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LOGIC IN WHITEHEAD'S UNIVERSAL ALGEBRA

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Abstract
In his *Treatise on Universal Algebra*, 1898, A. N. Whitehead intended to investigate all systems of symbolic reasoning related to ordinary algebra on the basis of the algebras of Grassmann and Hamilton, and on the basis of Boole's Symbolic Logic. We consider Whitehead's version of the algebra of symbolic logic presented there. Later on, in his last contribution to logic, he came back to questions and problems related to *Principia Mathematica*. This was the occasion for restating some of his positions on logic.

1. *Introduction*

Whitehead's *Treatise on Universal Algebra* [38] is generally known by its title only. But its importance and the role it played at the end of 19th century should not be underestimated. It was published at a time a unifying survey of the various systems of algebra that had started to develop in the works of G. Boole, W. R. Hamilton, and H. Grassmann was strongly needed.
These authors had generated what Whitehead calls 'extraordinary algebras', or, as they were also known, 'multiple algebras', algebras that deal with multiple quantity and do not respect all the laws of ordinary algebra. Hamilton's theory of Quaternions and Grassmann's theory of Extension were popular due to the hot debate surrounding the calculus of vector of which both theories could claim to constitute the basis. Their algebraic offsprings had already been considered in Benjamin Peirce's *Linear Associative Algebra*, an original investigation and presentation of multiple algebras.
Here, we only consider Boole's theory of the *Laws of Thought* and its corresponding algebraic theory as it is presented in the second Book of Whitehead's *Treatise*.

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"*The Algebra of Symbolic logic, viewed as a distinct algebra, is due to Boole* [who] *does not seem in this work to fully realise that he had discovered a system of symbols distinct from that of ordinary algebra.*" [38] (115) Boole's theory was later investigated by Venn, Jevons, and C. S. Peirce. "*These investigations of Peirce form the most important contribution to the subject of Symbolic Logic since Boole's work.*", (115) in particular, Peirce's *Logic of Relatives*. Nevertheless these are not taken into account in the *Treatise*.

Following the remarks of C. I. Lewis [21] related to the difficulty of precisely defining the notions and domains of logistic, symbolic logic, algebra of logic, algebraic logic and calculus, we consider as subject matter of logic the principles of general reasoning represented in a symbolic language and developed deductively. Since this symbolic representation can be treated according to the laws of algebra, such treatment of logic, as in Boole's *Investigation*, has been called "algebra of symbolic logic" or "Boolean algebra". Until this later notion acquired its current specific signification with Tarski in the 1930's, this sort of treatment could also be called "algebraic logic". "*The name Boolean algebra (or Boolean "algebras") for the calculus originated by Boole, extended by Schröder, and perfected by Whitehead seems to have been first suggered by Sheffer in 1913.*" [15] (278) Indeed, although C. S. Peirce referred to the algebra of symbolic logic as 'Boolian algebra', it is only with Sheffer that Boole's and related systems will be called Boolean algebra(s). Still, Boole's algebra is not exactly what we understand today under the name Boolean algebra which mainly concerns the underlying algebraic structure rather than the Boolean logic. [13] We will see that the underlying algebraic structure of Boole's system had already been clearly observed by Whitehead soon after publication of his *Treatise*.

We start with Whitehead's last contribution to logic, an article in which he reasserts and makes clear some of his positions on logic. [44] In particular, his formalist and structuralist positions which are deeply rooted in his *Treatise* and in the algebra of late 19th century, positions which are also in agreement with the axiomatic investigations of Boolean algebras, and with the developments of algebraic logic and of universal algebra of the time.

