carefully about two things at once, Morin has developed the ability to pass from outside to inside, or from one "room" to another. This kind of spatial imagination seems to be less dependent on visual experiences than on tactile ones. (Jackson 2002, 1248)

Now, the crucial point is that Morin, as is often the case with blind people, has enhanced tactile perception in comparison with sighted people.[30] So this 'edge' over sighted people may have been crucial for Morin to be able to reflect on the 'inside' as well as the 'outside' of a sphere.[31] At any rate, it is clear that it was *not* by means of 'cognitive manipulations' of systems of notations that Morin obtained his groundbreaking result on the eversion of the sphere. Thus, the point again (as with Padgett above) is that, while writing (broadly construed) appears to be central for mathematical reasoning in most cases, there is no doubt that there is more to mathematical discovery than the manipulation of notations. These two cases of individuals with what could be described as atypical cognitive apparatuses (a savant and a blind person) suggest precisely that. How exactly these alternative routes to mathematical insight function cognitively is still a poorly understood issue (in fact, the cognition of mathematical insight in general remains very poorly understood), but there is no doubt that they exist.

Another possible interpretation of these cases of (what could be described as) atypical mathematical insight is that they in fact outline an aspect of mathematical cognition that is present in everyone, but is made less conspicuous by the presence of symbol manipulation, namely an autonomous layer of mathematical intuition (Davis and Hersh 1981). Indeed, the main claim defended here is not that the manipulation of portions of writing is *sufficient* for mathematical knowledge and insight, but that it may well be at least to some extent *necessary*, at any rate in most cases (which is of course also related to the institutional aspect of how mathematics is taught). Moreover, in virtue of dissimilarities both in mathematical training and in personal inclination, one should expect to observe significant individual differences in the production of mathematical knowledge: some will rely more extensively on intuitions, while others generate new ideas predominantly by working 'hands-on' with the formalism.

[30] Indeed, there is empirical evidence to support the idea that the 'unused' parts of the brain in people with certain disabilities (blindness, deafness) are co-opted for other kinds of sensorial processing — see (Gougoux et al. 2005).

[31] Notice though that, after Morin's eversion, other eversions of a sphere have been discovered, including one by the very William Thurston mentioned in section 1 above, who found a way to make Smale's original proof constructive (Jackson 2002, 1248).

At any rate, cases such as Morin's and Padgett's are possible exceptions to the claim that manipulating portions of writing is constitutive (though not sufficient) for 'doing mathematics', and as such may be viewed as the exceptions confirming the rule.

4. *Conclusion*

I have defended the claim that the existence and manipulation of external symbolic systems (such as notations, words for numbers, abacuses etc.) are constitutive of mathematical reasoning and mathematical practice in the sense that, both from an ontogenetic and a phylogenetic point of view, the development of mathematical abilities is intimately related to acquiring mastery of such systems. Occasional circumstances of 'doing mathematics' may not require the actual 'paper-and-pencil' manipulation of notations, but from a diachronic point of view, notations are in fact a constant presence.

However, I have also argued that there seem to be other routes to mathematical insight, as suggested by the cases of individuals such as Jason Padgett and Bernard Morin. That is, in spite of the undeniable relevance of notations for mathematical practice, these cases remind us that any cognitive account of mathematical practices and mathematical reasoning must remain open to a multi-faceted approach.

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