PROPOSITIONAL ARITHMETIC

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One important development of symbolic logic is the extent to which the concepts and theorems of mathematics are expressible in purely logical form. The following is a system natural number arithmetic in a calculus of functions of propositions. The functorial calculus (abbreviated FC) contains,

a.) propositional variables p, q, r, ... etc.
b.) functorial variables f, g, h, ... etc.

These last take propositional arguments. Examples of functors over which the type b. variables range are,

'It is not the case that p',
'It is logically true that p',
'x believes that p',
'the ancients considered it to be more beneficial that p than that q'.

In axiomatizing FC we attach it to the propositional calculus and introduce quantification over the variables exactly as it is introduced over individual and predicate variables in the predicate calculus. Substitution for f$\hat{p}$ ($\hat{p}$ is a propositional argument place) may be made of any wff except a propositional variable where the argument p of fp replaces uniformly a distinct propositional variable. Because of the structural similarity, the theorems of the predicate calculus will in general (though not always) have FC analogues.

Identity of propositions is defined as follows,

Def = $\quad p = q :=\text{df} : (f)\ (fp \supset fq).$

This is analogous to Russell's definition of x = y, and states that two propositions may be said to be identical iff any function of the one is a function of the other. This is stronger than strict or provable equivalence. We cannot in FC prove that $p = \sim\sim p$, though we can prove $p \equiv \sim\sim p$. There are many functors within whose scope we might wish to deny the inter-substitutability of proved equivalents. One thinks e.g. of operators like 'x believes that p', since x might quite conceivably believe that p without believing that $[\{(\exists p)p \equiv (q \vee \sim q)\} \,.\, (r)\,(s)\,[frs \supset (gq \supset frs)] \,.\, p]$; yet this is demonstrably equivalent to p in FC. With this definition of identity it is possible to assume an infinity of distinct propositions. Following a suggestion of Professor Prior's (1) that no proposition can be a logical complication of itself, we add to FC the following axiom,

AxInf $p \neq fp$.

We call the system FC + *AxInf*, FC^i and prove the consistencey of FC^i as follows.

In the domain of natural numbers the even numbers are to be considered 'true' and the odd numbers 'false'. We give an interpretation under which $fp > p$ is always true.
The constant functors have the following assignment,

$\sim p = (p + 1)$ (Clearly $\sim p > p$ and the truth conditions for $\sim$ hold).

$(p \vee q) = [(p \times q) + 2]$ (Clearly $(p \vee q) > p$, $(p \vee q) > q$ and since $(p \times q)$ is only odd when p and q are both odd the truth conditions are fulfilled).

By the substitution rules, $\hat{p}$ is not a legitimate substitute for f, and so the axiom is true for all the constant truth functors. Since $(a)A(\hat{a})$ is never a legitimate substitution for f in B(f), the only case in which a quantified expression can be substituted will be cases built up from substitutions having the form $(a)\, A(a\hat{p}) / f\hat{p}$. Suppose A contains no quantifiers and for every a_n $A(a_n p) > p$. Then let $(a)\, A(ap)$ be evaluated as follows:

If every $A(a_n p)$ is even, then let $(a)A(ap) = A(a_1 p)$,
If some $A(a_n p)$ is odd, then, if a_1 is even, let $(a)\, A(ap) = A(a_1 p)+1$; if a_1 is odd, let $(a)\, A(ap) = A(ap) = A(a_1 p)$.

By the normal interpretation of the universal quantifier, this will be seen to fulfill the truth conditions, and since $(a)\, A(a_n p) > p$ *ex hypothesi*, then every expression constructed by means of quantification can be shown to satisfy the axiom.