2. *The business of logic: Propositions and Classes*

According to G. Boole, "*the business of Logic is with the relations of classes, and with the modes in which the mind contemplates those relations.*" [7] (184)

These 'relations of classes' were primarily discussed early last century in the context of the theory of "aggregates" or sets, and in relation to classes and to numbers defined extensionally and intensionally by Frege and by Russell;

for example, in the later's *Principles of Mathematics* [33] and in Whitehead's and Russell's *Principia Mathematica*. [43]

To some extent, two perspectives were then in opposition: that of mathematicians and that of philosophers. In *Principia*, this appears in the opposition of the calculus of propositions and the calculus of classes inherited from Boolean algebra and Boolean logic.

Much later, in his last substantial contribution to logic, Whitehead, the mathematician, will come back to these questions in order to clarify his own positions and to answer a few critiques addressed to *Principia*. He thus defined a class as follows: "*A class is a composite entity arising from the togetherness of many things in symmetrical connection with each other* [...] *A class is a class if its members are "together", it arises from that composition and any member is as good as any other with respect to membership.*" [44] (282)

There are various ways of 'being together', that is, various ways of building classes, and this makes '*togetherness*' an ambiguous notion. The extensionalist Whitehead notes that the particular mode of '*togetherness*' is already an intension that infects the composite entity. Therefore, in mathematics or in logic, logical '*togetherness*' must be logically defined in order to obtain a purely extensional composite entity.

There are also various logical ways of defining classes. Here is how Whitehead builds his extensional classes: "$\mathrm{Ec}\,!\,\mathrm{x}$" is selected as primary proposition about some object x. It is a true proposition whose subject is indicated by "x" and it reads as "that unique individual object x". Then, a mode of *togetherness* is selected; here, as in the *Principia*, it is the primitive idea "$p \vee q$", i.e., *p or q*, where "$\vee$" puts propositions together. And since Whitehead analyzes complex propositions only in terms of constitutive subordinate propositions in that same mode of *togetherness*, "$\vee$" is replaced by "$\cup$" as symbol of '*togetherness*' if constitutive propositions like "$(p \cdot q)$" i.e., *p and q*, have to be accommodated.

This restriction, Whitehead insists, shows that it is structure and not truth value that matters and that these structural relations apply to classes defined as special cases of propositions: assuming that the '*togetherness*' of two members of a class, "x" and "y", is defined in terms of '*togetherness*' of the propositions "$\mathrm{Ec}\,!\,\mathrm{x}$" and "$\mathrm{Ec}\,!\,\mathrm{y}$" and that it is expressed with the symbol "$\cup$", then "$\mathrm{Ec}\,!\,\mathrm{x} \;\cup\; \mathrm{Ec}\,!\,\mathrm{y}$" is the class with members "x" and "y". This is thus a sort of proposition whose structure alone and not the truth value matters. With a symbol "$=$" for equivalence of class membership and another symbol "$\equiv$" for truth value equivalence, a class is defined as a true proposition which can be reduced by equivalence transformations to a form like "$\mathrm{Ec}!\mathrm{x} \,\cup\, \mathrm{Ec}!\mathrm{y} \,\cup\, \mathrm{Ec}!\mathrm{z}\ldots$"

Adding appropriate definitions and postulates in elaborating his theory of classes, Whitehead displays a system essentially equivalent to a Boolean formulation of the system of Symbolic Logic of *Principia Mathematica*. Continuing, he reconsiders several aspects of this work, clarifying and introducing various corrections with respect to the notions of numbers, relations, arithmetical operations and derived numbers.

The larger logical and mathematical context is studied in much details in I. Grattan-Guinness [12] while the full Whiteheadian context is found in V. Lowe [22].

3. *The Scope of Logic: the Mingling of Forms*

These considerations of Whitehead correspond to "*a conception of the scope of Logic which was obscured by the dominant Aristotelian theory. The concept was adumbrated by Plato, ...* [who] *points out the importance of a science of the mingling of forms. This doctrine of the study of logical structures and of structures of structures, has been introduced into contemporary Logic by Prof. H. M. Sheffer. Mathematics (as currently understood) and the doctrine of classes form one preliminary division of it. In an enlarged sense of the term the whole topic may be termed 'mathematics'.* " [44] (294–295)

This study of logical structures amounts to the study of logical forms and their composition from subordinate forms; the general study of structures concerns the articulation of these logical forms and their inter-relations by means of inference rules.

