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THE CONSISTENCY OF ARITHMETIC, BASED ON A LOGIC OF MEANING CONTAINMENT*

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Dedicated to Richard Sylvan and Robert Meyer, who provided the inspiration for this work.

1. *Introduction*

The logic MC of meaning containment has been developed over a number of works, particularly in Brady [1996] and [UL], and recently modified in Brady and Meinander [2008]. This paper also utilizes some results and discussion from Brady and Rush [2008], in the application of this logic to arithmetic.

In order to understand the differences between this account of arithmetic and that of classical logic, we need to consider the conceptual differences between the logic MC and classical logic. The logic MC is basically conceptualized in terms of meaning rather than truth and falsity, and thus careful conceptual distinctions need to be drawn in the application of MC. This contrasts with classical logic, where numbers are identified with sets and the equivalents of the Axiom of Choice cover quite a range of pure and applied set-theory. We will conceptualize the entailment logic MC as the logic of meaning containment, and this idea will determine the axiomatization of the logic, and its quantificational and arithmetic extensions. This paper will serve to illustrate the impact this conceptual difference will have on the application and meta-theory of logic.

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The logic MC itself is a weak relevant logic containing neither of the key classical principles: the Law of Excluded Middle (LEM) and the Disjunctive Syllogism (DS). It also restricts distribution principles in order to maintain intensionality appropriate to a meaning-based logic. The content semantics of MC can be seen to satisfy the intensional set-theoretic containment properties, thus representing meaning containment without these extensional features. This is the preferred semantics as it pins down the logic rather than provide a semantics for a wide range of logics, as is the case with truth-theoretic semantics. (See Brady [1996] and [UL].)

We now consider the key technical results, achieved thus far, which illustrate the difference between this meaning-based logic and the truth-based classical logic. The principle one is the simple consistency of naive set theory and higher-order predicate theory, initiated in Brady [1971] and set out in full in [UL]. The simple consistency results in [UL] are wide-ranging in that they apply to a generalized comprehension axiom for classes and also for higher-order predicates that together suffice to solve the set-theoretic and semantic paradoxes in a non-ad hoc way. Another important result is the decidability of the predicate logic, obtained in Brady [2002–5] using a normalized natural deduction system.

As early as the mid-60's, Richard Sylvan (then Routley) had said "Gödel's proof would not go through with a decent logic". The rationale for this followed from discussion with Len Goddard about having to change the logic to solve the paradoxes, the idea being that Gödel's proof contains logical steps which also occur in paradox derivation. Indeed, Gödel's argument to his First Theorem runs close to the Liar Paradox, differing only in replacing truth by provability. Thus, Gödel's Theorems would be collateral damage in a logical solution to the paradoxes. This provides motivation for us to show that Gödel's Theorems do not apply when arithmetic is based on the logic of meaning containment.

In Section 4, we prove the simple consistency of arithmetic by finitary methods, where the arithmetic is based on the sentential logic MC. The method of proof of this result will use metavaluations, introduced by Meyer in [1976a] and extended by Slaney in [1984] and [1987]. This method will also enable us to show that if A $\vee$ B is provable then, for all its constant instances A$'$ $\vee$ B$'$, either A' or B' is provable. These results are in stark contrast to Gödel's second and first theorems, respectively. These are further key technical results for MC, demarcating it from classical logic, and hopefully heralding a string of property differences between MC and classical logic. In Section 5, we extend the results proved for the logic MC to most M1- and M2-metacomplete logics and, in Section 6, we will examine Mendelson [1964] to determine how this arithmetic will differ from classical Peano arithmetic. Indeed, we will show that all of Mendelson's theorems from sections 3.1 to 3.11 continue to hold, with almost all being in the same form and with some being in modified forms.

2. *The Logics MC, MCQ and MCQ$^-$*

We first set out the logic MC and its quantificational extension MCQ in its current modified form. Originally, the logic called DJ^d was presented in Brady [1996] with an axiomatization and content semantics, and subsequently also in the book [UL]. More recently in Brady and Rush [2008], consideration was given to dropping the two rules of quantified distribution, $\forall x(A \vee B) \Rightarrow A \vee \forall xB$ and $A \,\&\, \exists xB \Rightarrow \exists x(A \,\&\, B)$, the first of which prevents the Law of Excluded Middle $(A \vee {\sim}A)$ from applying to all quantified '$\rightarrow$'-free formulae in the context of arithmetic. [Of course, this also includes their formula forms, $\forall x(A \vee B) \rightarrow A \vee \forall xB$ and $A \,\&\, \exists xB \rightarrow \exists x(A \,\&\, B)$. Note too that x cannot occur free in A in each of these forms, as we take x to be only a bound variable.] It was argued in Brady and Rush [2008] that these two quantified distribution rules embrace a combination of the extensional and the intensional, this being their downfall. The extensional here refers to conjunction and disjunction, whilst the intensional refers to the two quantifiers, which can of course be applied to non-recursive domains, these not being establishable in any element-by-element fashion. Further, in Brady and Meinander [2008], the sentential distribution properties, $A \,\&\, (B \vee C) \rightarrow (A \,\&\, B) \vee (A \,\&\, C)$ and $(A \vee B) \,\&\, (A \vee C) \rightarrow A \vee (B \,\&\, C)$, are removed because it is argued there that sentential distribution is not an intensional property. Nevertheless, the sentential rule forms are still included in the logic as they follow from a widened form of the meta-rule of MC, viz. if $A, B \Rightarrow C$ then $D \vee A, D \vee B \Rightarrow D \vee C$.

However, some further considerations regarding these existential and universal distribution rules have come to be realized, requiring some adjustments to MCQ, these being also reported on in Brady and Meinander [2008]. The existential distribution rule, $A \,\&\, \exists xB \Rightarrow \exists x(A \,\&\, B)$, is easily derivable from the widened 2-premise meta-rule, if $A, B^{a}/_{x} \Rightarrow C^{a}/_{x}$ then $A, \exists xB \Rightarrow \exists xC$. This meta-rule holds as a result of the meaning of the existential quantifier as 'at least one of, but without necessarily including which one'. Since the variable a is common to premise and conclusion, the particular element is carried to the conclusion, where existential generalization then applies. Further, this 2-premise meta-rule is deductively equivalent to the single-premise version of the meta-rule, if $A^{a}/_{x} \Rightarrow B^{a}/_{x}$ then $\exists xA \Rightarrow \exists xB$, in combination with the existential distribution rule. So, we will replace the single premise meta-rule by the 2-premise one in the logic MCQ below.

Moreover, in Section 4, we will argue that the universal distribution rule, $\forall x(A \vee B) \Rightarrow A \vee \forall xB$, fails for a recursive interpretation of the universal quantifier. (Note that we will also argue that the existential distribution rule still holds for a recursive interpretation of the existential quantifier.) Since this interpretation is one of the interpretations within the ambit of the universal quantifier, generally construed, the universal distribution rule must also fail generally as well. Any specific interpretation can only add logical axioms and rules to the basic system MCQ, not subtract them. This is because the logic MCQ is based on properties of connectives and quantifiers, just determined by their fundamental meanings, without embellishments such as recursion. Nevertheless, regrettably, we will need to contract the system MCQ to the system MCQ^- in order for our consistency proof to work.

In the process, we do need to make a slight contraction to the meta-rules MR1 (if $A \Rightarrow B$ then $C \vee A \Rightarrow C \vee B$) and QMR1 (if $A^{a}/_{x} \Rightarrow B^{a}/_{x}$ then $\exists xA \Rightarrow \exists xB$) in the system MCQ in order to fit in with the rejection of universal distribution, as footnoted in Brady and Meinander [2008]. We restrict the application of QR1 ($D^{a}/_{x} \Rightarrow \forall xD$, where a does not occur in D), in the proof of the rule $A, B \Rightarrow C$ of MR1 (If $A, B \Rightarrow C$ then $D \vee A, D \vee B \Rightarrow D \vee C$) and in the proof of the rule $A, B^{a}/_{x} \Rightarrow C^{a}/_{x}$ of QMR1 (If $A, B^{a}/_{x} \Rightarrow C^{a}/_{x}$ then $A, \exists xB \Rightarrow \exists xC$), as follows: QR1 can only be used to prove $\forall xD$ as a theorem. This restriction makes sense as the meta-rules work through disjunctive or existential instantiations, which cannot then be universally generalized upon.

It is important to note here that these determinations are for the pure logics MC and MCQ and that in applied situations such as arithmetic to follow, strengthenings of these logics may well be appropriate. Indeed, it will be seen that the LEM should apply to the identity statements of arithmetic, and hence to formulae built up using ∼, & and ∨, and that the DS should also apply to such formulae, but we will nevertheless extend the DS to full use on the grounds that arithmetic is simply consistent.

So, we first set up the axiomatizations of MC and MCQ making these above adjustments regarding sentential and quantified distribution and the meta-rules.

MC.
Primitives: ∼, &, ∨, →.
Axioms:
1. $A \rightarrow A$.
2. $A \,\&\, B \rightarrow A$.
3. $A \,\&\, B \rightarrow B$.
4. $(A \rightarrow B) \,\&\, (A \rightarrow C) \rightarrow .A \rightarrow B \,\&\, C$.
5. $A \rightarrow A \vee B$.
6. $B \rightarrow A \vee B$.
7. $(A \rightarrow C) \,\&\, (B \rightarrow C) \rightarrow .A \vee B \rightarrow C$.
8. $\sim\sim A \rightarrow A$.

9. $A \to {\sim}B \to .B \to {\sim}A$.
10. $(A \to B) \,\&\, (B \to C) \to .A \to C$.
Rules:
1. $A, A \to B \Rightarrow B$.
2. $A, B \Rightarrow A \,\&\, B$.
3. $A \to B, C \to D \Rightarrow B \to C \to .A \to D$.
Meta-rule:
1. If $A, B \Rightarrow C$ then $D \vee A, D \vee B \Rightarrow D \vee C$.

