QUANTIFIERS IN MANY-VALUED LOGIC

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In the context of orthodox two-valued logic, A. Mostowski has introduced a very powerful generalization of the machinery of quantification typified in the familiar existential and universal quantifiers (1). On Mostowski's conception, a quantifier is, in effect, a relation (propositional function of two variables) of two *numbers* which are determined as follows: Given a one-place predicate (propositional function) P defined with respect to the individuals of a domain of discourse D, we are to conceive of a quantifier Q in such a way that

«$(Qx)\ Px$» amounts to: $Q(\alpha, \beta)$

where:

α = the cardinal number of the set of individuals x in D for which «Px» is *true*.

β = the cardinal number of the set of individuals x in D for which «Px» is *false*.

Thus, for example, to obtain the existential quantifier (∃-quantifier) by this scheme we simply take $Q(\alpha, \beta)$ as:

$\alpha \neq 0.$

Again, to obtain the universal quantifier (∀-quantifier) we take $Q(\alpha, \beta)$ as:

$\beta = 0.$

One further example: to obtain the «plurality-quantifier» (M-quantifier) which I have discussed elsewhere (2) we take $Q(\alpha, \beta)$ as:

$\alpha > \beta.$

This brief outline should suffice to exhibit the fundamental idea of Mostowski's approach.

The purpose of the present note is to present a *generalization of quantifiers* (in the sense of Mostowski) *for many-valued logics*. We suppose then that in place of the usual two truth-values T (for truth) and F (for falsity) we have a series of n truth values: $T_1, T_2, \ldots, T_n$.

(1) *Fundamenta Mathematica*, vol. 44 (1957), pp. 12-36.

(2) *Journal of Symbolic Logic*, vol. 27 (1962), pp. 373-374.

It seems natural to extend Mostowski's generalization of quantifiers as follows: Given a predicate P defined with respect to the individuals of a domain of discourse D, we are to conceive of a quantifier Q as a propositional function of n cardinal numbers (numerical parameters) in such a way that

«$(Qx)\ Px$» amounts to: $Q(\alpha_1, \alpha_2, \ldots, \alpha_n)$

where

α_i is the cardinal number of the set of individuals x in D for which «Px» assumes the truth-value T_i.

Thus, for example, consider a three-valued logic with the truth-values T (true), F (false), and N (neutral). We could now correspondingly obtain, for any predicate P, the three cardinal numbers:

α_1 (P) = the cardinal number of the set of all x in D for which «Px» takes the truth-value T;
α_2 (P) = the cardinal number of the set of all x in D for which «Px» takes the truth-value N;
α_3 (P) = the cardinal number of the set of all x in D for which «Px» takes the truth-value F;

and introduce such quantifiers as, for instance, the quantifier

$(Nx)\ Px$

to mean that $\alpha_2 \neq 0$, i.e., that for *some* element x of the domain D, Px is neutral (neither true nor false).

Or again, we could introduce the three-valued «mostly-quantifier,»

«$(Zx)\ Px$» amounting to: $\alpha_1 > \alpha_3$,

so that «Px» is more often true than false for elements x of the domain D. The three-valued version of the two-valued plurality quantifier «$(Mx)\ Px$» mentioned above would answer (most closely) to the relationship

$\alpha_1 > \alpha_2 + \alpha_3$.

It is clear that this extension of quantifiers affords highly flexible machinery for treating quantificational concepts in many-valued logics. The application of these ideas in special cases warrants further investigation. Interesting questions can be raised by its means — for example that of the minimum number of independent (non-interdefinable) quantifiers that can be specified by certain given means in various systems of many-valued logic.

For example, in a three-valued logic whose matrix (truth-table) for negation is given by

P	$\sim P$
T	F
N	N
F	T

there will be exactly six distinct definition-independent «*sum-inequality quantifiers,*» i.e., quantifiers specifiable in terms of inequalities ($<$ or $\leqslant$) among sums of the α_i. This is so because the defining relationship for such a quantifier, (Qx) Px, must have one of the six forms (where a, b, c stand for *distinct* a_i):

(1)	(2)
$a < b$	$a \leqslant b$
$a < b + c$	$a + b \leqslant c$
$a + b < c$	$a \leqslant b + c$

We can immediately cut the number of possibilities in half, since any type-(2) quantifier (say the quantifier (Qx) Px corresponding to $\alpha_1 \leqslant \alpha_2$) can be defined in terms of the negation of a type-(1) quantifier (in this case $\sim$(Q*x) Px, where (Q*x) Px corresponds to $\alpha_2 < \alpha_1$).

We are left with 12 possibilities, as follows: (i) for $a < b$, there are 3 choices for a and 2 for b, for a total of 6 possibilities, (ii) for $a < b + c$ there are three possibilities, fixed by the three choices of a, and (iii) the case of $a + b < c$ is exactly like the preceding.

But these twelve possibilities can again be cut in half because any quantifier in the group (say (Qx) Px corresponding to $\alpha_1 < \alpha_2$) gives rise to another quantifier of the group when we shift to (Qx) $\sim Px$ (and thus $\alpha_1 > \alpha_2$). For this shift — in virtue of the negation-matrix — effects the substitutions

α_1 to α_3
α_2 to α_2
α_3 to α_1

and thus gives rise to a second, different inequality within the same family. *In the three-valued logic at issue, there are thus exactly six* *distinct «sum-inequality» quantifiers on the basis of which all of the others can be introduced by definition.* This finding illustrates one kind of result that can be arrived at by the machinery here proposed for introducing quantifiers (of the Mostowski type) into many-valued logic.

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