A DEFINITION OF CONJUNCTION IN THE PURE IMPLICATIONAL CALCULUS WITH ONE VARIABLE

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It has been shown (1) that the following is a consistent and complete set of axioms for the fragment of the propositional calculus in which the only connective is '$\supset$' and the only variable is '*p*'.

Ax. 1. $*p$
Ax. 2. If $\vdash\alpha$ and $*\beta$, then $*(\alpha \supset \beta)$
Ax. 3 If $*(\alpha \supset \beta)$, then $\vdash\alpha$ and $*\beta$

The notation '$\vdash\alpha$' means that α is derivable and '$*\beta$' means that β is not derivable.

The word «calculus» is used here to mean simply a set of logical truths. «Propositional calculus,» «pure implicational calculus» (or more simply «implicational calculus»), and «pure implicational calculus with one variable» thus denote sets in order of increasing intension. A precedent for such usage is to be found, for example, in the title of Appendix I of Prior's *Formal Logic* (Oxford, 1955): «Postulate Sets for Logical Calculi.» But note that the axioms given above differ in principle as well as in detail from any of the postulate sets listed by Prior. They make essential use, for instance of the strictly metalogical predicates $\vdash$ and $*$. The role of these predicates is illustrated in the proof of $\vdash(p \supset p)$: If $*(p \supset p)$, then, by Ax. 3, $\vdash p$; but this contradicts Ax 1.. This proof and others like it presuppose that $\vdash\alpha$ if and only if not $*\alpha$, so that the metatheory must contain an intuitive negation. Another difference is that this calculus does not employ a rule of substitution, so that the very use of the word «variable» in the title may seem odd. However, the rule «If $\vdash\alpha$, and β is the result of substituting γ for p throughout α, then $\vdash\beta$» would be consistent with the axioms given above. Thus

(1) See Henry W. JOHNSTONE, Jr. and Robert PRICE, «Axioms for the Implicational Calculus with One Variable,» *Theoria*, Vol. XXX, 1964, pp. 1-4.

from $\vdash(p \supset p)$ one could infer $\vdash[(p \supset p) \supset (p \supset p)$ by the use of this rule. But since the axioms are complete without it, the rule is redundant.

It is obvious that within the system defined by Ax. 1, Ax. 2, and Ax. 3 we can define '$(\alpha \vee \beta)$' as '$(\alpha \supset \beta) \supset \beta$'. It now appears that we can also define conjunction. Hence the axioms given are sufficient not only for the implicational calculus of one variable, but for the whole negation-free calculus of one variable.

We define the conjunction 'α & β' as '$[\alpha \supset (\beta \supset p)] \supset p$'. We can verify this immediately by making use of a readily demonstrable metatheorem [2] to the effect that every wff in the system must either be a tautology or be truthfunctionally equivalent with p. Now if T and p are the only values α and β can have, 'α & β' must have the following «truth-table» :

α	&	β
T	T	T
p	p	T
T	p	p
p	p	p

But when we assign the same values to the 'α' and the 'β' in the proposed definition, we obtain

[α	$\supset$	(β	$\supset$	p)]	$\supset$	p
T	p	T	p	p	T	p
p	T	T	p	p	p	p
T	T	p	T	p	p	p
p	T	p	T	p	p	p

which shows that the definition is in fact equivalent with α & β.

We are able to generalize from an implicational calculus to a negation-free calculus because the set $\{T, p\}$ is closed with respect to the operations '&', 'v', and, for that matter, '$\equiv$', as well as '$\supset$'. In other words, no conditional, disjunction, conjunction, or biconditional having as constituents tautologies or statements equivalent p could truthfunctionally evaluate to anything beside a

(2) For a demonstration, see *ibid.*, p. 2.

tautology or a statement equivalent with p. If there were any other possibilities, Ax. 3 could not be maintained. For by Ax. 3, if $*\alpha$ and $*\beta$, then $\vdash(\alpha \supset \beta)$. But if the nontautologies α and β are permitted to assume non-equivalent values, then $\alpha \supset \beta$ is not generally a theorem.

The set $\{T, p\}$ is not closed, however, with respect to the operation '$\sim$', for $\sim T$ is a contradiction and $\sim p$ is something else again. Hence our negation-free calculus with one variable cannot be extended to a full calculus involving negation. Ax. 3 will break down because, for example, given the truths that $*p$, and that $*\sim p$, it will imply the falsehood that $\vdash(p \supset \sim p)$. Similar considerations prevent the extension of this system to a calculus of two or more variables.

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