As indicated above, we expand the standard one-premise meta-rule, if $A \Rightarrow B$ then $C \vee A \Rightarrow C \vee B$, to the above two-premise one, thus allowing a derivation of the distribution rules, $A \,\&\, (B \vee C) \Rightarrow (A \,\&\, B) \vee (A \,\&\, C)$ and $(A \vee B) \,\&\, (A \vee C) \Rightarrow A \vee (B \,\&\, C)$. This extends the use of the disjunctive meta-rule MR1 to include the above three two-premise rules R1–3 directly and generally expand its usefulness.[1]

MCQ.
Quantificational Primitives.
$\forall$, $\exists$ (quantifiers)
a, b, c, . . . (free individual variables)
x, y, z, . . . (bound individual variables)
f, g, h, . . . (predicate variables)
[m, n, . . . (individual constant schemes)]
[r, s, t, . . . (schemes for terms, which are variable or constant)]
Quantificational Axioms.
1. $\forall xA \to A^{t}/_{x}$, for any term t.
2. $\forall x(A \to B) \to .A \to \forall xB$.
3. $A^{t}/_{x} \to \exists xA$, for any term t.
4. $\forall x(A \to B) \to .\exists xA \to B$.

Note that, in distinguishing free and bound individual variables, x can only occur bound in the A of QA2 and in the B of QA4. Terms can be any free variable or individual constant.
Quantificational Rule.
1. $A^{a}/_{x} \Rightarrow \forall xA$, where a does not occur in A.

[1] The second sentential distribution rule is easily proved by applying MR1 to R2, after first applying A2 and A3. The first one requires a little more work. We first have: $A \,\&\, (B \vee C) \Rightarrow (A \,\&\, B) \vee A$, and then $A \,\&\, (B \vee C) \Rightarrow (C \vee A) \,\&\, (C \vee B) \Rightarrow C \vee (A \,\&\, B) \Rightarrow (A \,\&\, B) \vee C$. Hence, $A \,\&\, (B \vee C) \Rightarrow (A \,\&\, B) \vee (A \,\&\, C)$.
Also, MR1 can be completely generalized to: if $A_1, \ldots, A_n \Rightarrow B$ then $C \vee A_1, \ldots, C \vee A_n$, by repeated application of MR1 to R2.
Let $A_1, \ldots, A_n \Rightarrow B$. Then $A_1 \,\&\, \ldots \,\&\, A_{n-1}, A_n \Rightarrow B$. Let $C \vee A_1, \ldots, C \vee A_n$. Then, by repeated MR1, $C \vee (A_1 \,\&\, A_2), \ldots, C \vee A_n$, and hence $C \vee (A_1 \,\&\, \ldots \,\&\, A_{n-1}), C \vee A_n$ and $C \vee B$.

Quantificational Meta-rule.
1. If A, $B^{m}/_{x} \Rightarrow C^{m}/_{x}$ then A, $\exists$xB $\Rightarrow$ $\exists$xC.
QMR1 is subject to the proviso that, in the derivation A, $B^{m}/_{x} \Rightarrow C^{m}/_{x}$, QR1 cannot generalize on any variable free in either of the premises A and $B^{m}/_{x}$. Similarly, MR1 is subject to the same proviso concerning the derivation A, B $\Rightarrow$ C. For MR1, this proviso prevents $D \vee A^{a}/_{x} \Rightarrow D \vee \forall xA$ and hence $\forall x(D \vee A) \Rightarrow D \vee \forall xA$ from being derived. For QMR1, the proviso prevents the derivation of $\exists xB^{a}/_{y} \Rightarrow \exists x\forall yB$ and hence $\forall y\exists xB \Rightarrow \exists x\forall yB$. Of course, for both meta-rules, QR1 can still be used to prove a theorem in A, B $\Rightarrow$ C and in A, $B^{m}/_{x} \Rightarrow C^{m}/_{x}$, respectively, regardless of whether the generalized variable occurs free in the premises or not.

Though MCQ is ideal, the following method of metavaluations does simplify the modelling used to establish metacompleteness and hence the simple consistency. In so doing, the method cannot sufficiently distinguish $\forall x(A \to B) \Rightarrow A \to \forall xB$ from $\forall x(A \vee B) \Rightarrow A \vee \forall xB$, and unfortunately, at this stage, we need to exclude QA2, and its rule-form, and the rule-form of its associated QA4, as well as the already-excluded quantified distribution rule, $\forall x(A \vee B) \Rightarrow A \vee \forall xB$.

Since both QA2 and QA4 are used to prove $\exists xA \leftrightarrow \sim\forall x\sim A$, along with QA1 and QA3, we put $\exists$xA as a definition:
$\exists xA =_{df} \sim\forall x\sim A$.
We then drop QA3, which is now provable, but we replace QA2 by the much weaker:
QA2′.$A \to \forall xA$.
As above, x can only occur bound in A.
Thus, iterated '$\to$' 's are removed from the quantificational axioms.
This now forms the logic MCQ^{-}. This is the logic we will carry forward into the proof of metacompleteness.

3. *The Re-Shaping of Relevant Arithmetic*

The initial work on relevant arithmetic was done by Meyer in [1975] and [1975a], abstracted in [1976], where he set out its axiomatization based on the strong relevant logic R and proved that this arithmetic was non-trivial, i.e. not all formulae are provable in it.

We axiomatize his system, which he called R#, as is set out on p. 20 of [1975] and pp. 14–16 of [1975a]:
R#.
The definitions $A \vee B =_{df} \sim(\sim A \,\&\, \sim B)$ and $\exists xA =_{df} \sim\forall x\sim A$ were added.
x,y,z, ... (variables ranging over natural numbers).
0 (the number zero).
The functions $'$ (successor), $+$ (addition) and $\times$ (multiplication) were added.

0 and successor generate the natural numbers in the usual way.
a, b, c, ... (schemes for natural number constants).
r, s, t, ... (schemes for natural number terms)
Axioms.
1. $A \to B \to .B \to C \to .A \to C$.
2. $A \to .A \to B \to B$.
3. $A \,\&\, B \to A$.
4. $A \,\&\, B \to B$.
5. $(A \to B) \,\&\, (A \to C) \to .A \to B \,\&\, C$.
6. $A \,\&\, (B \vee C) \to (A \,\&\, B) \vee (A \,\&\, C)$.
7. $\sim\sim A \to A$.
8. $A \to \sim B \to .B \to \sim A$.
9. $A \to \sim A \to \sim A$.

Rules.
1. $A, A \to B \Rightarrow B$.
2. $A, B \Rightarrow A \,\&\, B$.

Quantificational Axiom.
10. $\forall xA \to A^t/_x$, where t is free for x in A.

Quantificational Rule.
3. $A \to B \vee C \Rightarrow A \to \forall xB \vee C$, where x is neither free in A nor in C.

Number-theoretic Axioms.
11. $x' = y' \to x = y$.
12. $x' \neq 0$.
13. $x = y \to x' = y'$.
14. $x = y \to .x = z \to y = z$.
15. $x + 0 = x$.
16. $x + y' = (x + y)'$.
17. $x \times 0 = 0$.
18. $x \times y' = (x \times y) + x$.

Number-theoretic Rule.
$A(0), A(x) \to A(x') \Rightarrow A(x)$. [Mathematical Induction]

Meyer in [1975a] set up a natural deduction system and used it to show that all the classically provable formulae of form $s = t$, for numerical terms s and t, are provable in R#. Over a period of time, he unsuccessfully tried to show that the rule γ: $A, \sim A \vee B \Rightarrow B$ (which is deductively equivalent to the DS), was an admissible rule of R#. Alas, he and Friedman subsequently showed in their [1988] that this was not the case, i.e. that γ is inadmissible.

An important result for R# is its non-triviality, proved by Meyer for the stronger system RM3#.[2] Indeed, what he proved was that $0 = 1$ is not provable in RM3#. To show this, he added the modelling condition $0 = 2$, yielding an inconsistent arithmetic model which is modulo 2. These inconsistencies are evaluated as 'both true and false' in RM3, whilst $0 = 1$ is maintained as just 'false' in the 3-valued modelling. Once soundness is shown, the unprovability of $0 = 1$ follows.[3] However, this is not the stronger and more familiar simple consistency, i.e., for every formula A, not both A and $\sim$A are provable.

Our present task is to re-axiomatize Peano arithmetic, based on our logic MC. We need to set up this axiomatization of arithmetic capturing the spirit of Peano's axioms in a form which reflects the ideals of MC, i.e. with its focus on entailment as a meaning containment. This was done in part on p. 205 in Brady and Rush [2008], where rule '$\Rightarrow$' 's were used to replace the '$\rightarrow$' 's in the axioms 11 and 13 above, in the process of examining the use of the Law of Excluded Middle in Peano arithmetic. This is because the entailment '$\rightarrow$' is an inappropriate relationship between statements involving distinct natural numbers. This can be seen as follows. By repeated application of axiom 11 above, we can easily prove that $100 = 100 \rightarrow 0 = 0$, and, by repeated application of axiom 13 above, $0 = 0 \rightarrow 100 = 100$ is provable. For numbers so far apart, these cannot be meaning containments, nor, putting them together, a meaning equivalence. Indeed, this would be so even for numbers 1 apart, as such numbers are based on distinct sets of objects. (This line of argument was taken a little further on p. 158 of Brady [1996].)

Additionally, on p. 205 of Brady and Rush [2008], the rule $\sim \mathrm{m} = \mathrm{n} \Rightarrow \sim \mathrm{m}' = \mathrm{n}'$ is added as rules are not generally contraposible. This illustrates De Morgan negation which, being essentially 4-valued, is captured using both positive and negative statements. Generally, such pairs of statements yield the four possible scenarios concerning the presence or absence of statements and their negations.

We do, however, take all the identity statements as classical, i.e. the LEM and the DS should both apply to them. As argued in Brady and Rush [2008], the scope of Boolean negation and hence classical logic is not universal but restricted at least to include a large part of the physical world and what can be mapped into it. It seems reasonable that identity statements in arithmetic should fall into this latter category, the reason being that identities between

[2] RM3 is a 3-valued logic, with values true, false and both true and false. RM3 extends the infinitely-valued logic RM, which is axiomatized as $\mathrm{R} + \mathrm{A} \rightarrow .\mathrm{A} \rightarrow \mathrm{A}$. An axiomatization for RM3 can be found in Brady [1982].

[3] This proof is available in abstracted form in Meyer [1976] and also in Meyer and Mortensen [1984].

two natural numbers either hold or they don't, and they can't do both. This relates back to the physical world through various sets of objects of the same and different sizes. Thus, we include the LEM as an assumption for identity statements, but we will show that the corresponding DS rule is admissible later on.

So, finally, based on MCQ$^-$, as set out in Section 2, we present the axiomatization for Peano arithmetic, which we will call MC#. We continue to use separate free and bound variables, together with the numerical constant schemes and schemes for terms.

Identity Axioms.
1. $\mathrm{a} = \mathrm{a}$.
2. $\mathrm{a} = \mathrm{b} \rightarrow \mathrm{b} = a$.
3. $\mathrm{a} = \mathrm{b} \,\&\, \mathrm{b} = \mathrm{c} \rightarrow \mathrm{a} = \mathrm{c}$.