Here, Whitehead sees the logical theory as "*the general study of structures which are definable by the use of the apparatus of notions which lie within its scope* [...] *classes, relations, number-systems*". And in this study, "*the notion of truth-value remains in the background.*" [44] (296–297)

Whitehead attributes to Sheffer alone the introduction of this study of structures in logic. There may be two reasons to this. In *Principia Mathematica*, Whitehead and Russell had introduced a logical theory in which elementary propositions were combined by means of two connectives, more precisely, two primitive propositional functions, negation and disjunction. In the introduction to the second edition in 1925, Russell remarks that the introduction by Sheffer [35] of a unique connective which replaces all the other connectives, the Sheffer stroke or 'incompatibility' defined as the negation of the logical conjunction ('*not both*'), had been "*The most definite improvement from work in mathematical logic*" [43] (xiii) since the first edition.

Earlier, in "*A Boolian algebra with one constant*", circa 1880, C. S. Peirce had anticipated Sheffer, and in 1902, in "*The simplest mathematics*", this idea reappears with the unique connective, the Peirce arrow ↓ ('*neither ... nor*'), the negation of the logical disjunction. Of course, this connective has

the same structural effect as the Sheffer stroke.
These works of C. S. Peirce were unpublished at the time of *Principia* and although Whitehead may have heard of them since they appeared shortly before his [44], he does not mention them.
But there is more than a new single connective behind Whitehead's attribution to Sheffer. Continuing his introduction, Russell mentioned that "*a new and very powerful method in mathematical logic has been invented by Dr. H. M. Sheffer.*" [43] (xv), but it was too late to take advantage of it because it would require a complete rewriting of the work, a task that is recommended to Sheffer himself! Indeed, although some of Sheffer's ideas had been circulated in a mimeographed paper, "*The General Theory of Notational Relativity*", 1921, hardly anything on his new method had been published.

Continuing with logical structures and logical forms, Whitehead recalls that in *Principia Mathematica*, logic amounts to the study of propositional forms like $a \vee b$ and $a \cdot b$ whose arguments are propositions, and to the investigation of the mingling of these forms. The propositions that appear in these forms are true or not depending on the value of their variables and on the characteristics of the form they exemplify. He thus distinguishes three characteristics that he calls "*validation-values*" (295): the propositional forms may be validating or invalidating, that is, *in virtue of their form*, any proposition that illustrates these forms is true or it is false. Or the propositional forms can be neutral, that is, the propositions that exemplify them can be true or false depending on their content. For example, a formula of propositional logic like "$a \cdot b \supset a$" is validating, "$a \cdot \sim a$" is not, and "$a \cdot b$" is neutral. The propositions of algebra share the same *validation-values*.

Whitehead remarks that the authors of the first edition of *Principia Mathematica* had not noticed this interpretation of what he calls 'real variables', while in the second edition, their use, and thus the 'validation-values', was prevented by the introduction of universal quantifiers. A consequence is that without validation-values, logical inference lost its justification. Indeed, as Whitehead shows, a proposition like "$(\forall a)(\forall b)(a \cdot b) \supset b$" is no longer treated differently from any other proposition like "$(\forall x)$, x is three months from Christmas $\supset$ x is September 25", which happens to be true. But it is not true in virtue of its form because the valuation-value of its logical form, $\phi x \supset \psi x$, as any other simple forms, is neutral. And "*It requires a 'mingling' of forms to produce validating, or invalidating, forms.*" (296)
Since logical analysis consists in the decomposition of propositions into subordinate constitutive propositional forms like $a \vee b$ and $a \cdot b$ whose validation-value is neutral, Whitehead calls the 'general question of implication' the question of knowing whether some validation-values of some of these constitutive propositional forms determine the validation-values of the other forms. "*Thus implication is primarily a relationship between propositional forms.*" (296) and the classical syllogism is an example of this relationship. We will

come back to the theory of syllogisms in considering Whitehead's algebra of symbolic logic as it is presented in his *Treatise*.

