PARADOX LOST AND REGAINED: PROFESSOR GEACH ON ENTAILMENT

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When the late C.I. Lewis took exception to the Russellian system of «material» implication [1] on the ground that it did not provide a satisfactory analysis of the «ordinary» meaning of *implies* (later *entails*), as was shown by the presence of such «paradoxical» theses as 'CpCqp' and 'CNpCpq' [2], he little knew that he was starting a hare that would still be actively chased more than half a century later. For his own «strict» implication was to be found by many people hardly less paradoxical than Russell's, since it admits of such counter-intuitive theses as «a necessary proposition is implied by any proposition whatever» and «an impossible proposition implies any proposition whatever».

One of the latest, and most gallant, attempts at slaying the paradoxical monsters has been Professor Geach's [3]. Though he had to pay dearly for it, by giving up the unrestricted transitivity of entailment (a step which may appear rather paradoxical too), he nonetheless thinks he has achieved full success. The purpose of this paper is to show that Professor Geach's victory is far less complete than he seems to believe and that the paradoxes, though admittedly absent from his system, are, so to speak, still lurking behind it.

Professor Geach characterizes his non-paradoxical entail-

(1) «Implication and the Algebra of Logic», *Mind*, n.s. 21(1912) 522-531; reprinted in *Collected Papers of Clarence Irving Lewis*, Stanford, 1970, pp. 35-359.

(2) I use a Polish-style notation: 'p', 'q' and 'r' are propositional variables, 'N', 'K', 'A' and 'C' are the functors of classical negation, conjunction, disjunction and (material) implication respectively.

(3) «Entailment Again», *The Philosophical Review*, 69(1970) 237-239, reprinted in *Logic Matters*, Oxford, 1972, pp. 186-188; see also «Entailment», *Aristotelian Society Supplementary Volumes*, 32(1958) 157-172.

ment, symbolized by 'W', by setting up what he calls «entailment extensions» of logical systems, such an extension being defined recursively, for any system S, by the following rules:

1. Any well-formed formula of S is a well-formed formula of ES (the entailment extension of S).
2. If A and B are well-formed formulae of S, so is ⌜WAB⌝.
3. There are no well-formed formulae except as provided in rules 1 and 2.
4. Any thesis of S is a thesis of ES.
5. If ⌜CAB⌝ is a thesis of S, and neither ⌜NA⌝ nor B is a thesis of S, then ⌜WAB⌝ is a thesis of ES.
6. Any substitution instance of a thesis of ES is a thesis of ES.
7. No well-formed formula is a thesis of ES except as provided in rules 4, 5 and 6.

In what follows I shall only consider the entailment extension (EPC) of the classical propositional calculus (PC), the inadequacy I am to show being bound to affect as well any system containing EPC.

Since Professor Geach's system has been devised explicitly as a means of capturing the properties of the relation *entails*, i.e. the converse of *follows from* or *is deducible from*, it stands to reason that any thesis of the form ⌜WAB⌝ must allow the deduction of its consequent from its antecedent. I am going to show that, though 'WKpNpq' is not a thesis of EPC, nevertheless EPC allows the deduction of any proposition from a unique premise of the form of 'KpNp' in at least two ways.

The First Deduction. Three theses are used in the first deduction; I shall first show that they are indeed theses of EPC (4).

A. WKpNpAKpNpq
   i. PC $\vdash$ CpApq
   ii. PC $\dashv$ Np
   iii. PC $\dashv$ Apq

(4) I write 'X $\vdash$ T' and 'X $\dashv$ T' for 'T is' and 'is not a thesis of system X' respectively.

iv.	EPC ⊢ WpApq	from i, ii and iii by rule 5
v.	EPC ⊢ WKpNpAKpNpq	from iv by rule 6 (p/KpNp)

B. WAKpNpqKApqANpq

i.	PC ⊢ CAKpqrKAprAqr	
ii.	PC ⊣ NAKpqr	
iii.	PC ⊣ KAprAqr	
iv.	EPC ⊢ WAKpqrKAprAqr	from i, ii and iii by rule 5
v.	EPC ⊢ WAKpNpqKApqANpq	from iv by rule 6 (q/Np, r/q)

C. WKApqANpqq

i.	PC ⊢ CKApqANpqq	
ii.	PC ⊣ NKApqANpqq	
iii.	PC ⊣ q	
iv.	EPC ⊢ WKApqANpqq	from i, ii and iii by rule 5

I can now proceed to the first deduction:

i.	KpNp	premise
ii.	AKpNpq	from i by thesis A
iii.	KApqANpq	from ii by thesis B
iv.	q	from iii by thesis C

The Second Deduction. The second deduction makes use of two theses of EPC.

D. WKpNpKpANpq

i.	PC ⊢ CKpqKpAqr	
ii.	PC ⊣ NKpq	
iii.	PC ⊣ KpAqr	
iv.	EPC ⊢ WKpqKpAqr	from i, ii and iii by rule 5
v.	EPC ⊢ WKpNpKpANpq	from iv by rule 6 (q/Np, r/q)

E. WKpANpqq

i.	PC ⊢ CKpANpqq	
ii.	PC ⊣ NKpANpq	
iii.	PC ⊣ q	
iv.	EPC ⊢ WKpANpqq	from i, ii and iii by rule 5

The second deduction follows:

i.	KpNp	premise
ii.	KpANpq	from i by thesis D
iii.	q	from ii by thesis E.

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