The entailments are appropriate here as identity is symmetric, and $\mathrm{a} = \mathrm{c}$ is contained within the conjunctive meaning of the two identities, $\mathrm{a} = \mathrm{b}$ and $\mathrm{b} = \mathrm{c}$.

Identity Rule.
1. $\mathrm{s} = \mathrm{t}, \mathrm{A(s)} \Rightarrow \mathrm{A(t)}$, where, for terms s and t, t is substituted for s in a single argument place.

Note that the entailment form of the identity rule, $\mathrm{s} = \mathrm{t} \Rightarrow \mathrm{A(s)} \rightarrow \mathrm{A(t)}$, can also be derived. Hence, $\mathrm{s} = \mathrm{t} \Rightarrow \mathrm{A(s)} \leftrightarrow \mathrm{A(t)}$, which shows up the very close relationship between identity and meaning equivalence. Note too that the converse holds, by universally quantifying over the A. Thus, the shapes of axioms and rules involving identity should mimic their equivalential shapes, as can be seen, for example, in IA3 above.

Number-theoretic Axioms.
1. $\sim\mathrm{a}' = 0$.
2. $\mathrm{a} + 0 = \mathrm{a}$.
3. $\mathrm{a} + \mathrm{b}' = (\mathrm{a} + \mathrm{b})'$.
4. $\mathrm{a} \times 0 = 0$.
5. $\mathrm{a} \times \mathrm{b}' = (\mathrm{a} \times \mathrm{b}) + \mathrm{a}$.

Number-theoretic Rules.
1. $\mathrm{s} = \mathrm{t} \Rightarrow \mathrm{s}' = \mathrm{t}'$.
2. $\mathrm{s}' = \mathrm{t}' \Rightarrow \mathrm{s} = \mathrm{t}$.
3. $\sim\mathrm{s} = \mathrm{t} \Rightarrow \sim\mathrm{s}' = \mathrm{t}'$.
4. $\sim\mathrm{s}' = \mathrm{t}' \Rightarrow \sim\mathrm{s} = \mathrm{t}$.

Number-theoretic Meta-Rule.
If $\mathrm{A(m)} \Rightarrow \mathrm{A(m')}$ then $\mathrm{A(0)} \Rightarrow \mathrm{A(t)}$, where t is an arbitrary numerical constant or variable. [Mathematical Induction]

Classicality Axiom.
1. $\mathrm{a} = \mathrm{b} \vee \sim\mathrm{a} = \mathrm{b}$. [The LEM]

We do not include the Classicality Rule (CR1), $\sim s = t, s = t \vee B \Rightarrow B$, here, as we would like, but wait until after the proof of consistency to establish the full DS admissibly and then add it on as a rule.

Note that we can establish the more familiar form of Mathematical Induction with conclusion $\forall xA(x)$ by putting t as a variable a, say, not occurring in $A(x)$, and by applying QR1.

Also, note that we do not restrict the application of NMR1 in the proofs of the rules, $A, B \Rightarrow C$ and $A, B^{m}/_{x} \Rightarrow C^{m}/_{x}$ of MR1 and QMR1, respectively, as $\forall xA(x)$ is not specifically introduced by NMR1.

As is shown by Theorem 1 on p. 204 of Brady and Rush [2008], the LEM extends to any $\rightarrow$-free formula A of arithmetic without quantifiers, viz. $A \vee \sim A$, for A built up using $\sim$, & and $\vee$ alone. Note that (sentential) distribution is used in rule form in the proof of this Theorem 1, but this is derivable from MR1.

It is also worth noting that if $\forall x(A \vee B) \Rightarrow A \vee \forall xB$ is included, then, by extending the induction argument of Theorem 1 of Brady and Rush [2008], the LEM extends to all quantificational formulae of classical logic. Moreover, we are including $A \,\&\, \exists xB \Rightarrow \exists x(A \,\&\, B)$, which, with the above Classicality Rule, CR1, by extending the induction argument of Theorem 2 of [2008], would enable the Disjunctive Syllogism to extend to all quantificational formulae of classical logic. These two results would then enable all of classical Peano arithmetic to be derived. In particular, for the Gödel sentence G, we would then have $G \vee \sim G$ provable, whilst, by Gödel's First Theorem, neither G nor $\sim G$ would be provable, on the assumption that Peano arithmetic is consistent. However, we will show in Section 4 that if a sentence $A \vee B$ is provable in our system, based on the above MCQ^{-}, then either A or B is provable. This will show that $\forall x(A \vee B) \Rightarrow A \vee \forall xB$ is not derivable in this system, as the LEM would then be provable for Gödel's G. Given that neither G nor $\sim G$ would be provable in the system, it being weaker than classical logic under the translation of '$\rightarrow$' into '$\supset$', the instances of the LEM must then stop short of $G \vee \sim G$.

4. *The Simple Consistency Proof*

We prove the simple consistency of the above re-shaped form of Peano arithmetic by using the method of metavaluations, which were introduced for quantified positive logics by Meyer in [1976a], and subsequently extended by Slaney in [1984] and [1987] to full sentential logics with negation. As can be seen from Brady [2010], in an applied setting containing classical formulae, this method works more smoothly for sentences, i.e. formulae with no free variables. So, we will modify their notion of metavaluation for formulae with free variables so that it is expressed in terms of its application to closed instances (i.e. constant instances). Further, these closed instances need to be arranged in such a way that they can be recursively constructed, not just because we are endeavouring to construct a finitary proof of consistency, but also because it is needed to prove metacompleteness. (Note, in Brady [2010], that recursion for closed formulae is introduced, but it is for axiomatic purposes.) It is important to realise here that in expressing the induction step, $A(m) \Rightarrow A(m')$, the two 'm' 's are treated as schematic for constants, and not evaluated as variables pertaining just to the $A(m)$ and to the $A(m')$.

Further, as recursive methods are used in the meta-theory, we need to pay particular attention to the logic of the meta-theory. Whilst intuitionist logic is espoused as the logic for constructive methods, our logic MCQ$^-$ with its focus on entailment and deduction can also be considered suitable for the meta-logic of recursive methods, which represents the extent of human reasoning in the arithmetic context. Indeed, MCQ$^-$ can be seen as a deductivist logic. (See Brady and Rush [2008] and Brady [2008] for some discussion of this, especially as it relates to negation in MC.) Also, any change in logic from the object- to the meta-language is hard to justify. The logic MCQ$^-$ of the object-language, especially given its universality, should also apply to the meta-language, though one would expect there to be some classical gain in the process. That is, the whole point of meta-theory is to examine the object-theory as a whole, from outside of itself, and make judgements on it, which would generally be classically evaluated. This is so in the assignment of T and F to the metavaluations below. However, the metavaluations for $\forall xA$ and for a formula A with free variables require recursive processes which may invoke a departure from classical logic, especially in cases where these processes are related to one another through the formula or rule.

Detlefsen [1979], on p. 309, makes this point re Gödel's Second Theorem:

> In order for a consistency formula to 'express' consistency in the appropriate sense the quantifiers and operators in it must be construed finitistically, and *not* classically, since it is the finitistic consistency of a classical system that is at issue. But a finitistic interpretation of the universal quantifier would seem to differ drastically from a classical interpretation of it, ...

Thus, he argues that a finitistic logic is needed for the meta-theory when finitary methods are being used to prove consistency, and that this logic is distinct from classical logic.

In our case, 'if $v(\forall x(A \vee B)) = T$ then $v(A \vee \forall xB) = T$' fails to hold in the meta-logic, as does $\forall x(A \vee B) \Rightarrow A \vee \forall xB$ in the object-logic.[4] Given the

[4] I owe this point to Graham Priest.

similarities of the positive first-degree for MCQ$^-$ and intuitionist predicate logic, the reasons for its failure are similar to those for intuitionist logic given in Dummett [1977], pp. 202–8, and Beall and Restall [2006], pp. 64–5. All one needs is an example of a recursive set expressed as a union of two sets, one of which is not recursive whilst the other is not satisfied for any particular recursively determined element. Indeed, if the Beall and Restall example on p. 65 was allowed to extend to the infinite, that would nicely suffice. They consider students in a philosophical logic class who have satisfied the prerequisite of doing an introductory logic subject or have been allowed to enrol by special permission. Say, if the class was denumerably infinite and each member was stipulated as being admitted in one of the two ways, then, since there was no rhyme or reason to their admission to the class within any enumeration of its members, it is quite possible for there to be no recursive process for determining that all have been admitted by prerequisite nor any recursive determination of someone admitted by special permission.[5]

Note that $\mathrm{A} \mathbin{\&} \exists \mathrm{xB} \Rightarrow \exists \mathrm{x}(\mathrm{A} \mathbin{\&} \mathrm{B})$ holds in intuitionist logic and does not present a problem for recursive processes, as any recursive process used to show $\exists\mathrm{xB}$ can also be used to show $\exists \mathrm{x}(\mathrm{A} \mathbin{\&} \mathrm{B})$, given that A holds. This provides the justification for maintaining it in the logic MCQ$^-$ for recursive arithmetic purposes.

To begin the proof, we first need the following:

Lemma 1.

(1) For any identity $\mathrm{s} = \mathrm{t}$, for constant terms s and t, either $\mathrm{s} = \mathrm{t}$ or $\sim\mathrm{s} = \mathrm{t}$ is provable in MC#.

(2) Further, $0 = \mathrm{s}'$, for any constant term s, is unprovable in MC#.

Proof. (1) Just apply the number-theoretic axioms and rules, for the appropriate proofs of the form $\mathrm{s} = \mathrm{t}$ or $\sim\mathrm{s} = \mathrm{t}$. If s and t represent the same number then use $\mathrm{a} = \mathrm{a}$. If s and t represent different numbers, use NA1 and NR3. Then, use the appropriate axioms and rules concerning successor, addition and multiplication to create the terms, for both the positive and negative cases. E.g. $2 \text{ x } (1' + 1) = 2 \text{ x } (1' + 0') = 2 \text{ x } (1' + 0)' = 2 \text{ x } 1'' = (2 \text{ x } 1') + 2 = (2 \text{ x } 1) + 2 + 2 = (2 \text{ x } 0) + 2 + 2 + 2 = 0 + 2 + 2 + 2$ (eliminating x) $= 0 + 2 + (2 + 1)' = 0 + 2 + (2 + 0)'' = 0 + 2 + 2'' = 0 + (2 + 2')' = 0 + (2 + 2)'' = 0 + (2 + 1)''' = 0 + (2 + 0)'''' = 0 + 2'''' = (0 + 2''')' = (0 + 2'')'' = (0 + 2')''' = (0 + 2)'''' = (0 + 1)''''' = (0 + 0)'''''' = 0''''''$ (eliminating $+$) $= 6$. (See Section 6 for more detail on this method.)