Since this interpretation uses the symbols = and >, we must assign interpretations. If $p = q$, then $(p = q) = (p + 2)$, otherwise if p is the larger and is even, $(p = q) = (p + 1)$ and, if odd, $= (p + 2)$,

(1) A. N. Prior, Is the Concept of Referential Opacity Really Necessary?, *Acta Philosophica Fennica*, fasc. 16 (1963), pp. 189-198, v. esp. p. 192.

similarly with q if it is the larger. For $(p > q)$, if $p > q$ and p is odd, then $(p > q) = p + 1$, if p is even, $(p > q) = p + 2$, if $(p \leqslant q)$, then, if q is odd, $(p > q) = q + 2$, if q is even, $(p > q) = q + 1$. These satisfy the axiom and the truth conditions.

Now the following is a theorem;

$$[(p)\ (fp > p)\ .\ (p)\ (gp > p)] \supset (p)\ (fgp > p).$$

Thus any combination of the constant functors will satisfy the axiom. We restrict the range of variable functors to combinations of the constant functors.

In FC^i we set up the following series N:

$(\exists p)p$, $\sim(\exists p)p$, $\sim\sim(\exists p)p$, $\sim\sim\sim(\exists p)p$, ... etc.

Clearly this will be an infinite series of distinct propositions for, given any p later in the series than q, $p = fq$ and hence $p \neq q$ by *AxInf*. Letting $p\,q =_{df} . p = \sim q$, we define f_* analogously with Russell's R_*. (cf PM*90)

*Def** $f^*pq =_{df} : (g)\ [\{gq\ .\ (r)\ (s)\ (gr\ .\ fsr) \supset gp]$.

We have the following definitions;

DefO $O =_{df} (\exists p)\ p$,
Def Seq: $Seq\ (pq) =_{df} [H\ (qO)\ .\ Hpq]$,
DefN: $N(p) =_{df} Seq_*(pO)$.

With these definitions we may prove the Peano axioms;

P1 $N(O)$ (from $f_*\,pp$)
P2.1 $[N(p)\ .\ Seq\ (qp)] \supset N(q)$ (from $(f_*pq\ .\ frp) \supset f_*rq$)
P2.2 $[Seq(pq)\ .\ Seq(rq)] \supset (p{=}r)$
(from $[(p = \sim q)\ .\ (r = \sim q)] \supset (p = r)$)
P2.3 $N(p) \supset (\exists q)\ Seq\ (qp)$ (from $N(p) \supset Seq\ (\sim pp)$)
P3 $[Seq(pq)\ .\ Seq\ (pr)] \supset (q = r)$
(this follows from $Seq_*(pq)\ .\ (\sim p = \sim q) \supset (p = q)$)
P4 $\sim Seq(Op)$
(from $H_*\,pq \supset (\exists f)\ (p = fq)$ and hence
$(H_*pq\ .\ Hqp) \supset (\exists f)\ (p = f\sim q)$ (since $Hpq = .\ p = \sim q$)
and so by Def Seq.)
P5 $[\{fO\ .\ (p)\ (q)\ ((fq\ .\ Seq\ (pq)) \supset fp)\} \supset \{N(r) \supset fr\}]$
(from Def* and Def N).

These results are not surprising (and therefore have not been proved in detail), since any suitable progression can be shown to satisfy the Peano axioms. We may hence define the usual arithmetical operations.

But if this system (ArithFC) can appropriately be called a logical analysis of arithmetic, we must show how it can be used to express the 'meaning' of arithmetical constants. In place of defining a predicate's being true of n individuals or a set's having n members, we define a functor's being true of n propositions.

Null(f) becomes $\sim(\exists p)fp$.
(An example of a null functor is $\hat{p} . \sim\hat{p}$.)

Unit(f) becomes $(\exists p)\,[fp\,.\,(q)\,\{fq \supset (p = q)\}]$
(An example of a unit functor is $\hat{p} = (\exists p)p$.)

The concept of a 1-1 functor is analogous to that of a 1-1 relation, (cf. PM *71.172).

