DECISION PROCEDURES AND SEMANTICS FOR C1, E1 AND SO.5°

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The very weak modal system **C1**, formulated with primitive connectives $\sim$, $\supset$, $\Box$, has as postulates:

I. Some formulation of sentential logic, with sole rule:

R1. A, $A \supset B \rightarrow B$

II. The modal postulates:

A5. $\Box(A \supset B) \supset .\Box A \supset \Box B$

R2[1]. $A \supset B \rightarrow \Box A \supset \Box B$, *provided* $A \supset B$ is a theorem of sentential logic I.

Lemmon's system **E1**, of [3], reformulated using axiom schemata, has as postulates the postulates of **C1** plus the scheme:
A4. $\Box A \supset A$

C1 and **E1** are closely related, respectively, to systems SO.5° and SO.5.

Lemmon's SO.5, of [3], has the same postulates as **E1**, except that **R2**[1] is replaced by

R2. $A \rightarrow \Box A$, provided A is a theorem of sentential logic I.

SO.5° is obtained from **C1** by replacing **R2′** by **R2**.

Decision procedures and semantics for SO.5 appear in Cresswell [1] and in [5]. Here an analogous development is sketched for **C1**, **E1** and SO.5°; and a different, but related, semantics is given for SO.5. Acquaintance with [5] is assumed.

The sequential system ***C1** has as postulates the schemes of Kleene's system **G1** (of [2]) and the following modal scheme:

$$\frac{\Gamma \rightarrow A}{\Box\Gamma \rightarrow \Box A}$$ ($\rightarrow \Box'$), *provided* (i) Γ is not empty,

and

(ii) the upper sequent is obtained by **G1** rules only (of course modal wff may appear).

The sequential system *SO.5° differs from ***C1** only in that Γ may be empty in (→□).

The sequential system ***E1** is obtained by adding to ***C1** the further modal scheme:

$$\frac{A, \Gamma \rightarrow \Theta}{\Box A, \Gamma \rightarrow \Theta} (\Box\rightarrow)$$

The cut-elimination theorem holds for ***C1**, *SO.5° and ***E1**. Proof is essentially given in Ohnishi's proof, in [4], for **S2***.

The equivalence theorems: **C1** is deductively equivalent to ***C1**, SO.5° to *SO.5°, and **E1** to ***E1**.
Proofs are special cases of that for SO.5 and *SO.5 given in [5].

The decidability theorem: **C1**, SO.5° and **E1** are Gentzen decidable.
Theorem: If $\vdash_{\mathrm{SO.5\,(SO.5^\circ)}} \Box A$ then $\vdash_{\mathbf{E1(C1)}} A$.
The converse does not hold: instead,
Theorem: If $\vdash_{\mathbf{E1(C1)}} A$ then $\vdash_{\mathbf{S2(S2^\circ)}} \Box A$.
Proof is by induction on the length of the proof of A in **E1** (**C1**).

A second decision procedure is provided by extended truth-table techniques.
Definition: Wff A is a **C1**-*tautology* iff every F-row of the truth-table $\mathfrak{T}$(A) for A satisfies the following requirement:
II′. Some constituents of the form $\Box C_1, \dots \Box C_n$ ($n \geqslant 1$) all have value T in r and some constituent of the form □B has the value F in row r, where C_1 & C_2 … & $C_n \supset B$ is a (substitution instance of a) tautology.
Definition: Wff A is an **E1**-*tautology* iff every F-row r of the truth-table $\mathfrak{T}$(A) for A satisfies at least one of these requirements:
I. Some constituent of the form □B has value T in r where B has value F in row r.
II′. As above.

Definition: A is an *SO.5°-tautology* iff every F-row r of the truth-table $\mathfrak{T}(A)$ for A satisfies the requirement II, where II differs from II′ only in that the provision n>0 replaces the provision of II′ that n>1.

Theorems (1) $_{C1}$A iff A is a C1-tautology
(2) $_{E1}$A iff A is a E1-tautology
(3) $_{SO.5°}$A iff A is an SO.5°-tautology

Proofs are special cases of those for SO.5, given in [5].

A **C1**-*model* is a structure K = <G, K, N, R, v> where K is a set, GεK, N$\subset$K, R is a binary relation on K, and v is a valuation function whose first domain is sentential variables and □-wff, whose second domain is elements of K (excluding G in the case of □-wff), and whose range is truth-values. A wff of the form □B is called a □-wff. An **E1**-*model* is a **C1**-model such that R is reflexive on N.

An *SO.5°-model* is a **C1**-model such that GεN.
An *SO.5-model* is an **E1**-model such that GεN.

The valuation v is extended so that its first domain is the set of all wff, as follows:

(i)) for all HεK, v($\sim$A, H) = T iff v(A, H) = F, and v(A$\supset$B, H) = T iff v(A, H) = F v v(B, H) = T;
(ii)) v(□A, G) = T iff (AH)(GRH $\supset$ v(A, H) = T) & GεN.

A is *true in* L-model K iff v(A, G) = T; *false in* L-model K iff v(A, G) = F; *L-valid* iff true in every L-model.

L-model K is a *countermodel* to A iff A is false in K.

Theorems: (1) If $_{C1}$A then A is C1-valid
(2) If $_{E1}$A then A is E1-valid
(3) If $_{SO.5°}$A then A is SO.5°-valid
(4) If $_{SO.5}$A then A is SO.5-valid

Proofs are by induction over the length of the derivation of A in the appropriate system.

Theorems: (1) If C1-valid then $_{C1}$A
(2) If E1-valid then $_{E1}$A
(3) If A is SO.5°-valid then $_{SO.5°}$A.
(4) If A is SO.5-valid then $_{SO.5}$A.

Proof of (1), (2), (3) and (4) are, respectively, similar to proofs of the corresponding theorems for **C2**, **E2**, **S2°** and **S2** given in [6]. But the valuation function v is further specified, as follows:

$$v(\Box A, H) = T \text{ iff } v_j(\Box A, H) = T, \text{ for } H \varepsilon K_j.$$

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