(2) We use Meyer's inconsistent modelling of relevant arithmetic to show that $0 = \mathrm{m}'$, for any numerical constant m, is unprovable in MC#, by choosing arithmetic modulo $\mathrm{m} + 2$. Then, $0 = \mathrm{s}'$ is also unprovable, for any

[5] It is easily shown that $\forall \mathrm{x}(\mathrm{A} \vee \mathrm{B}) \Rightarrow \mathrm{A} \vee \forall \mathrm{xB}$ (where x cannot be free in A) and $\forall \mathrm{x}(\mathrm{A} \vee \mathrm{B}) \Rightarrow \exists \mathrm{xA} \vee \forall \mathrm{xB}$ are deductively equivalent.

constant term s. His method, written up in Meyer and Mortensen [1984], uses the 3- valued logic RM3. Classical logic is not usable, as the DS must be invalid. (This is why the DS has been removed completely from the axiomatization of MC#.) □

We follow what is called the two-sorted approach for the quantified logic in Brady [2010], but the classical sort is initially restricted to atomic formulae of form $a = b$, which can be extended to formulae with $\sim$, & and $\vee$. We construct the metavaluations v and v* for the sentential part, as follows, bearing in mind that they take exactly one of the values T and F:

(i) $v(s = t) = T$ iff $s = t$ is a theorem of MC#, for *constant terms* s and t.
$v^*(s = t) = v(s = t)$, for *constant terms* s and t.

For the following metavaluations (ii)–(v), we let A and B be *sentences*.

(ii) $v(A \,\&\, B) = T$ iff $v(A) = T$ and $v(B) = T$.
$v^*(A \,\&\, B) = T$ iff $v^*(A) = T$ and $v^*(B) = T$.

(iii) $v(A \vee B) = T$ iff $v(A) = T$ or $v(B) = T$.
$v^*(A \vee B) = T$ iff $v^*(A) = T$ or $v^*(B) = T$.

(iv) $v(\sim A) = T$ iff $v^*(A) = F$.
$v^*(\sim A) = T$ iff $v(A) = F$.

(v) $v(A \to B) = T$ iff $A \to B$ is a theorem of MC#, if $v(A) = T$ then $v(B) = T$, and if $v^*(A) = T$ then $v^*(B) = T$.
$v^*(A \to B) = T$.

[Note that MC is M1-metacomplete, in Slaney's sense, in which case $v^*(A \to B)$ always takes the value T. (See Slaney [1987].)]

We add the following metavaluations v and v* to account for the quantifier $\forall$, where $\forall xA$ is a sentence and $A^n/_x$ is a constant instance of A, obtained by substituting the numerical constant scheme n for x in A.
Note that we use m, n, ..., as schemes for numerical constants.

(vi) $v(\forall xA) = T$ iff $v(A^n/_x) = T$, for all numerical constants n, recursively generated.
$v^*(\forall xA) = F$ iff $v^*(A^n/_x) = F$, for some numerical constant n, recursively determined.

In the case of vacuous quantification,
$v(\forall xA) = T$ iff $v(A) = T$ and $v^*(\forall xA) = F$ iff $v^*(A) = F$.

Then, to take account of *free variables*, we add the following metavaluation:

(vii) $v(A) = T$ iff $v(A_i) = T$, for all constant instances A_i of A, recursively generated.
$v^*(A) = F$ iff $v^*(A_i) = F$, for all constant instances A_i of A, recursively generated.

We need to explain 'recursive generation' and 'recursive determination' for the quantified valuations (and also for formulae with free variables). For recursive generation we start with the base case $v(A^0/_x) = T$. Then, an induction step of form, $v(A^m/_x) = T \Rightarrow v(A^{m'}/_x) = T$, is added, with the rule arrow understood as a classical rule and the natural number scheme m being interpreted as a universally quantified meta-logical variable over natural numbers. These will enable mathematical induction to apply, as a meta-logical principle, so as to derive the appropriate generality. So, we express '$v(A^n/_x) = T$, for all numerical constants n, recursively generated' as the conjunction: $v(A^0/_x) = T$ and, for all m, if $v(A^m/_x) = T$ then $v(A^{m'}/_x) = T$.

For (vii), the above also caters for one free variable. For further free variables, we use the same expression, but with replacement of $v(A(0))$, $v(A(m))$ and $v(A(m'))$ by metavaluations of formulae with one or more free variables, each of which are inductively evaluated. This enables variables to be added, one by one, in any particular order. Though mathematical inductions can fairly generally be carried out on any variable, it is conceivable that a formula might exist which requires induction on a particular variable. Also, we can use this method to show that universal statements are all provable via mathematical induction.

Unlike recursive generation, recursive determination is existential. Since existential quantification is equivalent to a negated universal and the negation is De Morgan, recursive determination is established by a process. (See Brady [2008].) Such a process would lead to a witness for the existential and, as such, is a complementary process to that of recursive generation of the universal. This is distinct from the Boolean negation of a universal, which is given by $v(\forall xA) = F$, which just negates the recursive generation, leaving open the possibility of $v(A^n/_x) = T$, for all n, but not recursively generated. And, this account of recursive determination is sufficient for the applications to be made of it in what follows.

We follow Meyer [1976a] and Slaney [1984] in establishing metacompleteness via the two following lemmas, but we need to account for variables:

Lemma 2.

If A is a theorem of MC# then $v(A) = T$, and hence:

If $\sim A$ is a theorem of MC# then $v^*(A) = F$.

Proof. Induction is on proof steps. At each step, we first consider replacing all uncoded variables by constant instances. We then build up the true metavaluations for the original formulae or original conclusions of rules, one variable at a time, in some order. Because this latter process is standard, we consider it separately in (ii).

(i) So, we start with *constant instances*. We give special attention to axioms and rules involving quantification and negated identity. We give some examples of these.

QA1. $\forall$xA $\rightarrow$ A$^t/_x$, for a term t.
Let v($\forall$xA) = T. Then, v(A$^n/_x$) = T, for all numerical constants n, recursively generated. Let t be a constant and let m be the numerical constant determined by t. But, any particular constant m, the value of term t, must be reached after a finite number of recursive steps. So, v(A$^t/_x$) = T. [We do the case where t is a variable in (ii).]
Let v*(A$^t/_x$) = F. Let m be as above. Then, v*(A$^t/_x$) = F, can be recursively determined in m steps, and hence v*($\forall$xA) = F.
QR1. A$^a/_x$ $\Rightarrow$ $\forall$xA, where a does not occur in A.
We consider this rule with just the free variable a, leaving the rest of the variables for (ii). Let v(A$^a/_x$) = T. Then, v(A$^n/_x$) = T, for all constant instances A$^n/_x$, recursively generated. Thus, v(A$^0/_x$) = T and v(A$^m/_x$) = T $\Rightarrow$ v(A$^{m'}/_x$) = T, and hence v($\forall$xA) = T, by definition.
QMR1. If A, B$^m/_x$ $\Rightarrow$ C$^m/_x$ then A, $\exists$xB $\Rightarrow$ $\exists$xC, where, in the derivation A, B$^m/_x$ $\Rightarrow$ C$^m/_x$, QR1 does not generalize on a free variable in A or B$^m/_x$.
Let, if v(A) = T and v(B$^m/_x$) = T then v(C$^m/_x$) = T, where m is a schematic numerical constant.
Let v(A) = T and v($\sim\forall$x$\sim$B) = T. Then, v*($\forall$x$\sim$B) = F and v*($\sim$B$^n/_x$) = F, for some numerical constant n, recursively determined. For this constant, v(B$^n/_x$) = T, and hence v(C$^n/_x$) = T by assumption, and then v*($\sim$C$^n/_x$) = F. So, v*($\forall$x$\sim$C) = F and v($\sim\forall$x$\sim$C) = T.
IR1. s = t, A(s) $\Rightarrow$ A(t).
Let v(s = t) = T and hence s = t is provable in MC#. By Lemma 1, if s and t represent distinct natural numbers then s = t is unprovable, and so s and t represent the same natural numbers. Then, v(A(s)) = v(A(t)).
NA1. $\sim$a$'$ = 0.
By Lemma 1, s$'$ = 0 is unprovable in MC#, and hence v(s$'$ = 0) = F, v*(s$'$ = 0) = F, and v($\sim$s$'$ = 0) = T.
NR1. s = t $\Rightarrow$ s$'$ = t$'$.
If v(s = t) = T then v(s$'$ = t$'$) = T, for constant terms s, t.
NR3. $\sim$s = t $\Rightarrow$ $\sim$s$'$ = t$'$.
Let v($\sim$s = t) = T, for constant terms s, t. Then, v*(s = t) = F and v(s = t) = F. By NR2, if v(s$'$ = t$'$) = T then v(s = t) = T. Hence, v(s$'$ = t$'$) = F, v*(s$'$ = t$'$) = F and v($\sim$s$'$ = t$'$) = T.
NMR1. If A(m) $\Rightarrow$ A(m$'$) then A(0) $\Rightarrow$ A(t).
Let, if v(A(m)) = T then v(A(m$'$)) = T. Then, as m is a schematic numerical constant it interprets as a meta-logical variable. Let v(A(0)) = T. Then, v($\forall$xA(x)) = T, as the constant instants are recursively generated, according to the definition. Hence, v(A(t)) = T, as for QA1.
CA1. a = b $\vee$ $\sim$a = b.
For constant terms s, t, v(s = t) = T or F. Then, v(s = t) = T or v*(s = t) = F, and v(s = t) = T or v($\sim$s = t) = T, in which case,

$v(s = t \vee \sim s = t) = T$.
Note that CR1 is left out at this point, to protect the proof of Lemma 1.

