A THREE-VALUED, NON-LEVELLED LOGIC CONSISTENT FOR ALL SELF-REFERENCE

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1. INTRODUCTION

In classical logic, some self-referring statements may be "liar-paradoxical" (such as S : ~S) and so violate the axioms of the logic (1).

Here a three valued logic is described in which all self-referring statements may be admitted. That is, any self-referring statement takes one and only one of the three values. We do not escape the non-definitive statement

S : S

which may consistently have any value. But we can avoid the liar-paradoxical situation where, in classical logic,

S : ~S

is both true and false : (if true then false, therefore false, and if false then true, therefore true).

In a three-valued logic there are 27 monadic operators including three constants. This logic admits eight monadic operators (including those derivable from dyadic ones) two of which are constants. The only other monadic operator which could be admitted without making some self-referring statements improper is the constant which always takes the value N.

This three-valued logic contains the classical one.

2. DEFINITIONS

Let statements have one and only one value from "True", "False",

(1) O'CARROL, Improper self-reference in classical logic (Logique et Analyse).

"Nonsense". Two statements are defined to be *equal* if and only if they are True, Nonsense or False together.

Now we define statements about Truth etc. using T, N and F as abbreviations for True, Nonsense and False.

Statement	Notation	Values as A is T, N and F respectively
It is true that A.	TA	T, N, F
It is nonsense that A.	NA	F, T, F
It is false that A.	FA	F, N, T

These operators generate a semigroup under succession. This semi-group is non-commutative and has six members. We list the operators of this semigroup with their values as the argument A isT, N and F respectively:

TA	T, N, F	NNA	F, F, F,
NA	F, T, F	FNA	T, F, T
FA	F, N, T	FNNA	T, T, T

That is, any combination of these operators produces another of them. Note that NNA and FNNA are constants. Thus if, in the conversation

"Nonsense!"
"Nonsense!"

between politicians, the latter remark refers to the former, then the latter is false in this logic. A possible conversation between a politician and a logician is

"Nonsense!"
"Nonsense!"
"False!"

Note that not all possible operators (combinations of T, N and F) appear in this logic. In two-valued logic all possible operators do appear; they are "it is true that", $\sim$ and constants.

Define conjunction and disjunction as follows:

Values of A.B

		when A is		
		T	N	F
and B is	T	T	N	F
	N	N	N	N
	F	F	N	F

Values of AvB

		when A is		
		T	N	F
and B is	T	T	N	T
	N	N	N	N
	F	T	N	F

Then A.B and AvB take value N if either A or B or both takes value N. If neither A nor B takes value N, then A.B and AvB are as in classical logic.

The purpose of this logic (namely, to admit all self-reference) will be satisfied if the implication A $\supset$ B is defined in terms of the above operators and connectives.

To be explicit, we *define A $\supset$ B to be FAvB.*
That is, the values of A $\supset$ B are

		when A is		
		T	N	F
and B is	T	T	N	T
	N	N	N	N
	F	F	N	T

It is easy to check that the only two monadic operators generated from these dyadic ones (in combination with the monadic operators listed above) are

A.NA (etc.) with values F, N, F as A is T, N, F.
AvFA (etc.) with values T, N, T as A is T, N, F.

3. ALGEBRA

The algebra of combinations of operators has already been discussed. It is easy to verify the algebraic laws:

Each of . and v is commutative, associative and distributive over the other.

For example, we easily check that A.(BvC) is (A.B)v(A.C), both having values:

	when C is T and A is T	N	F	when C is N and A is T	N	F	when C is F and A is T	N	F
and B is T	T	N	F	N	N	N	T	N	F
N	N	N	N	N	N	N	N	N	N
F	T	N	F	N	N	N	F	N	F

We also have for all A: A.A = A and AvA = A and

A.B = A if and only if B is T and
AvB = A if and only if B is F.

It follows directly from the definitions that operators combine with the connectives by

T(A.B) = TA.TB N(A.B) = NAvNB
T(AvB) = (A.B)v(A.FB)v(FA.B) N(AvB) = NAvNB
F(A.B) = (FA.B)v(FA.FB)v(A.FB)
F(AvB) = FA.FB

4. The general self-referring statement

From the laws of algebra above, any function f(S) involving any of the operators and connectives (including $\supset$), may be written as a polynomial in the quantities S, NS, FS, FNS and constants, where . is the "multiplication" and v the "addition". (The expressions NNS and FNNS are constants.) Further this polynomial is a (nonhomogeneous) multilinear form (linear in each of S, NS, FS and FNS) since A.A = A.

In considering the general self-referring statement S : f(S) we have then only to consider

S : L (S, NS, FS, FNS)

where L is a nonhomogeneous multilinear form. Now any form which includes terms in S or FS takes value N whenever S takes that value. [Any purely algebraic (i.e. not involving operators) function of a statement which is Nonsense is also Nonsense.] Then any self-referring statement S : L where L is such a form is consistent when S is Nonsense.

The only other self-referring statements are of the form

S : L (NS, FNS),
i.e. S : (NS.1)v(FNS.2)v(NS.FNS.3')v4

where 1, 2, 3* and 4 are constants. Now NS.FNS is always false, for any S, and so is a constant. Thus our statement is now of the form

S : (NS.1)v(FNS.2)v3

with 1, 2 and 3 constants. If any of 1, 2 and 3 is N, then the statement is consistent with S also N. If 2 and 3 are both F and 1 is not N, then the statement is consistent with S also F. In all other cases the statement is consistent with S true.

Thus we have shown that any self-referring statement S : f(S) is consistent for some (perhaps more than one) value of S. Unlike the case of classical logic we have no liar-paradoxical violation of axioms.

5. The operator "not"

If we try to reproduce a liar paradox we might consider the statements S : FS, S : NS and S : NSvFS. These are readily shown consistent with S taking values N, F and N respectively. The third of these statements attempts to say "S is not true". However, when S is N, NSvFS is also N. The statement "S is not true" should intuitively be F, T and T as S is T, N and F.

This logic has the operator "it is false that" but does not have an operator "it is not true that". If the logic had such an operator (denote it by ~) then the statement S : ~S would be inconsistent for any value of S.

I feel that this is not a major weakness of this logic, which in any case contains the classical logic. "False" says almost all we would like to say by saying "not". We can say "either NA or FA" in the sense NAvFA, and this reduces simply to FA. We have "or" in the sense in which v has been defined, and can say "NA or FA" in that sense. We do not have to say "NA or FA" in a sense outside the logic.

6. An imitation of the Goedel statement

If we try to reproduce, in a two-valued logic, the statement

G : *It is not provable that* G

G must state from what premiss G is to be not provable. In this two-valued logic G is provable as follows:

If $\sim$G, then G is provable, then G is true. Therefore G is true. Now the fact that G is provable implies G is false. Thus G is paradoxical. Here the premiss for provability of G is trivial (a tautology). If we write G now as

$$G : \sim(t \supset G)$$

this is immediately

$$G : \sim G$$

which is of course liar-paradoxical.

Now in our three-valued logic, we must have "false" in place of "not":

G : It is false that G is provable.

Try saying: if G is false, then G is provable, then G is true. It seems that we should now conclude that G is "not false" rather than G is true. However, here we are arguing intuitively. The intuitive statement "if G is false, then G is true" would be T, T and F as G is T, N and F. This operator does not exist in our logic. (It never takes value N so cannot have an algebraic dependence on A or FA, and there is a symmetry so that the other functions of A are the same when A is T or F.)

Keeping within our logic, we state G as

$$G : F(t \supset G)$$

which reduces to

$$G : FG$$

which is consistent with G Nonsense.

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