We have focused on the structural aspects emphasized by Whitehead in this article. Before turning to the context in which this article was produced, we may remark that in this paper, Whitehead's main goal was to found the notion of class on a purely logical basis that could be used in defining the mathematical notions. Indeed, *Principia Mathematica* had not solved all difficulties, in particular that of constructing arithmetics on such a logical basis. In relation to these difficulties, Whitehead mentions that his former Ph.D. student W. V. O. Quine had proposed another approach to the study of structure that would appear as his [31]. In his Forword to the book, Whitehead writes that it is "*A landmark in the history of the subject*" (ix). It originated in a research program which started with Carnap's *Logical Syntax of Language*. It will later give rise, in 1937, to the *New Foundations*, Quine's revision of the type theory of *Principia Mathematica* and his most important contribution to set theory.

4. *Logic and Universal Algebra at Harvard*

In a historical overview of the mathematics at Harvard, Garrett Birkhoff [5] describes in details the intellectual context in which Whitehead wrote the remarks exposed in the preceding paragraphs. Today's field of "*Universal Algebra*" originates in Birkhoff's work [3] who borrowed the name from the title of A. N. Whitehead's *Treatise*. If Birkhoff presented his work on lattices under that name, it is because the title of Whitehead seemed appropriate to him. Indeed, it was concerned with the logic of the symbolic method, and Birkhoff saw in symbol manipulation the essence of algebra. [3] [4] Related aspects of this history of universal algebra can be found in [32].

Earlier, while at Johns Hopkins, C. S. Peirce had based Boolean algebra on the concept of partial order or containment. This had influenced E. Schröder's work which, in turn, influenced Dedekind and his notion of lattice that Birkhoff rediscovered in the early thirties.

Peirce's interests in logic also influenced E. V. Huntington who had studied in Germany and who worked on systems of postulates for various mathematical structures, including what would be soon today's Boolean algebra. His early work clearly anticipated the modern notions of relational structure and algebraic structure.

4.1. *Axioms and postulates*

The Euclidean axiomatic method often presented as a model of formal rigor had already been introduced into Logic at the end of the 17^{th} century. Although the axioms of elementary geometry subjected to scrutiny had given rise to non-Euclidean geometries and shaken the foundations of geometry, Hilbert's investigations did consider the axiomatic method as a way out of the foundational crisis in mathematics.
Before him, Peano [24] already relied on this method to provide an early axiomatization of arithmetics. In Peano's work, the influence of H. Grassmann as well as that of Dedekind who followed Frege to reduce arithmetics to logic were not negligible.

C. S. Peirce was also influential through his various articles on the algebra of logic [27], [29], [30], in which he develops algebraically the deductive apparatus of logic. He sees the main problem of logic as that of producing a method for the discovery of methods in mathematics. "*The algebra of logic should be self-developed, and arithmetic should spring out of logic instead of reverting to it.*" [29] (186) In his [28], he thus sketches his arithmetics founded on the natural order of the integers and the operations '$+$' and '$-$', with the aim of showing that elementary arithmetic propositions are "*strictly syllogistic consequences from a few primary propositions.*" (85) As remarked by a reviewer, the axiomatic system presented there has been proved equivalent to both Peano's and Dedekind's systems in [36].

Around the turn of the century, in American mathematics, E. H. Moore defended the axiomatic method under the name "*postulational method*". [23] Moore had studied under Weierstrass in Berlin and his method originates in the works of Peano and Hilbert on the foundation of Geometry. Moore emphasized the notion of process in mathematics and considered his work as an application of Boole's conceptions, as well as of logical analysis based on Peano's formalism and on Cantor's theory of classes, to the theory of continuous functions.