(ii) We next deal with *formulae with variables*. Here, the method is quite general, but we consider a typical axiom, rule and meta-rule.
A1. $A \to A$.
To prove $v(A \to A) = T$, we start with $v(A_i \to A_i) = T$, where A_i is A with all of its variables replaced by constants. This will hold unilaterally, as above, for any choice of constants, through a simple piece of general (finitary) argument, in the manner of Meyer and Slaney. If we instead replace all variables bar a, say, in A_i by constants, then $v(A_i \to A_i) = T$ since $v(A_i \to A_i{}^0/a) = T$ and $v(A_i \to A_i{}^{m'}/a) = T$, yielding 'if $v(A_i \to A_i{}^m/a) = T$ then $v(A_i \to A_i{}^{m'}/a) = T$', as the rule is classical. [We keep using the same symbolism 'A_i' here.]
Note that $v(A_i \to A_i) = T$, for any choice of constants substituted for all variables bar a. We next (instead) replace all variables bar a and b in A_i by constants. Then, $v(A_i \to A_i{}^0/b) = T$ and $v(A_i \to A_i{}^{m'}/b) = T$, where the variable a still occurs in A_i, and again: if $v(A_i \to A_i{}^m/b) = T$ then $v(A_i \to A_i{}^{m'}/b) = T$, yielding $v(A_i \to A_i) = T$. The procedure continues inductively until all variables are fully re-instated, variable by variable, and hence $v(A \to A) = T$.

This method is based on a two-step deductive process to establish the property for all constant substitutions, which is taken to be an alternative form of Mathematical Induction, classically created out of NMR1. That is, the property holds for 0 and also for m', together covering all natural numbers, without the assumption of the property for m. Again, the order of variable reinstatement does not matter, as long as there is some order.
R1. $A, A \to B \Rightarrow B$.
Let $v(A) = T$ and $v(A \to B) = T$, created as above for A1. We similarly create $v(B) = T$ from the constant instances $v(B_i) = T$, obtained from the constant instances of $v(A) = T$ and $v(A \to B) = T$.
MR1. If $A, B \Rightarrow C$ then $D \vee A, D \vee B \Rightarrow D \vee C$, with the proviso.
Let $v(D \vee A) = T$ and $v(D \vee B) = T$, both created as above for A1. Let, if $v(A) = T$ and $v(B) = T$ then $v(C) = T$, each true metavaluation v being created as for A1 above. So, each of these true metavaluations apply to all their constant instances, from which we can similarly create $v(D \vee C) = T$ from its constant instances.

(iii) Lastly, we need to show that if $\vdash \sim A$ then $v^*(A) = F$. Let $\vdash \sim A$. Clearly, from (i) and (ii), $v(\sim A) = T$. Whenever A is a sentence, equivalently, $v^*(A) = F$. Let A have variables. Then $v(\sim A_i) = T$, for all constant instances $\sim A_i$ of $\sim A$. At the level of constant instances, $v^*(A_i) = F$, and the same recursive generation as used above, but for v* and F, can be used to yield v*(A) = F. □

Corollary.
For any identity s = t, for constant terms s and t, not both s = t and ∼s = t are provable in MC#.
Proof. Let s = t and ∼s = t be provable in MC#. By Lemma 2, v(s = t) = T and v*(s = t) = F. But, then, v(s = t) = F, which is a contradiction, as only one value of T and F can be taken. Thus, not both s = t and ∼s = t are provable in MC#. □

In the proof of Lemma 2, it is important that the rule, ∀x(A ∨ B) ⇒ A∨∀xB, does not preserve T for the metavaluation v, as otherwise, as pointed out in Section 2 and Section 3, this would mean that Corollary (1) to Theorem 1 below would contradict Gödel's Incompleteness Theorem, especially given the consistency of MC# (Theorem 2). The fact that the above rule does not preserve T can be seen from the need of that same rule in the meta-logic. However, as argued at the beginning of this section, this does not hold here.

Lemma 3.
(1) If v(A) = T then A is a theorem of MC#, and:
(2) If v*(A) = F then ∼A is a theorem of MC#.
Proof. We prove both (1) and (2) together by double induction on formula construction and on replacement of constants by variables, making use of some simple derived rules of MCQ$^-$. The overarching induction is on formulae and at each stage we employ an induction on the number of variables introduced by Mathematical Induction. The induction on formulae is similar to that of Meyer in [1976a] and of Slaney in [1984].
(i) The base case concerns the atoms s = t, for terms s and t. Consider the constant terms s_i and t_i. If $v(s_i = t_i) = T$ then $\vdash s_i = t_i$, and if $v^*(s_i = t_i) = F$ then $v(s_i = t_i) = F$ and $s_i = t_i$ is unprovable. Then, by Lemma 1, $\vdash \sim s_i = t_i$.
Consider terms s and t with one variable a between them, and let v(s = t) = T. Then, $v(s = t^0/_a) = T$ and if $v(s = t^m/_a) = T$ then $v(s = t^{m'}/_a) = T$. By the base case and Lemma 2, $\vdash s = t^0/_a$ and if $\vdash s = t^m/_a$ then $\vdash s = t^{m'}/_a$, and, by Mathematical Induction, ⊢ s = t, introducing the variable a. Similarly, by letting v*(s = t) = F for this s = t with the variable a, we obtain ⊢ ∼s = t, by again using the base case, Lemma 2, and Mathematical Induction.
Now consider s = t with just the variables a and b, letting v(s = t) = T. Each of $v(s = t^0/_b) = T$ and if $v(s = t^m/_b) = T$ then $v(s = t^{m'}/_b) = T$ holds, by definition of v(s = t) = T. By induction hypothesis and Lemma 2, $\vdash s = t^0/_b$ and if $\vdash s = t^m/_b$ then $\vdash s = t^{m'}/_b$, and, by Mathematical

Induction, $\vdash s = t$. Similarly, by letting $v^*(s = t) = F$, we obtain $\vdash \sim s = t$. We continue to inductively introduce the remainder of the variables in this fashion until the final $s = t$ with all its free variables is derived.
(ii) The induction steps for $\sim A$, $A \,\&\, B$, $A \vee B$ and $A \rightarrow B$, for the constant instances $\sim A_i$, $A_i \,\&\, B_i$, $A_i \vee B_i$ and $A_i \rightarrow B_i$, are straightforward, as in Meyer [1976a] and Slaney [1984]. The induction on the number of introduced variables proceeds as given for the base case of formula construction.
(iii) For the induction step for $\forall xA$, for the constant instance $v(\forall xA_i) = T$, we use the recursive generation: $v(A_i{}^{0}\!/\!_{x}) = T$, if $v(A_i{}^{m}\!/\!_{x}) = T$ then $v(A_i{}^{m'}\!/\!_{x}) = T$, as follows. By the base case and Lemma 2, $\vdash A_i{}^{0}\!/\!_{x}$ and if $\vdash A_i{}^{m}\!/\!_{x}$ then $\vdash A_i{}^{m'}\!/\!_{x}$. By Mathematical Induction and QR1, $\vdash \forall xA_i$. Similarly for $v^*(\forall xA_i) = F$, we use the recursive determination of a numerical constant m such that $v^*(A_i{}^{m}\!/\!_{x}) = F$. By induction hypothesis, $\vdash \sim A_i{}^{m}\!/\!_{x}$, for this m, and hence $\vdash \sim\forall xA_i$. The induction on the number of introduced variables can now proceed as above. □

Thus, by Lemmas 2 and 3:
Theorem 1.
The system MC# is metacomplete, i.e. $v(A) = T$ iff A is provable in MC#, and also $v^*(A) = F$ iff $\sim A$ is provable in MC#.

Corollaries.
For MC#:

(1) If $\vdash A \vee B$ then $\vdash A$ or $\vdash B$, for sentences A, B. [Priming Property]
(2) If $\vdash A \vee B$ then, for all constant instances $A_i \vee B_i$ of $A \vee B$, $\vdash A_i$ or $\vdash B_i$. [Extended Priming Property]
(3) If $\vdash \exists xA$ then $\vdash A^{m}\!/\!_{x}$, for some numerical constant m, for sentence $\exists xA$. [Satisfaction Property]
(4) If $\vdash \exists xA$ then, for all constant instances $\exists xA_i$ of $\exists xA$, $\vdash A_i{}^{m}\!/\!_{x}$, for some numerical constant m. [Extended Satisfaction Property]

Corollary (2) is a nice yardstick by which applied logics can be judged. It extends Theorem 3 of Brady and Rush [2008]: For any formula A built up from atoms of form $a = b$, using $\sim$, & and $\vee$ only, if $A \vee \sim A$ is provable then, for each of its constant instances $A' \vee \sim A'$, either A' or $\sim A'$ is provable. It ensures that the $\vee$-I rule introduces all constant disjunctions. Next we have the proof of simple consistency, which, given the nature of the methods we have been using, is obtained by finitary methods.

Theorem 2.
If $v(A) = T$ then $v^*(A) = T$. Thus, MC# is simply consistent.
Proof. By induction on formula construction.
(i) The base case concerns the atoms of form $s = t$, for terms s and t. For

constant terms s_i and t_i, $v(s_i = t_i) = v^*(s_i = t_i)$, and hence the theorem holds. For terms s and t with variables, $v(s = t) = T$ and $v^*(s = t) = F$ clearly cannot both hold.
(ii) For sentences A and B, if $v(A \& B) = T$ then, by using the induction hypothesis for A and B, $v^*(A \& B) = T$. Similarly, if $v(A \vee B) = T$ then $v^*(A \vee B) = T$, and if $v(\sim A) = T$ then $v^*(\sim A) = T$, also by induction hypothesis. For any sentence $A \rightarrow B$, if $v(A \rightarrow B) = T$ then $v^*(A \rightarrow B) = T$, since the latter always holds. For formulae A and/or B with variables, the theorem clearly applies to these cases.
(iii) For a sentence $\forall xA$, if $v(\forall xA) = T$ then, by induction hypothesis, $v^*(\forall xA)$ cannot be F. Again, for A with other variables, $v(\forall xA) = T$ and $v^*(\forall xA) = F$ cannot both hold. □

Theorem 3.
For all sentences A, built up from atoms using $\sim$, & and $\vee$ only, $v(A) = v^*(A)$. Thus, these sentences are negation-complete, as well as consistent.
Proof. By induction on formula construction. The base case is as for Theorem 2. The induction steps for $\sim$, & and $\vee$, follow by converse reasoning to that of Theorem 2. □

Theorem 4.
The rule CR1, $\sim s = t, s = t \vee B \Rightarrow B$, is an admissible rule of MC#.
Proof. Let $\vdash \sim s = t$ and $\vdash s = t \vee B$. Then, for all their respective constant instances, $\vdash \sim s_i = t_i$ and $\vdash s_i = t_i \vee B_i$, and, by Corollary (1) above, $\vdash s_i = t_i$ or $\vdash B_i$. By Theorem 2, $\vdash B_i$, and by Lemma 2, $v(B_i) = T$, for all constant instances. By the proof method of Lemma 2, $v(B) = T$, and, by Theorem 1, $\vdash B$. □

Corollary.
The DS, $\sim A, A \vee B \Rightarrow B$, is also an admissible rule of MC#.
Proof. The proof of Theorem 4 equally applies to general formulae A in place of $s = t$. □

This corollary allows us to add the full DS as a rule of the system MC#, as each of its rules just preserve true metavaluations and hence theoremhood in MC#, and thus the DS cannot cause a non-theorem to be derived. Then, the unprovability arguments of Lemma 1 will still apply, as the true metavaluations do characterize a simply consistent subtheory of the inconsistent modular arithmetic used in the proof of this lemma.

