ANOTHER BASIS FOR S4

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In [1] (p. 190 *et seq*) Professor A. N. Prior discusses a system of logic in which there are variables for functions taking propositional arguments. Where f, g,... etc are these variables we may either define propositional identity as $(\alpha = \beta) =_{df} (f)(f\alpha \supset f\beta)$ (α and β wffs) or introduce identity as primitive with the two axioms:

I1 $p = p$,

I2 $(p = q) \supset (fp \supset fq)$

(or by the equivalent schemata if we do not have rules of uniform substitution for variables).

In this system one cannot, without additional axioms, prove $(p \equiv q) \supset (p = q)$ or even $\vdash \alpha \equiv \beta \rightarrow \vdash \alpha = \beta$. If we add the former we obtain the Leśniewskian protothetic (v. [2] pp. 152, 153). If we add the latter propositional identity would seem to amount to provable equivalence. This identification suggests the possibility of defining the operator 'It is logically necessary that p' (Lp) as 'p is identical with some provable truth' where we choose some theorem (call it 1) and define $L\alpha =_{df} (\alpha = 1)$.

If we take identity as primitive then by using schemata we can dispense with the variables for functions of propositions. We set up the following system, called PCR:

Primitive symbols:
Constants, $=$, $\supset$, (,), 0,
propositional variables p, q, r,... etc.

Formation rules: 1.) a propositional variable is a wff, 2.) 0 is a wff, 3.) If α and β are wffs then $(\alpha = \beta)$, $(\alpha \supset \beta)$ are wffs.

Definitions:
Def 1 $1 =_{df} (0 \supset 0)$,
Def L $L\alpha =_{df} (\alpha = 1)$ (α any wff),
Usual definitions of $\sim$, ., v, $\equiv$.

Axioms: PC, some set sufficient for a propositional calculus based on $\supset$ and 0 with uniform substitution and Modus Ponens.

Axiom Schema: I $(p=q) \supset (\alpha \supset \beta)$ where α and β are any wffs, α differing from β only in having p in some of the places where β has q.

Rule of Transformation:

R $\quad \vdash \alpha \equiv \beta \rightarrow \vdash \alpha = \beta$.

(By R we may prove $\vdash p=p$ and so only need one identity schema. By R and I also follows in PCR a rule of inter-substitutability of proved equivalents.)

We shew that PCR is deductively equivalent to S4. We use the following axiomatization of S4 (cf. [3] § 63.3 pp. 98, 99): PC with uniform substitution for propositional variables, modus ponens, L primitive and the following axioms and rules (our numbering):

LA1 $\quad Lp \supset p$,
LA2 $\quad L(p \supset q) \supset (Lp \supset Lq)$,
LA3 $\quad Lp \supset LLp$,
LR1 $\quad \vdash \alpha \rightarrow \vdash L\alpha$,

Def = $\alpha = \beta : =_{df} : L(\alpha \equiv \beta)$.

We prove in PCR each axiom and rule of S4 and the identity
LD $\quad (p = q) = [(p \equiv q) = 1]$

LA1 $\quad Lp \supset p$.

Proof:

I	(1)	$(p = 1) \supset [(p \supset p) \supset (1 \supset p)]$
(1) PC	(2)	$(p = 1) \supset (1 \supset p)$
PC	(3)	$(1 \supset p) \supset p$
(2) (3) PC	(4)	$(p = 1) \supset p$
(4) *Def L*	(5)	$Lp \supset p$ QED

LA2 $\quad L(p \supset q) \supset (Lp \supset Lq)$.

Proof:

I, PC	(1)	$((p \supset q) = 1) \supset [(p = 1) \supset ((1 \supset q) = 1)]$
PC	(2)	$(1 \supset q) \equiv q$
(2) R	(3)	$(1 \supset q) = q$
(1) (3) I	(4)	$((p \supset q) = 1) \supset [(p = 1) \supset (q = 1)]$
(4) *Def L*	(5)	$L(p \supset q) \supset (Lp \supset Lq)$ QED

LA3 $Lp \supset LLp$.

Proof:

I	(1)	$(p = 1) \supset [((1 = 1) = 1) \supset ((p = 1) = 1)]$
PC	(2)	$1 \equiv 1$
(2) R	(3)	$1 = 1$
(3) PC	(4)	$(1 = 1) \equiv 1$
(4) R	(5)	$(1 = 1) = 1$
(1) (5) PC	(6)	$(p = 1) \supset [(p = 1) = 1]$
(6) *Def L*	(7)	$Lp \supset LLp$ QED

LR1 $\vdash \alpha \rightarrow \vdash L\alpha$.