At the time of Moore's writings, various sets of independent properties or postulates for various mathematical theories like groups, fields, and geometry had already been found. Moore demanded not only to provide their existential theories, that is, an interpretation of their postulates, but also to determine sets of completely independent fundamental properties or postulates of these theories.
Although Moore's method did not attract much interest from mathematicians and was mainly developed in his "Chicago School of Mathematics", it influenced several of his pupils who would be prominent later on.

Indeed, Huntington who had also been influenced by C. S. Peirce was one of early proponents of the method. [14] Later on, in the 1930s, he inspired the new generation of algebraists — the generation of Garrett Birkhoff —

who took a renewed interest in the postulational method. Thus, Huntington [16] tried to encourage "*further work in the direction of a definitive set of postulates for the theory of deduction.*" (92), a problem that he found mathematically as justified as that of finding postulates for Boolean algebra.
In [15], he observed that more than 30 systems of postulates had been proposed for Boolean algebra. Then, for comparison with that of *Principia Mathematica*, he put forward a new system of postulates that fulfills Moore's requirements. Diamond's set of postulates [11] is another example of such a system. There, calling Huntington's original system of postulates for the algebra of logic [14] the "*Whitehead-Huntington set of postulates for the Boole-Schröder algebra of logic.*", (940) he reminded that A. N. Whitehead was probably the first to exhibit these postulates and he followed Bernstein's earlier opinion: "*Of the various sets of postulates that have been given for Boolean logic the most elegant and natural is the set of Huntington's based on Whitehead's 'formal laws'.*" [1] (458) Huntington will advertise again the method of postulates in [18], a presentation based on his earlier [14]. In order to support his claims in favor of the postulational method in logic and in mathematics, he starts with the comparison of two concrete and intuitive systems, one geometrical, the other propositional, and he shows how to build an abstract system which encompasses the main features of both. As we will see now, this abstract system happens to be a Boolean algebra, an algebra originally devised to investigate problems in the logic of classes and propositions.

4.2. *The algebra and the logic*

Let thus $A, B, C, \ldots$ be *regions* in a square; let A' be the region outside region A and call it its complement. Let AB be the region common to A and to B. Then, looking at the initial square, it is easy to see that

1: If A and B are regions, AB is a region.
2: If A is a region, A' is a region.
3: $AB = BA$
4: $(AB)C = A(BC)$
5: $(A'B)'(A'B')' = A$
6: If Z is the *null* region, $ZA = Z$
7: If U is the *whole* square, $AU = A$
8: If $AB = A$ then A is in B and inversely.
9: etc...

Now, let $P, Q, R, \ldots$ be *propositions* or *statements* assumed to be true. Let PQ be the joint proposition "$P \text{ and Q}$" asserting that P and Q are true and let P' be the contradictory of proposition P, asserting that P is false. Then, as in the preceding geometrical case, one can see that

1: If P and Q are propositions, PQ is a proposition.
2: If P is a proposition, P' is a proposition.
3: $PQ = QP$
4: $(PQ)R = P(QR)$
5: $(P'Q)'(P'Q')' = P$
6: If Z is the contradiction PP', $ZP = Z$
7: If U is any necessarily true proposition, $PU = P$
8: If $PQ = P$ then P implies Q and inversely.
9: etc...

From these intuitive symbolic representations of concrete situations, an abstract (algebraic and logical) system is easily constructed. First, replace the notion of region by that of class (K); then, give some relation that tells when members of the class represented by some symbols $(a, b, c, \ldots)$ are equivalent or not; next, give some operations $(+, \cdot, \ldots)$ on those symbols that allow to associate them in certain ways. The conditions imposed on the symbols are now the postulates or the axioms. And these have to respect some conditions of consistency and independence. (Rather than the symbol "·", we use concatenation.)