Ex Falso Quodlibet, $A, \sim A \Rightarrow B$, can be derived from the DS by the familiar Lewis argument. This rule proves useful in deriving rules in arithmetic, as it can ensure that a rule $A \Rightarrow B$ continues to hold when $\sim A$ is a theorem. Rules are always stated in general terms even though they only apply in the case when all their premises are theorems, and thus EFQ will aid us in showing that a rule continues to hold even when the negation of one of its premises is a theorem. We will use the DS and EFQ in this way in the proofs of Section 6 below.

The DS is also added to the derivations, A, B $\Rightarrow$ C and A, $B^{m}/_{x} \Rightarrow C^{m}/_{x}$ of MR1 and QMR1, respectively. When we apply MR1 to the DS, we get $D \vee \sim A, D \vee A \vee B \Rightarrow D \vee B$, a strengthened form of the DS, which would still be admissible in the manner of Theorem 4. Further, if we apply QMR1 to the DS to yield $\sim A, \exists x(A \vee B) \Rightarrow \exists xB$, then this can be transformed to $\sim A, A \vee \exists xB \Rightarrow \exists xB$, an instance of the DS and thus admissible. Note that the form $\exists x \sim A, A \vee B \Rightarrow \exists xB$ does not apply as x should be free in $A \vee B$.[6]

5. *Extending these Results to Metacomplete Logics*

For which logics do the above proofs of results go through? Clearly, since metavaluations are used to model the logic, the logics will need to be metacomplete. At the sentential level, M1- and M2-metacomplete logics, as defined in Slaney [1987], would normally suffice for Lemmas 2 and 3, and we can choose either to include or exclude the distribution axiom, $A \,\&\, (B \vee C) \rightarrow (A \,\&\, B) \vee (A \,\&\, C)$, and choose either to include the meta-rule 'if A, B $\Rightarrow$ C then $D \vee A, D \vee B \Rightarrow D \vee C$', the meta-rule 'if $A \Rightarrow B$ then $C \vee A \Rightarrow C \vee B$', or no meta-rule at all.

However, since we have changed the usual order of application, i.e. completeness before soundness, and we have proved soundness first before completeness, some of the soundness proofs that rely on completeness will not be able to be proved. These include the key axioms, $A \rightarrow .A \rightarrow B \rightarrow B$, $(A \rightarrow .B \rightarrow C) \rightarrow .B \rightarrow .A \rightarrow C$ and $A \rightarrow .B \rightarrow A$, where 'if $v(A) = T$ then $\vdash A$' or the same with B are needed, and so we cannot include these axiom-forms in our logics. Nevertheless, we can include some

[6] Nevertheless, there is a proof of $\forall x(A \vee B) \Rightarrow A \vee \forall xB$, in the special case where A is classical, i.e. the LEM, $A \vee \sim A$, and the DS, $\sim A, A \vee B \Rightarrow B$, both hold for A. Here, if A is quantifier-free and '$\rightarrow$'-free, A will satisfy the LEM. For such an A, apply the DS: $\sim A$, $A \vee B^{a}/_{x} \Rightarrow B^{a}/_{x}$. Then, by QA1 and QR1, $\sim A, \forall x(A \vee B) \Rightarrow \forall xB$, and so by MR1, $A \vee \sim A$, $A \vee \forall x(A \vee B) \Rightarrow A \vee \forall xB$. Hence, $\forall x(A \vee B) \Rightarrow A \vee \forall xB$.
Note that the derivation of $\forall x(A \vee B) \Rightarrow \exists xA \vee \forall xB$, mentioned in footnote 5, requires the replacement of A in $\forall x(A \vee B) \Rightarrow A \vee \forall xB$ by $\exists xA$. Since $\exists xA$ need not satisfy the LEM, this existential form need not follow here. Also, note that the counter-example given at the beginning of Section 4 involved this existential form and thus will rely on the non-classicality of $\exists xA$ in order to work. Moreover, it can be shown that $\exists xA \vee \sim \exists xA$ and $\forall x(A \vee B) \Rightarrow \exists xA \vee \forall xB$ are inter-derivable, given the DS for $\exists xA$ and the LEM for $A^{b}/_{x}$ (b not free in A).

weakened forms, obtained by replacing an '$\rightarrow$' by a rule '$\Rightarrow$' or by replacing its antecedent by an '$\rightarrow$'-formula. Thus, we could include the rules, $A \Rightarrow A \rightarrow B \rightarrow B$, $A \rightarrow .B \rightarrow C, B \Rightarrow A \rightarrow C$ and $A \Rightarrow B \rightarrow A$, and the axioms, $A \rightarrow B \rightarrow .A \rightarrow B \rightarrow C \rightarrow C$, $(A \rightarrow .B \rightarrow C \rightarrow D) \rightarrow .B \rightarrow C \rightarrow .A \rightarrow D$ and $A \rightarrow B \rightarrow .C \rightarrow .A \rightarrow B$. We could also include the weakened axiom $A \rightarrow .A \rightarrow A$.

However, there is still one more consideration. All these logics, bar those with the rule, $A \Rightarrow B \rightarrow A$, or with the axiom, $A \rightarrow B \rightarrow .C \rightarrow .A \rightarrow B$, are contained in RM3 and do not include the DS, and so suffice for Lemma 1. Again, the DS can be admissibly added to the arithmetic, once it is shown to be consistent. The quantificational axioms and rules are just those of MCQ$^-$.

We first present just the sentential axioms and rules that are not included in MC, and then point out which combinations yield M1-logics and which yield M2-logics, all of which will suffice as the sentential basis for the results of this paper.

Additional Axioms.

11. $A \,\&\, (B \vee C) \rightarrow (A \,\&\, B) \vee (A \,\&\, C)$.
12. $A \rightarrow B \rightarrow .C \rightarrow A \rightarrow .C \rightarrow B$.
13. $A \rightarrow B \rightarrow .B \rightarrow C \rightarrow .A \rightarrow C$.
14. $(A \rightarrow B \vee C) \,\&\, (A \,\&\, B \rightarrow C) \rightarrow .A \rightarrow C$.
15. $A \rightarrow B \rightarrow .A \rightarrow B \rightarrow C \rightarrow C$.
16. $(A \rightarrow .B \rightarrow C \rightarrow D) \rightarrow .B \rightarrow C \rightarrow .A \rightarrow D$.
17. $A \rightarrow .A \rightarrow A$.

Additional Rules.

5. $A \Rightarrow A \rightarrow B \rightarrow B$.
6. $A \rightarrow .B \rightarrow C, B \Rightarrow A \rightarrow C$.

The basic system B of Routley and Meyer in [RLR1] is MC without A10 and with A9 replaced by the rule:

4. $A \rightarrow \sim B \Rightarrow B \rightarrow \sim A$.

A suitable M1-logic is obtained by adding to the system B, zero or more of the axioms A9–14,17. A suitable M2-logic can be obtained by adding rule R5, and then adding to B, zero or more of the axioms A9,11–13,15–17 and the rule R6. For these M2-logics, in the assignment for the metavaluation v*, replace '$v^*(A \rightarrow B) = T$' by '$v^*(A \rightarrow B) = T$ iff, if $v(A) = T$ then $v^*(B) = T$'. The assignment $v^*(A \rightarrow B) = T$ applies to the M1-logics only. We also note that for M2-logics the induction step for formulae $A \rightarrow B$ still applies in the proof of Theorem 2. Since $v(A \rightarrow B) = T$ and $v(A) = T$, $v(B) = T$ follows, and also $v^*(B) = T$ by the induction hypothesis.

6. *Comparison with Classical Peano Arithmetic*

Lastly, we will examine the development of arithmetic in Mendelson's [1964][7] to compare what our system can and cannot do in relation to the classical Peano arithmetic. The inclusion of existential distribution, $A \mathbin{\&} \exists xB \Rightarrow \exists x(A \mathbin{\&} B)$, will make quite a difference to the formal development of arithmetic as $s < t$ and $t \mid s$ (t is a divisor of s) are defined in terms of an existential quantifier, as can $s \leq t$ and $\sim s = t$. Thus, existential quantifiers can be moved around in the standard classical way, even if universal quantifiers are more restricted in their movements.

One advantage of arithmetic is that all negative classical statements can be rendered positive. Just push negations through the formula until they attach to identity statements of form, $\sim s = t$, and replace them by $\exists x(s + x' = t) \vee \exists x(t + x' = s)$. Then, it would leave the only restriction on classical Peano arithmetic as the restriction on distribution of universal quantifiers over disjunction.

We now proceed to examine Chapter 3 'Formal Number Theory' of Mendelson [1964] in some detail. We start with the axioms of his system S of arithmetic and then explore each of his propositions from 3.1 to 3.11. We should first note that the inferential propositions amongst these are stated in '$\supset$'-form. Most often, these will be proved in rule-form and converted to '$\supset$'-form using the LEM and MR1, as follows:
Given $A \Rightarrow B$, $\sim A \vee A \Rightarrow \sim A \vee B$, by MR1, and hence $A \supset B$, due to the LEM.
Occasionally, we will prove $A \supset B$ in the form $\sim A \vee B$, in which case the rule-form easily follows by the DS. Recall that the LEM holds for all formulae of form $s = t$ and for all formulae built from these by using the connectives $\sim$, $\&$, $\vee$. Moreover, we will prove the LEM for $s < t$ and $t \mid s$ along the way, enhancing the range of available '$\supset$'-forms of inferential propositions.

Mendelson lists the axioms of S, as follows:
[We convert the symbolism, as appropriate.]
(S1) $a = b \supset (a = c \supset b = c)$.
(S2) $a = b \supset a' = b'$.
(S3) $0 \neq a'$.
(S4) $a' = b' \supset a = b$.
(S5) $a + 0 = a$.
(S6) $a + b' = (a + b)'$.
(S7) $a \times 0 = 0$.

