VREDENDUIN'S SYSTEM OF STRICT IMPLICATION

R. Routley

Vredenduin claims ([3], p. 76) that his system **N** of strict implication 'is in conformity with real deducibility' whereas the system **S2** of Lewis and Langford is not. But, as will be shown, the essential part of Vredenduin's system **N** coincides with **S2**; and therefore, since **N** contains **S2**, **N** is just as paradoxical as **S2**.

N is got by dropping Lewis's definition, $A \dashv B =_{Df} \sim\Diamond(A \,\&\, \sim B)$, of strict implication, and taking '$\dashv$', as well as '$\sim$', '&' and '$\Diamond$', as a primitive connective; by taking over intact Lewis & Langford's postulates ([1], pp. 124-126, p. 166) for **S2**; and by supplementing these postulates by the following axioms:

13. $\sim p \dashv q \dashv . \sim q \dashv p$
14. $p \,\&\, q \dashv r \dashv . p \,\&\, \sim r \dashv \sim q$
15. $p \dashv q \dashv . p \,\&\, r \dashv q \,\&\, r$
16. $p \dashv q \,\&\, r \dashv s \dashv . p \,\&\, r \dashv q \,\&\, s$
17. $p \dashv \Diamond p$
18. $p \dashv q \,\&\, \Diamond p \dashv . \Diamond q$
19. $p \dashv q \dashv . \sim \Diamond(p \,\&\, \sim q)$.

The axiomatisation in fact contains redundancies, *e.g.* 16. is derivable using 15.; and it could also be shortened by minor modifications of axioms, *e.g.* Lewis's axiom $p \,\&\, q \dashv q \,\&\, p$ is readily derived from a permuted form of 15.

Theorem 1. **N contains S2.**

Proof: The possibility connective 'M', of **S2**, can be defined, as usual:

$$M\,A =_{Df} \sim(A \dashv \sim A).$$

It remains to show that M has the correct properties; and for this it suffices to prove:

(1) $p ⥽ q ⥽ . \sim M(p \,\&\, \sim q)$
(2) $\sim M(p \,\&\, \sim q) ⥽ . p ⥽ q$
(3) $M(p \,\&\, q) ⥽ M p$

For (1) and (2) together provide Lewis's definition of strict implication, and (3) gives the only postulate of **S2** that **N** does not provide.

ad (1): It is enough to prove:

$$p ⥽ q ⥽ . p \,\&\, \sim q ⥽ . \sim (p \,\&\, \sim q).$$

Now $p ⥽ q ⥽ . \sim q ⥽ \sim p$
$⥽ . \sim q \,\&\, \sim q ⥽ \sim p$ by Substitutivity
$⥽ . p \,\&\, \sim q ⥽ q$ by Antilogism
$⥽ . p \,\&\, (p \,\&\, \sim q) ⥽ q$ by Substitutivity
$⥽ . p \,\&\, \sim q ⥽ \sim (p \,\&\, q)$ by Antilogism

ad (2): $p \,\&\, \sim q ⥽ \sim (p \,\&\, \sim q) ⥽ . p \,\&\, (p \,\&\, \sim q) ⥽ q$
by Antilogism
$⥽ . p \,\&\, \sim q ⥽ q$
$⥽ . \sim q \,\&\, \sim q ⥽ \sim p$
$⥽ . \sim q ⥽ \sim p$
$⥽ . p ⥽ q$

ad (3): Proof uses the principle: $p ⥽ q \,\&\, r ⥽ . p ⥽ q$ ([1], 19.62 and, as a T-principle, 16.5; these principles are conceded by Vredenduin [3], footnote 8).

Then,

$p ⥽ \sim p ⥽ . p \,\&\, q ⥽ \sim p \,\&\, q$ by Factor
$⥽ . p \,\&\, q ⥽ \sim p$
$⥽ . q \,\&\, (p \,\&\, q) ⥽ \sim p$
$⥽ . p \,\&\, q ⥽ \sim (p \,\&\, \sim q)$ by Antilogism

Hence $\sim Mp ⥽ \sim M(p \,\&\, q)$.

N thus amounts to **S2**, formulated with connective set $\{\sim, \&, ⥽\}$ supplemented by an additional modal functor '$\Diamond$'.

Hence every paradox of **S2** reappears in **N**; only the notation is changed so that $\sim \Diamond p \prec . p \prec q$ of **S2** reappears as $\sim Mp \prec . p \prec q$ in **N**.

Nor does **N** outrun **S2**. Call the subsystem of **N** with all wff and theorems of **N**, formulated just using connective set $\{\sim, \&, \prec\}$, system **V**.

Then:

Theorem 2. $\vdash_{S2} A$ iff $\vdash_V A$.

Proof: It follows from theorem 1 that if $\vdash_{S2} A$ then $\vdash_V A$. For the converse, define an **N**-structure $\mathfrak{N} = \langle G, K, N, R, S, v \rangle$, where $\langle G, K, N, R, v \rangle$, is an **S2**-model (as defined in [2]) and S is a further relation on K, such that S is reflexive and $S \subseteq R$, i.e. for every $H_1, H_2 \varepsilon K$, $H_1 S H_2 \supset H_1 R H_2$. The valuation function v is extended in the usual **S2** way for connectives of **V**; and for the connective '$\Diamond$', for H such that G R H,
$v(\Diamond A,H) = T$ iff (SH_1) $(HSH_1 \& v(A,H_1) = T) \vee H \notin N$;
and for other H, $v(\Diamond A,H)$ is assigned arbitrarily (or as part of the model). Then
(a) Every theorem of **N** is true in every **N**-structure.
But, since in assessing a wff of **V**, relation S is never used, the separability result
(b) Every theorem of **V** is true in every **S2**-model, follows.
Hence since **S2** is complete with respect to **S2**-models (see [2]), every theorem of **V** is a theorem of **S2**.

Finally, the N-modelling shows that the further modal connective '$\Diamond$', added to **V** to yield **N**, is of less thant **S2** strength, since it does not guarantee unlimited substitutivity of strict equivalence.

R. ROUTLEY

REFERENCES

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