THE COMPLETENESS OF SO.5

M. J. CRESSWELL

In [1] and [2] E. J. Lemmon sets out a system of modal logic in which the necessity operator L is interpreted as "It is tautological (by truth-table) that" and calls it SO.5.

SO.5 has the following basis:

PCL: If α is a PC tautology, then $\vdash L\alpha$ ([1]);
LA1: $Lp \supset p$;
LA2: $L(p \supset q) \supset (Lp \supset Lq)$;
MP: $\vdash \alpha, \vdash \alpha \supset \beta \rightarrow \vdash \beta$;

with uniform substitution for propositional variables.

We shew that a semantics can be given for SO.5 analogous to those of [3] and [4] for other modal systems. Adopting the terminology of [5] we define an SO.5 model as an ordered triple $\langle V\ W\ x_1 \rangle$ where W is a set of objects (worlds), $x_1 \varepsilon W$, and V is an assignment from formulae and worlds to the set $\{1, 0\}$ of truth values. The basic assumption is that x_1 is the real world and in it necessity is evaluated as in the models of [3], while the rest are worlds in which only PC tautologies are true ([2]). This is ensured by letting $V(L\alpha\, x_i) = 1$ or 0 independently of the value of α (for $x_1 \neq x_i$).

We can set this out formally as follows:
$\langle V\ W\ x_1 \rangle$ is an SO.5 model iff:
W is a set of worlds, $x_1 \varepsilon W$ and V is an assignment satisfying:
1.1: For propositional variable p and $x_i \varepsilon W$, $V(p\, x_i) = 1$ or 0;
1.2: For wff α and $x_i \varepsilon W$, $V(\sim\alpha\, x_i) = 1$ if $V(\alpha\, x_i) = 0$, otherwise 0;

([1]) The numbering is ours. Lemmon actually has two rules (v. [1] p. 31), PC: If α is a PC tautology then $\vdash \alpha$, and R1: If α is a PC tautology then $\vdash L\alpha$. Clearly by LA1 and MP the first of these follows from the second.

([2]) These worlds are somewhat like the 'non-normal' worlds of [4] p. 211 where $L\alpha$ is always false.

1.3: For wffs α and β and $x_i \varepsilon W$, $V((\alpha v \beta)\, x_i) = 1$ iff either $V(\alpha\, x_i) = 1$ or $V(\beta\, x_i) = 1$, otherwise 0;

1.4: For wff α and $x_i \varepsilon W$ $(x_i \neq x_1)$, $V(L\alpha\, x_i) = 1$ or 0; for x_1 $V(L\alpha\, x_1) = 1$ iff for every $x_j \varepsilon W$, $V(\alpha\, x_j) = 1$, otherwise 0.

α is true in an SO.5 model $\langle V\, W\, x_1 \rangle$ iff $V(\alpha\, x_1) = 1$.

α is SO.5 valid iff α is true in every SO.5 model.

We shew that every theorem is valid:

1.) If α is a PC tautology then by 1.1-1.3, for every $x_i \varepsilon W$, in every SO.5 model, $V(\alpha\, x_i) = 1$, hence $V(L\alpha\, x_1) = 1$ (in every model), hence $L\alpha$ is valid.

2.) Suppose for some SO.5 model $\langle V\, W\, x_1 \rangle$, $V((Lp \supset p)\, x_1) = 0$, then $V(p\, x_1) = 0$ and $V(Lp\, x_1) = 1$. But $x_1 \varepsilon W$, hence $V(p\, x_1) = 1$, contrary to reductio hypothesis.

3.) Suppose that, for some SO.5 model $\langle V\, W\, x_1 \rangle$, $V((L(p \supset q) \supset (Lp \supset Lq))\, x_1) = 0$. Then $V(Lq\, x_1) = 0$; hence for some $x_i \varepsilon W$, $V(q\, x_i) = 0$; But $V(Lp\, x_1) = 1$, hence $V(p\, x_i) = 1$, hence $V((p \supset q)\, x_i) = 0$, hence $V(L(p \supset q)\, x_1) = 0$, contrary to reductio hypothesis.

4.) Uniform substitution for propositional variables is clearly validity preserving.

5.) Modus Ponens is validity-preserving for, if α is true in every SO.5 model and $\alpha \supset \beta$ is true in every SO.5 model, then for every model $\langle V\, W\, x_1 \rangle$, $V(\alpha\, x_1) = 1$ and $V((\alpha \supset \beta)\, x_1) = 1$, hence $V(\beta\, x_1) = 1$ (in every model), hence β is valid.

Hence every theorem of SO.5 is valid.

To prove completeness we use a method analogous to the adaptation in [5] of the decision procedure of [6] for T.

Every SO.5 formula will have the form of a truth-function whose constituents are:

a.) propositional variables

or b.) L followed by a wff.

We call these latter *L-constituents*. We draw up the *modal truth table* of α by assigning 1's and 0's to each constituent, as if they were all propositional variables. Obviously every wf part of α will have an assigned or calculated value in each row of the table. We call rows for which α's calculated value is 0, *F-rows*. To shew that α

is a theorem it suffices to shew that each F-row is inconsistent; i.e. that when we have the conjunction of all the members having 1 in the row and the negations of all the members having 0 we can prove the negation of the whole conjunction. This can always be done if one of the following conditions holds of each F-row (where β, γ are wf parts of α):

I: Some $L\beta$ has 1 while β has 0;

II: Some $L\gamma_1, \ldots, L\gamma_n$ have 1 while $L\beta$ has 0 where $(\gamma_1 . \ldots . \gamma_n) \supset \beta$ is a PC tautology (or substitution instance of one),

III: β has 0 where β is a substitution instance of a PC tautology.