[7] More recent editions such as the 5th, dated 2010, are little different in this chapter.

(S8) $a \times b' = a \times b + a$.
(S9) $A(0) \supset (\forall x(A(x) \supset A(x')) \supset \forall xA(x))$, for any $A(x)$.

(S3), (S5), (S6), (S7) and (S8) are directly provable in our system MC#, whilst (S1), (S2) and (S4) are provable from their rule-forms with help from the LEM. Because of the generality of the formula A in (S9), the LEM will not be available here and the meta-rule-form NMR1: if $A(m) \Rightarrow A(m')$ then $A(0) \Rightarrow A(t)$, will be left as it stands, replacing (S9). In any case, this form is stronger than (S9) in that the rule $A(m) \Rightarrow A(m')$ is easier to show than the '$\supset$'-form. And, Mathematical Induction is usually applied in rule-form anyway, in deducing a universal conclusion from the Base Case and the Induction Step, which was established by deducing $A(x')$ from $A(x)$.

Mendelson's (S1′)–(S8′) are simply (S1)–(S8) with terms substituted for variables, easily done via QR1 and QA1 in MC#. His Proposition 3.2(a)–(o) is as follows:

(a) $t = t$.
(b) $t = r \supset r = t$.
(c) $t = r \supset (r = s \supset t = s)$.
(d) $r = t \supset (s = t \supset r = s)$.
(e) $t = r \supset t + s = r + s$.
(f) $t = 0 + t$.
(g) $t' + r = (t + r)'$.
(h) $t + r = r + t$.
(i) $t = r \supset s + t = s + r$.
(j) $(t + r) + s = t + (r + s)$.
(k) $t = r \supset t \times s = r \times s$.
(l) $0 \times t = 0$.
(m) $t' \times r = t \times r + r$.
(n) $t \times r = r \times t$.
(o) $t = r \supset s \times t = s \times r$.

The proofs in MC# of (a)–(o) can be carried out in rule-form in the style of Mendelson, with appropriate '$\supset$'-forms introduced by MR1 and the LEM, instead of using the Deduction Theorem. However, we can use the identity axioms and rule to expedite the proofs of (a)–(e),(i),(k),(o). Note that if mathematical induction is used to prove (e) or (k), induction on 's' can be carried out in the respective consequent formulae, under assumption, with the '$\supset$' added afterwards.

Mendelson's Corollary 3.3 essentially consists of the identity axiom IA1 and the identity rule IR1. Proposition 3.4(a)–(d) is as follows:

(a) $t \times (r + s) = (t \times r) + (t \times s)$.
(b) $(r + s) \times t = (r \times t) + (s \times t)$.
(c) $(t \times r) \times s = t \times (r \times s)$.
(d) $t + s = r + s \supset t = r$.

The proofs of (a)–(c) are as in Mendelson, but the mathematical induction proof of (d) is applied to ‘s’ in the whole ‘$\supset$’-formula. And, as given above, all the previous ‘$\supset$’-forms have been introduced by use of the LEM and MR1.

The numerals $\overline{1}$, $\overline{2}$, ... are defined for Proposition 3.5 as $0'$, $1'$, ... We set out 3.5(a)–(j), as follows:
(a) $t + \overline{1} = t'$.
(b) $t \times \overline{1} = t$.
(c) $t \times \overline{2} = t + t$. (Etc. for 3, 4, ...)
(d) $t + s = 0 \supset t = 0 \,\&\, s = 0$.
(e) $t \neq 0 \supset (s \times t = 0 \supset s = 0)$.
(f) $t + s = \overline{1} \supset (t = 0 \,\&\, s = \overline{1}) \vee (t = \overline{1} \,\&\, s = 0)$.
(g) $t \times s = \overline{1} \supset (t = \overline{1} \,\&\, s = \overline{1})$.
(h) $t \neq 0 \supset \exists y(t = y')$.
(i) $s \neq 0 \supset (t \times s = r \times s \supset t = r)$.
(j) $t \neq 0 \supset (t \neq \overline{1} \supset \exists y(t = y''))$.
Again, we follow Mendelson’s style of proof. For (d), the induction is applied to ‘s’ in the whole formula, and, for the induction step, we use the theorem, $\sim(t + m)' = 0$, applying $\sim A \Rightarrow \sim A \vee B$ to establish $t + m' = 0 \supset t = 0 \,\&\, m' = 0$, without using the induction hypothesis. For (e), induction is on ‘s’ in the consequent ‘$\supset$’-formula, while the main ‘$\supset$’ is added later. Again, the induction hypothesis is not used. Also for (f), induction is on ‘s’ and the induction hypothesis is not used. For (g), induction is also on ‘s’, and, for (h), induction is on ‘t’ with no use of induction hypothesis. For (i), induction is on ‘r’ and follows the proof as laid out in Mendelson, the introduction of the universal quantifier over x in the ‘t’-position being used to allow a different substitution from ‘t’ to be made. For (j), we use induction on ‘t’.

We set out only Proposition 3.6(a), as (b) and (c) are outside the scope of the arithmetic derivations we are exploring in this paper.
(a) For any natural numbers m and n, if $m \neq n$ then $\overline{m} \neq \overline{n}$, and also, $\overline{m + n} = \overline{m} + \overline{n}$ and $\overline{m \times n} = \overline{m} \times \overline{n}$.
The proofs follow Mendelson, the latter two by induction on n in the metalanguage.

We introduce the following definitions for use in Proposition 3.7.
$t < s$ for $\exists w(w \neq 0 \,\&\, t + w = s)$.
$t \leq s$ for $t < s \vee t = s$.
$t > s$ for $s < t$.
$t \geq s$ for $s \leq t$.
$t \not< s$ for $\sim(t < s)$.
Proposition 3.7(a)–(z) is as follows:
(a) $t \not< t$.
(b) $t < s \supset (s < r \supset t < r)$.

(c) $t < s \supset s \nless t$.
(d) $t < s \equiv t + r < s + r$.
(e) $t \leq t$.
(f) $t \leq s \supset (s \leq r \supset t \leq r)$.
(g) $t \leq s \equiv (t + r \leq s + r)$.
(h) $t \leq s \supset (s < r \supset t < r)$.
(i) $0 \leq t$.
(j) $0 < t'$.
(k) $t < r \equiv t' \leq r$.
(l) $t \leq r \equiv t < r'$.
(m) $t < t'$.
(n) $(0 < \overline{1})$ & $(\overline{1} < \overline{2})$ & $(\overline{2} < \overline{3})$ & $\ldots$
(o) $t \neq r \supset (t < r \vee r < t)$.
(o′) $t = r \vee t < r \vee r < t$.
(p) $t \leq r \vee r \leq t$.
(q) $t + r \geq t$.
(r) $r \neq 0 \supset t + r > t$.
(s) $r \neq 0 \supset t \times r \geq t$.
(t) $r \neq 0 \equiv r > 0$.
(u) $r > 0 \supset (t > 0 \supset r \times t > 0)$.
(v) $r \neq 0 \supset (t > \overline{1} \supset t \times r > r)$.
(w) $r \neq 0 \supset (t < s \equiv t \times r < s \times r)$.
(x) $r \neq 0 \supset (t \leq s \equiv t \times r \leq s \times r)$.
(y) $t \nless 0$.
(z) $t \leq r$ & $r \leq t \supset t = r$.

Now, the LEM does not initially hold for any of the above definitions which are based on that for $t < s$. This is because the existential quantifier in the definitions goes beyond the scope of our previous proof of the LEM. So, each of the 3.7 '$\supset$'-forms are first proved in rule-form, i.e. with '$\Rightarrow$' replacing each '$\supset$' occurring within, and hence with '$\Leftrightarrow$' replacing '$\equiv$'. Then, the LEM will be proved for $t < s$, using some of these rule-forms, which will then enable each of the above '$\supset$'-forms to be proved. Of course, those '$\supset$'-forms with '$t \neq r$' or '$r \neq 0$' as an antecedent can be obtained without this result.

The proofs of the rule-forms of 3.7(a)–(z) will still follow Mendelson's style of proof. For 3.7(b) and subsequently (and also previously for 3.5(i)), Mendelson's use of his Rule C is replaced by our use of the meta-rule QMR1. After an existential statement $\exists wA(w)$ is reached, we introduce $A^{a}/_{w}$ as an assumption, with 'a' new to the proof. Then, we prove a conclusion, usually without the 'a', and apply QMR1. For 3.7(b), we note the role of the existential distribution rule, A & $\exists wB \Rightarrow \exists w(A$ & $B)$ in carrying this out. In 3.7(f) and subsequently, we use the rule version of disjunction elimination:

if $A \Rightarrow C$ and $B \Rightarrow C$ then $A \vee B \Rightarrow C$. [This is easily proved using MR1.] Note that 3.7(m) is best proved before (k), (o′) before (o), and (y) before (w).

Since 3.7(o′) is akin to the LEM for $t < r$, we will set out the proof. We use mathematical induction on 'r' in the formula $\forall x(x = r \vee x < r \vee r < x)$. For the base case, $0 \leq a$, by (i), and hence $a = 0 \vee a < 0 \vee 0 < a$ and $\forall x(x = 0 \vee x < 0 \vee 0 < x)$. For the induction step, let $\forall x(x = m \vee x < m \vee m < x)$. Now, by 3.5(h), $a = 0 \vee \exists y(a = y')$. By (j), $a = 0 \Rightarrow a = m' \vee a < m' \vee m' < a$. Let $a = b'$. Then, by induction hypothesis, $b = m \vee b < m \vee m < b$, and then $b' = m' \vee b' < m' \vee m' < b'$, by using (k) and (l). Substituting a/b', $a = m' \vee a < m' \vee m' < a$, and, by QMR1, $\exists y(a = y') \Rightarrow a = m' \vee a < m' \vee m' < a$. By MR1, $a = 0 \vee \exists y(a = y') \Rightarrow a = m' \vee a < m' \vee m' < a$. Finally, $a = m' \vee a < m' \vee m' < a$ and $\forall x(x = m' \vee x < m' \vee m' < x)$. Thus, $\forall x(x = r \vee x < r \vee r < x)$ and $t = r \vee t < r \vee r < t$.

Further, 3.7(o′) can be used to prove the LEM for $s < t$, as follows. By 3.7(o′), $s = t \vee s < t \vee t < s$. By (c), $t < s \Rightarrow s \not< t$. By (a), $s = t \Rightarrow s \not< t$. By MR1, $s = t \vee t < s \Rightarrow s \not< t$. So, by MR1, $s < t \vee s \not< t$. The LEM then extends to the other definitions: $t \leq s$, $t > s$, $t \geq s$, $t \not< s$. Thus, each of the rule-forms in 3.7 can be replaced by their '$\supset$'-forms.

