ON TWO OF PROFESSOR RESCHER'S MODAL THESES

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Ten years ago, Professor Nicholas Rescher published (1) an attempt to formalize two well-known theses of traditional modal logic, viz.

(1) *a posse ad esse non ualet consequentia*

and

(2) *ab esse ad necesse esse non ualet consequentia.* It is plain that the contradictories of these theses, viz.

(3) *a posse ad esse ualet consequentia*

and

(4) *ab esse ad necesse esse ualet consequentia* must be rejected. For it is obvious that (3) and (4) must be rendered, respectively, as

(5) $Mp.\Longrightarrow.p$ (2)

and

(6) $p.\Longrightarrow.Lp.$

But the addition of either (5) or (6) to a modal system would reduce it to assertoric propositional logic (with some redundant notations), since the following formulas would be theses:

(7) $Mp.\Longleftrightarrow.p,$

(8) $Lp.\Longleftrightarrow.p,$

(1) Notre Dame Journal of Formal Logic, 2 (1961), 154-157.

(2) I use the following symbols: '$\vdash\alpha$' for 'α is a theorem'; '&' and '$\rightarrow$' for classical conjunction and (material) implication; 'M' and 'L' for the functors of possibility and necessity; '$\Rightarrow$' and '$\Longleftrightarrow$' for strict implication and strict equivalence; '—' for classical negation, this sign being written above the principal functor of its scope; 'Π' and 'Σ' for the universal and particular quantifiers.

and thus

(9) $Mp.\Longleftrightarrow.Lp.$

The problem is thus to find some theses whose addition to a modal system would prevent the further addition of either (3) or (4) on pain of inconsistency.

Prof. Rescher considered only what he called 'normal' systems, i.e. systems possessing, in addition to the usual rules of *modus ponens* and substitution, the following definition and theses:

(10) $Lp.=df.\bar{M}\bar{p}$

(11) $Lp.\Longrightarrow.p$

(12) $p\&q.\Longrightarrow.p$

(13) $p\Longrightarrow q.\Longrightarrow.\bar{q}\Longrightarrow\bar{p}$

(14) $p\Longrightarrow q.\Longrightarrow:\bar{M}.p\&\bar{q},$

all of which obtain in S2.
It is further assumed that the systems considered are enlarged with quantifiers binding propositional variables, subject to the usual laws and rules.

The first formula which is put forward as a formalization of (1) is

(15) $\bar{\Sigma}p:Mp.\Longrightarrow.p,$

which is rejected on the ground that in any system which has the thesis

(16) $\bar{M}.p\&\bar{q}:\Longrightarrow.p\Longrightarrow q,$

i.e. the converse of 14, it would allow the derivation of

(17) $\bar{\Sigma}p.Lp,$

which denies the existence of any necessary proposition.

It may be added that the presence of (15) in any system in which the rule

(18) *If* $\vdash\alpha$ and $\vdash\alpha\&\beta.\Longrightarrow.\gamma$, then $\vdash\beta\Longrightarrow\gamma$.

which is necessary to solve the Lewis Carroll paradox, and the thesis

(19) $p \Longrightarrow q. \Longrightarrow: p\&r. \Longrightarrow q$,

which may be derived from (12) unless we restrict the transitivity of entailment, obtain leads to contradiction, The proof is easy: let 'p' be any necessary proposition such that 'Lp' is a thesis; then

i. Lp	by hypothesis,
ii. $Lp. \Longrightarrow .p$	thesis (11),
iii. $Lp.\&.Mp: \Longrightarrow .p$	from (ii) by thesis (19),
iv. $Mp. \Longrightarrow .p$	from (i) and (iii) by rule (18),
v. $\Sigma p:Mp. \Longrightarrow .p$	from (v) by Existential Generalization.

But (v) is the contradictory of (15).

According to Prof. Rescher, the reason why (15) fails is that it applies even to 'degenerate' cases of entailment, that is those cases which only obtain because the antecedent is impossible or the consequent necessary. To exclude those cases, he defines a new constant '$\longrightarrow$' by

(20) $p \longrightarrow q. = df:.p \Longrightarrow q.\&:Mp.\&.M\bar{q}$

and formalizes (1) as

(21) $\bar{\Sigma} p:Mp. \longrightarrow .p$.

Even in a system in which (16) is a thesis, Prof. Rescher says, the formalization of (1) as (21) "does not entail untenable consequences". I will attempt to show that this is not the case.

The contention that (21) is an adequate formalization of (1) does entail a most unpalatable consequence indeed for one of the most interesting of the classical Lewis systems, viz. S4. For the addition of (5) to S4 would reduce it to assertoric propositional logic. But the addition to the same of (21), which is intended to prohibit the addition of (5), would reduce it to a system at least as strong as S5. So that, so to speak, S4 only enjoys independent existence as long as we do not ask embarrassing questions.

