



YABLO’S PARADOX AND THE OMITTING TYPES  
 THEOREM FOR PROPOSITIONAL LANGUAGES

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We start by recapitulating Yablo’s paradox from [1].

We have infinitely many assertions  $\{p_i : i \in \mathbb{N}\}$  and each  $p_i$  is equivalent to the assertion that all subsequent  $p_j$  are false. A contradiction follows.

There is a wealth of literature on this delightful puzzle, and I have been guilty of a minor contribution to it myself. This literature places Yablo’s paradox in the *semantical* column of Ramsey’s division of the paradoxes into *semantical* versus *logical* paradoxes. However — as I hope to show below — there is merit to be gained by regarding it as a purely logical puzzle.

*Yablo’s Paradox in Propositional Logic*

If we are to treat Yablo’s paradox as a purely logical puzzle we should try to capture it entirely within a first-order language with no special predicates. In fact we can even make progress while using nothing more than a *propositional* language; the obvious language  $\mathcal{L}$  to use has infinitely many propositional letters  $\{p_i : i \in \mathbb{N}\}$ . Next we want a propositional theory with axioms

$$p_i \leftrightarrow \bigwedge_{j>i} \neg p_j \tag{1}$$

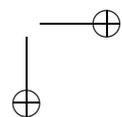
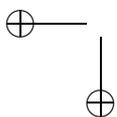
for each  $i \in \mathbb{N}$ ,

...but of course we cannot do this in a finitary language. However, one thing we can do in a finitary language is capture the left-to-right direction of these biconditionals, and we do that with the simple scheme

$$p_i \rightarrow \neg p_j \tag{2}$$

for all  $i < j \in \mathbb{N}$ .

It can be seen that this is equivalent to the even simpler scheme



$$\neg p_i \vee \neg p_j \tag{3}$$

for all  $i \neq j \in \mathbb{N}$ .

Let us call this theory  $Y$ .  $Y$  says that at most one  $p_i$  can be true.

It is the right-to-left direction of the biconditionals that gives us trouble . . .

$$\left(\bigwedge_{j>i} \neg p_j\right) \rightarrow p_i \tag{4}$$

for each  $i \in \mathbb{N}$ .

For each  $i$  the right-to-left direction of the  $i$ th biconditional (4) asserts that at least one of the formulæ in the set  $\Sigma(i)$  is false:

$$\{\neg p_j : j \geq i\} \tag{(\Sigma(i))}$$

$\Sigma(i)$  is an example of what model theorists call a *0-type*, a *type* being nothing more than a set of formulæ<sup>1</sup>. The ‘0’ means that the formulæ in the type have no free variables. Our desire that at least one thing in a type should be false is — in the terminology of model theory — a desire to *omit* that type. What we need is a theorem that tells us that a theory can have models that omit a specified type. There is such a theorem, and it is known as the *Omitting Types Theorem*. We say a theory  $T$  in a language  $\mathcal{L}$  *locally omits* a type  $\Sigma$  if, whenever  $\phi \in \mathcal{L}$  is a formula such that  $T$  proves  $\phi \rightarrow \sigma$  for every  $\sigma \in \Sigma$ , then  $T \vdash \neg\phi$ . The omitting types theorem for propositional languages now says:

*Theorem 1: Let  $T$  be a consistent theory in a propositional language  $\mathcal{L}$ . If  $T$  locally omits a type  $\Sigma$  then there is an  $\mathcal{L}$ -valuation  $v$  that satisfies every theorem of  $T$  but falsifies at least one  $\sigma$  in  $\Sigma$ .*

We say in these circumstances that  $v$  *omits*  $\Sigma$ .