We conclude 3.7 with a proof of the converse form of 3.7(w): $r \neq 0 \supset (t \times r < s \times r \supset t < s)$. We use mathematical induction on 's' in the formula $\forall x(r \neq 0 \supset (x \times r < s \times r \supset x < s))$. For the base case, $\sim a \times r < 0 \times r$, by (y), and hence $\forall x(r \neq 0 \supset (x \times r < 0 \times r \supset x < 0))$. For the induction step, let $\forall x(r \neq 0 \supset (x \times r < m \times r \supset x < m))$. Let $r \neq 0$. Now, $a = 0 \vee a \neq 0$. By (j), $a = 0 \Rightarrow a < m'$ and hence $a = 0 \Rightarrow a \times r < m' \times r \supset a < m'$. By 3.5(h), $a \neq 0 \Rightarrow \exists y(a = y')$. Let $a = b'$. By induction hypothesis, $b \times r < m \times r \supset b < m$ and hence, by (k) and (l), $\forall w(w = 0 \vee b \times r + w \neq m \times r) \vee b' < m'$. Then, by 3.4(d), $\forall w(w = 0 \vee b' \times r + w \neq m' \times r) \vee b' < m'$. Hence, $b' \times r < m' \times r \supset b' < m'$ and, substituting a/b', $a = b' \Rightarrow a \times r < m' \times r \supset a < m'$. So, by QMR1, $\exists y(a = y') \Rightarrow a \times r < m' \times r \supset a < m'$ and $a \neq 0 \Rightarrow a \times r < m' \times r \supset a < m'$. Since $a = 0 \vee a \neq 0$, $a \times r < m' \times r \supset a < m'$, by MR1. So, $r \neq 0 \supset (a \times r < m' \times r \supset a < m')$, by LEM, and $\forall x(r \neq 0 \supset (x \times r < m' \times r \supset x < m'))$.

Proposition 3.8(a)–(c) is as follows:

(a) $t = 0 \vee \ldots \vee t = \overline{k} \equiv t \leq \overline{k}$, for any natural number k.

(a′) $A(0)\ \&\ \ldots\ \&\ A(\overline{k}) \equiv \forall x(x \leq \overline{k} \supset A(x))$, for any k and formula A.

(b) $t = 0 \vee \ldots \vee t = \overline{k-1} \equiv t < \overline{k}$, for any $k > 0$.

(b′) $A(0)\ \&\ \ldots\ \&\ A(\overline{k-1}) \equiv \forall x(x < \overline{k} \supset A(x))$, for any $k > 0$ and formula A.

(c) $(\forall x(x < y \supset A(x))\ \&\ \forall x(x \geq y \supset B(x))) \supset \forall x(A(x) \vee B(x))$.

3.8(a) can be proved in its '$\equiv$'-form by induction on k in the metalanguage, as set out in Mendelson, using the LEM. (a′) in its rule-form, $A(0)\ \&\ \ldots\ \&\ A(\overline{k}) \Leftrightarrow \forall x(x \leq \overline{k} \supset A(x))$, can be proved using (a). This is not extended to its '≡'- form above, since we do not have the LEM in the generality required for A(x). (b) follows as for (a) and (b′) is proved in the form, $A(0)\ \&\ \ldots\ \&\ A(\overline{k-1}) \Leftrightarrow \forall x(x < \overline{k} \supset A(x))$, and again this is not extended to its '≡'-form above. (c) also follows in its rule-form, $(\forall x(x < y \supset A(x))\ \&\ \forall x(x \geq y \supset B(x))) \Rightarrow \forall x(A(x) \vee B(x))$, and is not extended to its above '⊃'-form.

Proposition 3.9(a)–(b), with Exercise:
(a) (Complete Induction) $\forall x(\forall z(z < x \supset A(z)) \supset A(x)) \supset \forall xA(x)$.
(b) (Least Number Principle) $A(x) \supset \exists y(A(y)\ \&\ \forall z(z < y \supset \sim A(z)))$.
(Ex.) (Method of Infinite Descent): $\forall x(A(x) \supset \exists y(y < x\ \&\ A(y))) \supset \forall x\sim A(x)$.
3.9(a) can be proved in the meta-rule formulation:
$(\forall z(z < t \supset A(z) \Rightarrow A(t)) \Rightarrow \forall xA(x)$.
The proof follows that of Mendelson, with B(s) as $\forall z(z \leq s \supset A(z))$, by mathematical induction on 's' in B(s).
However, 3.9(b) is provable in the contraposed rule-form:
$\sim\exists y(A(y)\ \&\ \forall z(z < y \supset \sim A(z))) \Rightarrow \forall y\sim A(y)$.
The proof follows Mendelson, using (a), but the rule does not contrapose due to the lack of LEM for the premise. This is mainly because of the generality of the formula A, but there could also be a problem with the two quantifiers. There is a reasonable prospect, for some specific A such that the LEM holds for it, of obtaining the least number satisfying A and thus being able to derive the least-number principle in rule-form and hence in '⊃'- form.
(Ex.) is provable in rule-form: $\forall x(A(x) \supset \exists y(y < x\ \&\ A(y))) \Rightarrow \forall x\sim A(x)$, using 3.9(a).

We then introduce the notion of divisibility.
$t \mid s$ for $\exists z(s = t \times z)$. (t divides s, or s is divisible by t.)
Proposition 3.10(a)–(h), with Exercises 1 and 2:
(a) $t \mid t$.
(b) $\overline{1} \mid t$.
(c) $t \mid 0$.
(d) $t \mid s\ \&\ s \mid r \supset t \mid r$.
(e) $s \neq 0\ \&\ t \mid s \supset t \leq s$.
(f) $t \mid s\ \&\ s \mid t \supset s = t$.
(g) $t \mid s \supset t \mid (r \times s)$.
(h) $t \mid s\ \&\ t \mid r \supset t \mid (s + r)$.
(Ex.1) $t \mid \overline{1} \supset t = \overline{1}$.
(Ex.2) $t \mid s\ \&\ t \mid s' \supset t = \overline{1}$.
We prove (a)–(c) and the rule-forms of (d)–(h). In the proof of the rule-form of (f), we consider the three cases, $s \neq 0$, $t \neq 0$ and $s = t = 0$. Exercise 1 is provable in '⊃'-form starting with $t \neq \overline{1} \vee t = \overline{1}$, whereas Exercise 2 is proved in rule-form, as follows. Let $s = t \times a$ and $s' = t \times b$, for some a and

b, and so $t \times b = t \times a + \overline{1}$. By letting $b < a$, prove $\overline{1} = 0$, and similarly, by letting $a = b$, prove $\overline{1} = 0$. Hence, $a < b$ and by letting $a + d = b$, $t \times d = \overline{1}$ and $t = \overline{1}$.

Proposition 3.11 gives the existence of a unique quotient and remainder upon division:

$y \neq 0 \supset \exists!u\exists!v(x = y \times u + v \,\&\, v < y)$.

The proof follows Mendelson.

Proposition 3.11 plays an important role in the proof of the LEM for $t \mid s$, which is presented as follows:

$s = 0 \Rightarrow 0 \mid s$, by 3.10(c). $s \neq 0 \Rightarrow \forall w(s \neq 0 \text{ x } w) \Rightarrow \sim 0 \mid s$. Hence, $t = 0 \Rightarrow t \mid s \vee \sim t \mid s$.

Let $t \neq 0$. Then, by 3.11, $\exists!u\exists!v(s = t \times u + v \,\&\, v < t)$. So, let $s = t \times a + b \,\&\, b < t$ and $s = t \times c + d \,\&\, d < t \Rightarrow a = c \,\&\, b = d$. $b = 0 \Rightarrow s = t \times a \Rightarrow t \mid s$. So, it remains to consider $b \neq 0$. Let $s = t \times c \,\&\, 0 < t$. By uniqueness, $a = c$ and $b = 0$. So, $s = t \times c \Rightarrow b = 0$, $s = t \times c \supset b = 0$, $b \neq 0 \supset s \neq t \times c$, and $b \neq 0 \Rightarrow \forall z(s \neq t \times z) \Rightarrow \sim t \mid s$. Since $b = 0 \vee b \neq 0$, $t \mid s \vee \sim t \mid s$. So, $t \neq 0 \Rightarrow \exists!u\exists!v(s = t \times u + v \,\&\, v < t) \Rightarrow t \mid s \vee \sim t \mid s$, and, since $t = 0 \vee t \neq 0$, $t \mid s \vee \sim t \mid s$ follows.

As suggested by Mendelson, we can recursively introduce the functions a^b and a!. The recursive definition for a^b is:

$a^0 = \overline{1}$.

$a^{b'} = a^b \times a$.

From this, we can easily prove:

$a^{b+c} = a^b \times a^c$, by induction on b, and $(a^b)^c = a^{b \times c}$, by induction on c.

The recursive definition of a! is:

$0! = \overline{1}$.

$a'! = a' \times a!$

These just expand the range of terms and thus the LEM is unaffected.

In general, mathematical induction enabled us to prove the properties concerning identity of terms, due to the use of the LEM to convert rules to their corresponding '$\supset$'-formulae. The properties concerning the ordered relation '$<$' and the divisibility relation '$|$' relied upon the removal of the existential quantifier of their definitions and the use of previous identity properties.

In conclusion, we appear to be able to prove the expected properties for existentially defined concepts. The problems occur with the generality of an arbitrary formula A, where the LEM cannot be assumed, or with nonconstructive methods such as occurs with the Least Number Principle with this generality. On this latter point, I suspect that the Gödel representation of provability would not satisfy the LEM, due to the probable lack of recursive procedure for determining the set of all non-theorems. Also, leading up to this, the failure to prove the Least Number Principle makes it hard to see how general recursion can be formulated, the lack of which would restrict us to primitive recursion. And, our use of recursion in determining the metavaluations of quantified formulae and those with free variables reflects primitive recursion rather than general recursion. Further, the logic MC, together with all M1- and M2-metacomplete logics of Section 5, are constructive logics in that their positive fragments are contained in intuitionist logic and their negated formulae still need to be established, but in a corresponding mirror-image way to that of unnegated formulae.[8] This contrasts with the falsity of classical logic, which acts as a fallback when truth does not apply. The clarification of all this will have to await another paper, which details the Gödel argument.

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