A NOTE ON COMPACTNESS AND DECIDABILITY

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This note is a continuation of the investigations reported in [5] and [6]. It was suggested in [5] that two standard measures of semantic complexity: compactness and the recursive enumerability of logical truths — might be related. In both [5] and [6] non-compact modal logics were exhibited whose logical truths were recursively enumerable. Here results are obtained which imply the existence of both non-compact, decidable and non-recursively enumerable compacts logics.

The Gödel-Bernays set theory [1] is assumed throughout (in particular the distinction between sets and proper classes p. 3). For any set S, $|S|$ denotes the cardinality of S.

By a *language* we mean any set L. A *semantics for L* is a four-tuple $S(L) = \langle I,V,T,D \rangle$: where I is a non-empty class called *interpretations* for L: $V : L \times I \rightarrow T$, called *the valuation of S(L)*: T is a non-empty set called *the truth values of S(L)*: and D is a non-empty proper subset of T called the *designated values of S(L)*.

For S(L) a semantics and A a sentence in L, *A is logically true in S(L)* provided that V(A,i) assumes a designated value for each interpretation i in I; for S a set a sentences, and A a sentence, the argument *$\langle S,A \rangle$ is valid in S(L)* provided for every interpretation i, if V(B,i) is a member of D for every member B of S, then V(A,i) is a member of D. For all i, j in I, *i is equivalent to j in S(L)* provided $V(A,i) = V(A,j)$ for every sentence A in L.

Let $S(L) = \langle I,V,T,D \rangle$ and $S'(L) = \langle I',V',T',D' \rangle$ be any pair of semantics for L, we say (i) that S(L) *is weakly equivalent to* S'(L) provided that a sentence is logically true in S(L) iff logically true in S'(L); that S(L) *is strongly equivalent* to S'(L) provided that an argument is valid in S(L) iff valid in S'(L); (iii) that S(L) *is a subsystem of* S'(L) provided $I \subseteq I'$, $T' = T$, D' = D and V is the restriction of V' to I × L (in this case we sometimes also say that S'(L) *is an extension of* S(L)); (iv) that S(L) is *compact* provided for every set of sentences S and every sentence A, if $<S,A>$ is valid in S(L), then there exists S' a finite subset of S so that $<S',A>$ (a finite subargument of $<S,A>$) is valid in S(L); (In a compact semantics every infinite valid argument is infinitely redundant.); (v) that S(L) *is decidable* provided the logical truths of S(L) are decidable; (vi) that S(L) is *recursively enumerable* provided the logical truths of S(L) are recursively enumerable.

Obviously, any semantics weakly equivalent to a recursively-enumerable (decidable) semantics is recursively enumerable (decidable); and any semantics strongly equivalent to a compact semantics is compact.

Theorem I: Every semantics S(L) has (i) a weakly equivalent subsystem having no more interpretations than sentences; and (ii) a strongly equivalent subsystem having no more than $2^{|L|}$ interpretations.

Proof: Let S(L) = $<I,V,T,D>$. Let W contain all those sentences L which are not logically true in S(L); for each A in W, let $\bar{A}$ denote the set of interpretations in I which make A "false" (i.e. V(A,i) does not belong to D). Let $\overline{W}$ be the collection of these non-empty sets and let C be the choice function for $\overline{W}$. Let I' = C($\overline{W}$). Since $|\overline{W}| \leqslant |L|, |I'| \leqslant |L|$. Let S'(L) = $<I',V',T,D>$ where V' is the restriction of V to L × I'. We can easily verify that S'(L) is a weakly equivalent subsystem of S(L). A similar argument can be used to verify (ii).

Corollary 1: There exists decidable non-compact, and recursively enumerable non-compact semantics.

Proof: Let L be a sentential language containing a countably infinite set of sentential constants and closed under the usual sentencial connectives, including "and" and "not." Let S(L) be the standard semantics for L. The logical truths of S(L) are decidable and there are $2^{\aleph_0}$ non-equivalent interpretations in S(L). By theorem 1, S(L) has a weakly equivalent subsystem, S'(L) having no more than $\aleph_0$ many interpretations; and a simple cardinality argument shows that S'(L) is not compact.

The same argument can be applied to first order logic to give a recursively enumerable non-compact semantics.

S(L) is called *normal* provided that every interpretation gives some sentence in L a non-designated value. The above argument can easily be generalized to prove the following.

Theorem 2: If S(L) is normal semantics having at least $2^{|L|}$ many non-equivalent interpretations, then no weakly equivalent subsystem of S(L) having fewer than $2^{|L|}$ many interpretations is strongly equivalent to S(L).

Theorem 3: Every semantics S(L), for which the class of sets not belonging to I is a set of cardinality at least $2^{|L|}$ has a weakly equivalent compact extension.

Proof: Let S(L) be any semantics. Suppose the class of sets not belonging to I is a set of cardinality at least $2^{|L|}$. Let W contain those arguments $<S,A>$ valid in S(L) where S is infinite. Let R be that set of sets not belonging to I; there is a 1 - 1 function ,f, from W into R. For each $<S,A>$ in W set $<S,A>' = f(<S,A>)$. Let $I' = I \cup f(W)$.

Let V': $L \times I' \rightarrow T$ be s.t. for all A in L and i in I, $V'(A,i) = V(A,i)$. Let d be a designated value (in S(L)) and $\bar{d}$ be a non-designated value (in S(L)); for A' in L and $<S,A>$ in W, $V'(A',<S,A>') = d$ if A' is logically true in S(L) or A' is in S and $V'(A', <S,A>') = d$, otherwise. Let $S'(L) = <I',V',T,D>$. We can easily verify that S'(L) is a compact weakly equivalent extension of S(L).

We can use the above theorem to show that any second order language having at least one binary predicate symbol among its non-logical constants can be given a semantics which is weakly equivalent to the standard semantics but which is compact. Let L^2 be such a second order language and $S(L^2)$ be its standard semantics. It has been shown [7] that there is an infinite cardinal β s.t. every interpretation in $S(L^2)$ is equivalent to an interpretation of cardinality β or less and that $\beta > \aleph_0$ (β is called the *weak Löwenheim-Skolem number of* $S(L^2)$). It follows trivially from a result of Tarski [2] (p. 712) that there are $2^{\aleph_0}$ many non-equivalent interpretations in every infinite cardinal.

Let $LS(L^2)$ (called the *weak Löwenheim-Skolem* semantics for L^2) be that subsystem of $S(L^2)$ containing only those interpretations of $S(L^2)$ of cardinality $\leqslant$ the weak Löwenheim-Skolem number of $S(L^2)$. We can easily establish the following.

Theorem 4: $LS(L^2)$ is a strongly equivalent subsystem of $S(L^2)$ and there is a set of at least $2^{\aleph_0}$ sets not among the interpretations for $LS(L^2)$.

Corollary 2: There exists a compact non-recusively enumerable semantics.

Proof: $S(L^2)$ is not compact ([3] p. 124) nor is its set of logical truths recusively enumerable ([2] p. 174). By theorems 3 and 4, $LS(L^2)$ has a compact extension that is weakly equivalent to $S(L^2)$.

The notion of semantics articulated above seems general enough to encompass all of the semantics encountered in the literature. One might wonder, however, whether there is a more "reasonable" notion which accounts for known semantics and for which compactness implies enumerability.

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