If one of I-III hold of every F-row then $\vdash_{S0.5} \alpha$.

Suppose I holds. Then from LA1 we have (by PC) $\vdash \sim(L\beta . \sim\beta)$, and so the whole conjunction is inconsistent.

For II we observe that if $(\gamma_1 . \ldots . \gamma_n) \supset \beta$ is a PC-tautology, then by PCL $\vdash L((\gamma_1 . \ldots . \gamma_n) \supset \beta)$, hence by LA2 $\vdash L(\gamma_1 . \ldots . \gamma_n) \supset L\beta$. Now from $p \supset (q \supset (p . q))$ we may (by PCL and two applications of LA2) prove $Lp \supset (Lq \supset L(p . q))$, and by successive applications of this we have:

$$\vdash (L\gamma_1 . \ldots . L\gamma_n) \supset L\beta, \text{ hence } \vdash \sim(L\gamma_1 . \ldots . L\gamma_n . \sim L\beta),$$

and so the whole conjunction is inconsistent.

If III holds then by PCL $\vdash L\beta$ and hence any conjunction containing $\sim L\beta$ is inconsistent.

Suppose that for some F-row none of I-III hold. We define an SO.5 model in which α is false. Take the first F-row for which none of I-III hold and, for propositional variables, let $V(p\, x_1) = 1$ or 0 according as p has 1 or 0 in the table.

Where $L\gamma_1, \ldots, L\gamma_n$ are all the L-constituents having 1 in the table then, for each $L\beta_i$ having 0 form, $(L\gamma_1 . \ldots . L\gamma_n) \supset L\beta_i$. Now $(\gamma_1 . \ldots . \gamma_n) \supset \beta_i$ is not a substitution instance of a PC tautology (if it were condition II would obtain). This means that we can make some PC assignment to the variables (where L-constituents are regarded as variables) such that $(\gamma_1 . \ldots .)\gamma_n \supset \beta_i$ has 0. With each such β_i we associate a world x_i and, for propositional variables and L-constituents δ of α, we let $V(\delta\, x_i) = 1$ or 0 according as the PC assignment to $(\gamma_1 . \ldots . \gamma_n) \supset \beta_i$ gives them 1 or 0. From this we have that $V(\gamma_1\, x_i) = 1, \ldots, V(\gamma_n\, x_i) = 1$ and $V(\beta_i\, x_i) = 0$. (If there are no $L\gamma$'s having 1 in the table, then $V(\beta\, x_i)$ still $= 0$ or condition III would obtain). Let W be the set of x_1 and all x_i associated with each β_i. Clearly $\langle V\, W\, x_1 \rangle$ can be extended to an SO.5 model. Now for each $\gamma_k (1 \leq k \leq n)$, $V(\gamma_k\, x_i) = 1$. Further $V(\gamma_k\, x_1) = 1$ (or condition I would obtain (3)). Hence for every $x_j \varepsilon W$, $V(\gamma_k\, x_j) = 1$, hence $V(L\gamma_k\, x_1) = 1$. And since $V(\beta_i\, x_i) = 0$, then $V(L\beta_i\, x_1) = 0$. Hence every L-constituent is true or false in the model according as it has 1 or 0 in the F-row of the table. Hence the whole row is false in the model, i.e. α is false in the model, hence α is not valid.

Thus either α is an SO.5 theorem or it is false in some SO.5 model. I.e. SO.5 is complete. Further the method gives a decision procedure for SO.5.

Victoria University of Wellington M. J. CRESSWELL

REFERENCES

[1] E. J. LEMMON, 'Is there only one correct System of Modal Logic?', *Aristotelian Society Supplementary Volumes*, Vol. XXXIII (1959), pp. 23-40.

[2] ——— 'New Foundations for Lewis Modal Systems', *The Journal of Symbolic Logic*, Vol. 22 (1957), pp. 176-186.

[3] Saul A. KRIPKE, 'Semantical Analysis of Modal Logic I. Normal modal propositional calculi' *Zeitschrift fur mathematische Logik und Grundlagen der Mathematik*, Vol. 9 (1963) pp. 67-96.

[4] Saul A. KRIPKE, 'Semantical Analysis of Modal Logic II. Non-normal modal propositional calculi' in *The Theory of Models*, Amsterdam, North Holland Publishing Co, 1965, pp. 206-220.

[5] M. J. CRESSWELL, 'Alternative Completeness Theorems for Modal Systems' *Notre Dame Journal of Formal Logic* (forthcoming).

[6] A. R. ANDERSON, 'Improved Decision Procedures for Lewis's Calculus S4 and Von Wright's Calculus M', *The Journal of Symbolic Logic*, Vol., 19 (1954), (pp. 201-214).

(3) Strictly we should add here that this is an induction hypothesis, since what we are shewing is that $L\gamma$ has 1 or 0 in x_i according as it has it in the table if γ has 1 or 0 in x_i according as it has it in the table.