Proof:

ex hypothesi	(1)	α
PC	(2)	$\alpha \supset (\alpha \equiv 1)$
(1) (2) MP	(3)	$\alpha \equiv 1$
(3) R	(4)	$\alpha = 1$
(4) *Def L*	(5)	$L\alpha$ QED

LD $(p = q) = [(p \equiv q) = 1]$.

Proof:

I	(1)	$(p = q) \supset [((p \equiv p) = 1) \supset ((p \equiv q) = 1)]$
PC	(2)	$(p \equiv p) \equiv 1$
(2) R	(3)	$(p \equiv p) = 1$
(1) (3) PC	(4)	$(p = q) \supset [(p \equiv q) = 1]$
PC	(5)	$[(p \equiv q).p] \equiv [(p \equiv q).q]$
(5) R	(6)	$[(p \equiv q).p] = [(p \equiv q).q]$
I	(7)	$[(p \equiv q) = 1] \supset : (6) \supset [(1.p) = (1.q)]$
(6) (7) PC	(8)	$[(p \equiv q) = 1] \supset [(1.p) = (1.q)]$
PC	(9)	$(1.p) \equiv p$
(9) R	(10)	$(1.p) = p$
(10) q/p	(11)	$(1.q) = q$
(8) (10) (11) I	(12)	$[(p \equiv q) = 1] \supset (p = q)$
(4) (12) PC	(13)	$(p = q) \equiv [(p \equiv q) = 1]$
(13) R	(14)	$(p = q) = [(p \equiv q) = 1]$ QED

Hence PCR $\rightarrow$ S4.

We prove that S4 $\rightarrow$ PCR.

It suffices to prove the following in S4:

L $\quad Lp = (p = 1)$,

I $\quad (p = q) \supset (\alpha \supset \beta)$ where α and β are any wffs α differing from β only in having p in some of the places where β has q,

R $\quad \vdash \alpha \equiv \beta \rightarrow \vdash \alpha = \beta$.

R follows directly by LR1 and Def=. L is known theorem of S4 (v. [3] Th. 45.24 p. 73 or Th. 62.56 p. 97, bearing in mind that we have to prove I and R before we can substitute 1 for Vp). To prove I it suffices to prove the following:

I.1 $\quad (p = q) \supset [(p = r) = (q = r)]$,

I.2 $\quad (p = q) \supset [(r = p) = (r = q)]$,

I.3 $\quad (p = q) \supset [(p \supset r) = (q \supset r)]$,

I.4 $\quad (p = q) \supset [(r \supset p) = (r \supset q)]$,

since every formula is made up of propositional variables and 0 combined by $\supset$ and $=$. Thus we apply successively I.1 — I.4 as often as necessary and use $(p = q) \supset (p \supset q)$ (an obvious theorem of S4).

Clearly S4 based on $\supset$ and 0 is equivalent to S4 based on any other complete PC. Using the following rule:

B $\quad \vdash \alpha \supset \beta \rightarrow \; \vdash L\alpha \supset L\beta$

(an obvious consequence of LR1 and LA2) we prove I.1 as follows:

PC, B	(1)	$L(p \equiv q) \supset L[(p \equiv r) \supset (q \equiv r)]$
(1) LA2 PC	(2)	$L(p \equiv q) \supset : L(p \equiv r) \supset L(q \equiv r)$
similarly	(3)	$L(p \equiv q) \supset : L(q \equiv r) \supset L(p \equiv r)$
(2) (3) PC	(4)	$L(p \equiv q) \supset : L(p \equiv r) \equiv L(q \equiv r)$
(4) B Def =	(5)	$LL(p \equiv q) \supset : (p = r) = (q = r)$
LA3	(6)	$L(p \equiv q) \supset LL(p \equiv q)$
(5) (6) PC Def =	(7)	$(p = q) \supset [(p = r) = (q = r)]$ QED

I.2 may be proved similarly.

I.3, I.4 follow easily by PC and B.

Hence S4 $\rightarrow$ PCR.

Obviously I.1 — I.4 could have been taken as axioms instead of the schema I. In such a case our PCR would be more like an equivalential version of Prior's S4 for strict implication ([4] p. 3). The interesting point about PCR however is that I is simply the identity schema and R also seems to contain no reference to modality. The equivalence of PCR and S4 would seem to give further evidence for the view (v.e.g. [5] pp. 32-33) that where L means, 'It is informally provable that' then S4 is the system which captures its meaning.

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REFERENCES

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