SOME REMARKS ON RUSSELL'S TREATMENT OF DEFINITE DESCRIPTIONS

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Our language contains the following symbols:
(1) the logical constants $\sim$ («not»), $\rightarrow$ («only if»), $\wedge$ («and»), $\vee$ («or»), $\leftrightarrow$ («if and only if»), $\bigwedge$ («for all»), $\bigvee$ («for some»), I («is identical with»), ι («the»), and V^1 («for exactly one»); we call the first five of these sentential connectives and all of the rest except I variable binders;
(2) a denumerable infinity of distinct
  (a) individual variables,
  (b) individual constants, and
  (c) predicates of any positive number of places.

We use «$\langle$» and «$\rangle$» in the metalanguage to mark the boundaries of non-empty finite sequences. Terms and formulas will be understood as follows:
(1) all variables and individual constants are terms;
(2) for any positive integer m, m-place predicate p, and m-term sequence of terms t, $\langle pt \rangle$ is a formula;
(3) for any terms t and u, $\langle tIu \rangle$ is a formula;
(4) for any variable v and formulas f and g,
  (a) $\langle \iota vf \rangle$ is a term and
  (b) $\langle \sim f \rangle$, $\langle f \rightarrow g \rangle$, $\langle f \wedge g \rangle$, $\langle f \vee g \rangle$, $\langle f \leftrightarrow g \rangle$, $\langle \bigwedge vf \rangle$, $\langle \bigvee vf \rangle$, and $\langle V^1 vfg \rangle$ are formulas.

In what follows, we omit sequence marks according to the usual conventions for the omission of parentheses. Given terms t and u and a term of formula f, we understand freedom and PStuf (the result of properly substituting t for u in f) as follows:
(1) if u = f, then u is free in f and PStuf = t;
(2) if u $\neq$ f, then
  (a) if f is a variable or individual constant, then u is not free in f and PStuf = f;
  (b) for any positive integer m, m-place predicate p, and m-term sequence of terms v, if f = $\langle pv \rangle$, then u is free in f just in case u is free in some member of the range of v and PStuf = $\langle$p the m-term sequence w such that w(i) = PStuv(i) for any i in the domain of w$\rangle$; also, for any terms v and w, if f = $\langle vIw \rangle$, then u is free in f just in case u is free in either v or w and PStuf = $\langle$PStuv I PStuw$\rangle$;
  (c) for any sentential connective c and formulas g and h,

(1) if f = ⟨cg⟩, then u is free in f just in case u is free in g and PStuf = ⟨c PStug⟩; and

(2) if f = ⟨gch⟩, then u is free in f just in case u is free in g or h and PStuf = ⟨PStug c PStuh⟩; and

(d) for any variable binder b, variable v, and formulas g and h,

(1) if f = ⟨bvg⟩, then u is free in f just in case u is free in g and v is not free in u; also, if f = ⟨bvgh⟩, then u is free in f just in case u is free in g or h and v is not free in u;

(2) if u is not free in f and either f = ⟨bvg⟩ or f = ⟨bvgh⟩, then PStuf = f;

(3) if u is free in f and v is not free in t, then PStuf = ⟨bv PStug⟩ if f = ⟨bvg⟩ and PStuf = ⟨bv PStug PStuh⟩ if f = ⟨bvgh⟩; and

(4) if u is free in f, v is free in t, and w is the first variable not occurring in either f or t, then PStuf = ⟨bw PStuPSwvg⟩ if f = ⟨bvg⟩ and PStuf = ⟨bw PStuPSwvg PStuPSwvh⟩ if f = ⟨bvgh⟩.

By F (weak first order quantifier logic with identity and descriptions), we mean the deductive system whose inference rules are modus ponens and universal generalization and whose axioms are the e such that, for some individual constant c, formulas f and g, terms t and u, and distinct variables v and w such that w is not free in t or f, e is one of the following:

(1) a tautology
(2) $\bigwedge v f \wedge \bigvee w\ wIt \rightarrow PStvf$
(3) $\bigwedge w \langle f \rightarrow g \rangle \wedge f \rightarrow \bigwedge w g$
(4) $\bigvee v f \leftrightarrow \sim \bigwedge v \sim f$
(5) $\bigvee v\ vIw$
(6) $\bigvee v\ vIc$
(7) $\bigvee w\ wIt \rightarrow tIt$
(8) $tIu \wedge PStuf \rightarrow f$
(9) $\bigvee w\ wI\ \iota vf \leftrightarrow \bigvee w \bigwedge v \langle f \leftrightarrow vIw \rangle$.