That the addition of (5) would reduce S4 to assertoric propositional logic is obvious. That the addition of (21) would reduce it to a system containing S5 I will prove by deriving

(22) $Mp.\Longrightarrow.LMP$,

a proper axiom of S5, from that basis. In addition to theses and rules already mentioned, I will use the following ones:

(23) $p\bar{\&}q.\Longleftrightarrow.p\longrightarrow\bar{q}$

(24) $p\Longrightarrow q.\Longrightarrow{:}q\Longrightarrow r.\Longrightarrow.p\Longrightarrow r$

(25) $p.\Longrightarrow.Mp$

(26) If $\vdash\alpha\longrightarrow\beta$, then $\vdash\alpha\Longrightarrow\beta$

(27) $MMp.\Longrightarrow.Mp$.

(28) Substitution of definitional equivalents

(29) Substitution of strict equivalents.

The derivation follows:

i.	$\Pi p{:}Mp.\overline{\longrightarrow}.p$	from 21,
ii.	$Mp.\overline{\longrightarrow}.p$	from (i) by Universal Instantiation,
iii.	$Mp.\Longrightarrow.p{:}\bar{\&}{:}MMp.\&.M\bar{p}$	from (ii) by definition (20),
iv.	$Mp.\Longrightarrow.p{:}\longrightarrow{:}MMp.\bar{\&}.M\bar{p}$	from (iii) by (23) and (29),
v.	$Mp.\Longrightarrow.p{:}\longrightarrow{:}MMp.\longrightarrow.\overline{M\bar{p}}$	from (iv) by (23) and (29),
vi.	$Mp.\Longrightarrow.p{:}\longrightarrow{:}MMp.\longrightarrow.Lp$	from (v) by definition (10),
vii.	$MMp.\Longrightarrow.Mp{:}\longrightarrow{:}MMMp.\longrightarrow.LMp$	from (vi) by substitution,
viii.	$MMMp.\longrightarrow.LMp$	from (27) and (vii) by *modus ponens*,
ix.	$MMMp.\Longrightarrow.LMp$	from (viii) by (26),
x.	$MMp.\Longrightarrow.MMMp$	from (25) by substitution,
xi.	$MMp.\Longrightarrow.LMp$	from (x) and (ix) by (24),
xii.	$Mp.\Longrightarrow.MMp$	from (25) by substitution,
xiii.	$Mp.\Longrightarrow.LMp$	from (xii) and (xi) by (24).

I have thus shown that (21) is not an adequate formalization of (1). As for (2), after having rejected

(30) $\Pi p \Pi q{:}.p \Longrightarrow q.\&.\bar{L}p{:}\Longrightarrow.\bar{L}q$,

which makes the system inconsistent, Prof. Rescher proposes

(31) $\Pi p \Pi q{:}.p \longrightarrow q.\&.\bar{L}p{:}\Longrightarrow.\bar{L}q$,

or, equivalently,

(32) $\Pi p \Pi q{::}.p \Longrightarrow q.\&{:}Mp.\&.M\bar{q}{:}.\&.\bar{L}p{::}\Longrightarrow \bar{L}q$.

That (32) is no more an adequate formalization of (2) than (15) is of (1) is easily seen. For there is a consistent system which includes both (6), the contradictory of (2), and (32), viz. the classical assertoric propositional calculus. The addition of (6) to a modal system reduces it to the assertoric one. But from (32), by Universal Instantiation, we get

(33) $p \Longrightarrow q.\&{:}Mp.\&.M\bar{q}{:}.\&.\bar{L}p{::}\Longrightarrow.\bar{L}q$,

from which, as soon as we add quantifiers, we again get (32), by Generalization. But is plain that, in the presence of (6), (33) is deductively equivalent to

(34) $p \longrightarrow q.\&.p\&\bar{q}{:}\&.\bar{p}{:}.\longrightarrow.\bar{q}$,

which is indeed a thesis of the assertoric propositional logic.

I have shown so far that the formalizations proposed by Prof. Rescher for the two traditional theses (1) and (2) are not adequate. It may be asked: is it possible to formalize them adequately ? Indeed it is, by using no other apparatus than the one used by Prof. Rescher, viz. classical modal logic enlarged by quantifiers binding propositional variables.

For suppose that (3) and (4) were valid. They could then be adequately formalized as, respectively, (5) and (6). But, since we have quantifiers binding propositional variables, with the usual theses and rules, (5) and (6) are deductively equivalent, respectively, to

(35) $\Pi p{:}Mp.\Longrightarrow.p$

and

(36) $\Pi p{:}p.\Longrightarrow.Lp$.

If we agree that (5) and (6) are adequate formalizations of (3) and (4), which we reject while accepting their contradictories (1) and (2), and that (35) and (36) are deductively equivalent, respectively, to (5) and (6), I think it is plain that (1) and (2) may be adequately formalized, respectively, as

(37) $\bar{\Pi}p{:}Mp.\Longrightarrow.p$

and

(38) $\bar{\Pi}p{:}p.\Longrightarrow.Lp$,

or, equivalently, as

(39) $\Sigma p{:}Mp.\overline{\Longrightarrow}.p$

and

(40) $\Sigma p{:}p.\overline{\Longrightarrow}.Lp$.

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