Def 1-1 $h \varepsilon 1\text{-}1 . =_{df} (p)\,(q)\,(r)\,[\{(hpr\,.\,hqr) \supset (p=q)\}\,.\,\{(hpq\,.\,hpr) \supset (q = r)\}]$.

Similarity of functors is defined (cf PM*73),

Def Siml: $\mathrm{Siml}\,(fg) =_{df} (\exists h)\;[(h \varepsilon 1\text{-}1)\,.\,(p)\,(\exists q)\,(fp \supset (gq\,.\,hpq))\,.\,(q)\,(\exists p)\,(gp \supset (fp\,.\,hpq))]$.

This of course is the concept of a 1-1 correspondence between the propositions of which f is true and those of which g is true.

In PM a number is the class of classes equivalent to a given class. The FC analogue to this would mean the introduction of functors which take functorial arguments. This would require a type hierarchy. Instead we associate similar functors with a proposition. In particular each n-membered functor is associated with the n'th proposition in the series N characterized above, i.e.
Def Nc: $\mathrm{Nc}(pf) =_{df} [\mathrm{Seq}_*(pO)\,.\,\mathrm{Siml}(f\{\mathrm{Seq}_*(\hat{q}O)\,.\,\sim(\mathrm{Seq}_*\,(\hat{q}p)\}]$

Nc(pf) can be read 'p is the cardinal number of f'.

The following theorems hold in virtue of Def Nc:

T1 $(\mathrm{Nc}(pf)\,.\,\mathrm{Nc}(qf)) \supset (p=q)$.

T2 $N(p) \supset (\exists f)\mathrm{Nc}(pf)$
(Since $N(p) \supset \mathrm{Nc}(p\{\mathrm{Seq}_*(\hat{q}O).\ \sim \mathrm{Seq}_*(\hat{q}p))\})$,

T3 $(\exists p)\,[\mathrm{Nc}(pf).\ \mathrm{Nc}(pg)] \supset \mathrm{Siml}(fg)$
(by $(\exists h)\;[\mathrm{Siml}(fh).\ \mathrm{Siml}(gh)] \supset \mathrm{Siml}(fg)$,

T4 Nc(O, $\hat{p}$. ~$\hat{p}$),

T5 Nc (1, $\hat{p}$ = (∃p)p),

T6 Nc(2, $\hat{p}$ = (∃p) p . v . $\hat{p}$ = ~(∃p)p), etc.

T7 [Nc(pf) . Nc(qg) . ~(∃r) (fr.gr)] ⊃ Nc((p+q) ($f\hat{r}$. v . $g\hat{r}$)),
and so on for the other operations.

Since by *AxInf* there are an infinite number of propositions and hence, a functor such as $\hat{p}$v ~ $\hat{p}$ holding of an infinite number of propositions, there will be some functors which have no cardinal in the series N. Further, although *AxInf* is satisfiable in a denumerable domain, it is compatible with a non-denumerable infinity of propositions. The formula (∃p) Nc (pf) v Siml (f, $\hat{p}$v ~ $\hat{p}$) is not a theorem. The system ArithFC developed above is sufficient for the finite arithmetic of natural numbers. Transfinite arithmetic and real number theory will be partly a matter of legislating in favour of certain new definitions and axioms.

Since there is nothing intrinsically *arithmetical* about the series N, we might very well have chosen another series satisfying the Peano axioms. The only reason for taking N is that it is definable in terms of the primitive logical symbols of FC. A series of e.g. individuals would require the predicate calculus augmented by individual and predicate constants. And since all logics have propositions and proposition-forming operators on propositions (though not always variables for them), FC may be regarded as the most economical system in this respect having variables of this type only.

ArithFC may be extended to predicate and set arithmetic by the definitions, Nc(pø) and Nc(pα), and in general, where δ is an operator of which numerical statements are appropriate, we can define Nc(pδ). Thus an adequate natural number arithmetic can be defined in a logical system without type theory.

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