1. *Russell's Theory of Descriptive Terms*

On page 178 of his *Introduction to Mathematical Philosophy* (London, 1919), Bertrand Russell says

> And generally: «the term satisfying Φx satisfies Ψx» is defined as meaning:
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> «There is a term c such that (1) Φx is always equivalent to 'x is c,' (2) Ψc is true.»

Similarly, on page 173 of volume I of the second edition of *Principia Mathematica* (Cambridge, 1925), Alfred Whitehead and Bertrand Russell say

> Thus when we say: «The term x which satisfies $\Phi\hat{x}$ satisfies $\Psi\hat{x}$,» we shall mean: «There is a term b such that Φx is true when, and only when, x is b, and Ψb is true.» That is, writing «$(\imath x)(\Phi x)$» for «the term x which satisfies $\Phi\hat{x}$», $\Psi(\imath x)(\Phi x)$ is to mean
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> $$(\exists b) : \Phi x \,.\, \equiv_x x=b : \Psi b.$$

The deductive system which is being referred to in these passages may be called Russell's theory of descriptive terms. Its first order analogue D with respect to our conventions results from adding to the axioms of F the set of all e such that, for some formulas f and g and distinct variables v and w such that w is not free in either f or g, e is the Russell description equivalence $PS\langle \imath vf\rangle vg \leftrightarrow \bigvee w \langle \bigwedge v \langle f \leftrightarrow vIw\rangle \wedge PSwvg\rangle$.

Unfortunately, D is inconsistent. This can be shown by taking f to be the negation of some tautology, taking g to be some tautology, and then applying the Russell description equivalence.

2. *Russell's Theory of Uniqueness*

Russell did feel that something was wrong with his version of D. Thus, on the page of *Principia Mathematica* that was mentioned in the preceding section, Whitehead and Russell say that their version of the Russell description equivalence

> ...is not yet quite adequate as a definition, for when $(\imath x)(\Phi x)$ occurs in a proposition which is part of a larger proposition, there is doubt whether the smaller or the larger proposition is to be taken as the «$\Psi(\imath x)(\Phi x)$.» ...The proposition which is to be treated as the «$\Psi(\imath x)(\Phi x)$» will be called the *scope* of $(\imath x)(\Phi x)$... In order to avoid ambiguities as to scope, we shall indicate the scope by writing «$[(\imath x)(\Phi x)]$» at the beginning of the scope, followed by enough dots to extend to the end of the scope... Thus we arrive at the following definition:
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> *14.01. $[(\imath x)(\Phi x)] \,.\, \Psi(\imath x)(\Phi x) \,.\, = \,:\, (\exists b) : \Phi x \,.\, \equiv_x x=b : \Psi b$ Df

Evidently, the scope marked definiendum of this definition stands for a formula; moreover, it bears one bound variable and two subformulas which may be replaced by other formulas, but not the identity constant of the object language. Thus, if it is thought of as a constant of the object language rather than of the metalanguage, then the symbol defined in this definition must be a 1-place 0-term 2-formula formula making variable binder and so cannot be the 1-place 0-term 1-formula term making variable binder ı. We shall use V^1 as the variable binder concerned. Also, as our analogue to what may be called Russell's theory of uniqueness, we use the deductive system U which results from adding to the axioms of F the set of all e such that, for some formulas f and g and distinct variables v and w such that w is not free in either f or g, e is the Russell uniqueness definition $V^1vfg \leftrightarrow Vw\langle \bigwedge v\langle f \leftrightarrow vIw\rangle . \wedge \bigwedge v\langle vIw \rightarrow g\rangle\rangle$.

U is consistent; however, it is very incomplete since it has no special rules for definite descriptions beyond those supplied by F. For a related deductive system which is both sound and semantically complete, the reader is referred to the author's 'Contributions to syntax, semantics, and the philosophy of science' (*Notre Dame Journal of Formal Logic,* forthcoming).

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