However, what we need here is the slightly stronger:

*Theorem 2: (Extended Omitting Types Theorem)*

*Let  $T$  be a consistent theory in a propositional language  $\mathcal{L}$ . If  $T$  locally omits each type  $\Sigma$  in a countable class  $\mathfrak{S}$  of types then there is an  $\mathcal{L}$ -valuation that satisfies every theorem of  $T$  but, for each  $\Sigma \in \mathfrak{S}$ , falsifies at least one  $\sigma$  in  $\Sigma$ .*

<sup>1</sup> A countably infinite set unless otherwise specified.

I will omit a proof of this result, since it is standard in the model-theoretic literature.

In asserting the right-to-left directions (4) of the biconditionals we are restricting ourselves to  $\mathcal{L}$ -valuations that omit all the types  $\Sigma(i)$ . There are countably many of these types so it would be natural to reach for the extended omitting types theorem, theorem 2. Now if we are to exploit theorem 2 we want our theory  $Y$  to locally omit each  $\Sigma(i)$ . But it doesn't. The formula  $p_0$ , in conjunction with the axioms of  $Y$ , implies  $\neg p_i$  for every  $i > 0$  and thereby implies everything in  $\Sigma(1)$ . If  $Y$  were to locally omit  $\Sigma(1)$  as we desire then we would have to have  $Y \vdash \neg p_0$ . But  $Y$  clearly does not prove  $\neg p_0$ . If we were to add  $\neg p_0$  as part of a project of adding axioms to  $Y$  to obtain a theory that did omit  $\Sigma(1)$  we would find by the same token that we would have to add  $\neg p_i$  for all other  $i \in \mathbb{N}$  as well, and then we end up realising all the  $\Sigma(i)$ .

Thus  $Y$  does not locally omit even one of the  $\Sigma(i)$ , let alone all of them. So we cannot invoke theorem 2. However, for each  $i$  the valuation that makes  $p_i$  true and everything else false satisfies  $Y$  all right, and it omits all  $\Sigma(j)$  for all  $j < i$ . This illustrates how a theory  $T$  can sometimes have a model that omits a type  $\Sigma$  even though  $T$  does not locally omit  $\Sigma$ .

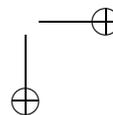
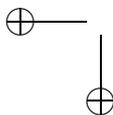
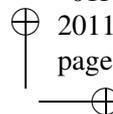
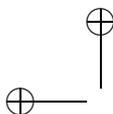
Very well: for each  $i$  there is an  $\mathcal{L}$ -valuation that satisfies  $Y$  and omits  $\Sigma(j)$  for all  $j < i$ . Can we find a  $\mathcal{L}$ -valuation that satisfies  $Y$  and omits all the  $\Sigma(i)$ ? No! Such a valuation would satisfy all the right-to-left directions of the biconditionals in (1), namely the conditionals in (4) and thereby manifest Yablo's paradox!

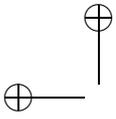
Conclusion

Yablo's paradox provides us with an illustration of a setting where there is a theory  $Y$  and an infinite family  $\{\Sigma(i) : i \in \mathbb{N}\}$  of types where, although  $Y$  does not locally omit any of the  $\Sigma(i)$ , it nevertheless has valuations that omit any finite set of them. Further, it has no valuation that omits them all. That last fact illustrates how the condition in theorem 2 — namely that  $T$  locally omit every  $\Sigma \in \mathfrak{S}$  — really is necessary, so the extended omitting types theorem for propositional logic really is best possible.

For  $T$  to have a model omitting all the  $\Sigma_i$  it is not sufficient for it to have models omitting any given finite family of them; we really do need the stronger condition that  $T$  should locally omit every finite subset of  $\Sigma_i$ .

It illustrates that for  $T$  to have a model that omits a type  $\Sigma$  is sufficient but not necessary for  $T$  to locally omit  $\Sigma$ .





This is pædagogically quite instructive!

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#### REFERENCES

- [1] Steve Yablo, “Paradox without self-reference”. *Analysis* 53.4 (1993) pp. 251–52.