The following set of postulates, the *Whitehead-Huntington postulates* of [11] and [14] is then easy to understand:

0: There is an a and there is a b in K s.t. $a \neq b$
1a: $a + b$ is in K if a, b are in K
1b: ab is in K if a, b are in K
2a: For all a in K, there is a Z such that $a + Z = a$
2b: For all a in K, there is a U such that $aU = a$
3a: $a + b = b + a$ whenever a, b, $a + b$ and $b + a$ are in K
3b: $ab = ba$ whenever a, b, ab and ba are in K
4a: $a + bc = (a + b)(a + c)$ whenever a, b, c, bc, $a + b$, $a + c$, etc. are in K
4b: $a(b + c) = ab + ac$ whenever a, b, c, $b + c$, ab, ac, etc. are in K
5: If U and Z exist and are unique, there are a, a' s.t. $a + a' = U$ and $aa' = Z$

We will see that in his *Treatise* [38], in the second Book devoted to the algebra of logic, Whitehead gave an essentially similar set of postulates, as well as a similar intuitive interpretation. First, he gave a set of postulates or formal laws that adds to the preceding set the following postulates (the numbering continues that of the intuitive postulates):

8a: $a + a = a$ (idempotence)
8b: $aa = a$ (idem)
10a: $a + (ab) = a$ (absorption)
13a: $(a + b) + c = a + (b + c)$ (associativity of addition)
13b: $(ab)c = a(bc)$ (associativity of multiplication)

Next, Whitehead proposed a second set of postulates that contains, in addition to [2ab], [3ab], [4ab], [10a] and [13ab], the following four:

9a: $a + U = U$
9b: $aZ = Z$
10b: $a(a + b) = a$
11: if $a' \in Z$, $a + a' = U$ and $aa' = Z$

Z and U stand for 'null' and 'universe' which can also be represented by Peano's $\bigwedge$ and $\bigvee$ or by Boole 0 and 1. Whitehead used 0 and i.

4.3. *Leftover Problems*

Following that original work of Whitehead and Huntington, various attempts at formalizing the postulates of Boolean algebra as well as those of logic were made by B. A. Bernstein, P. Henle, and several others, like Sheffer who was mentioned earlier. The goal was to obtain satisfactory sets of postulates for logic that also respect the conditions of independence and consistency, conditions already fulfilled in Huntington's [14].

For example, although Whitehead and Huntington had given their axiomatic or postulational foundation to the logic of classes, none had been given to the logic of propositions. This will be done by B. A. Bernstein in [2]. There, he remarks that Boole who had created "*the mathematical sciences now known as the logic of classes and the logic of propositions.*" (472) had developed these logics as theories of primary or concrete propositions (like "snow is white") and secondary or abstract propositions (like "it is true that snow is white"). These theories, *as improved by Peirce and Schröder, may be called, after Sheffer, Boolean algebras.*" (472) But, Bernstein continues in notes, Boole was wrong in stating that the formal laws of his primary propositions are identical to those of the secondary propositions. Schröder made another mistake in thinking that it sufficed to add to any set of postulates for the logic of classes a postulate saying that the logic consists of two elements (i.e., the logic of propositions being thus the algebra of truth-values) to obtain a set of independent postulates for the logic of propositions. And, even *Principia Mathematica* was defective with respect to the independence of the set of primitives of its theory of deduction. As we know today, failure in interpretation and in distinguishing the object language and the metalanguage can explain these problems.

C. S. Peirce had already been introduced to logicians by C. I. Lewis, [21] but the publication of his work had only started at the end of the 1920's. We have seen with G. Birkhoff, that the rediscovery of Peirce's work by the new generation of American algebraists in the early thirties also raised their interest in foundational issues related to logic.

For example, Huntington, again, will make use of the method of postulates to show the equivalence of the Hilbert-Bernays system with that of the *Principia*. [17] During this time, numerous results in universal algebra were coming out, all in a perspective that was definitely structural. At that time also, Whitehead had been a faculty member at Harvard, and it is in that context that he wrote his remarks. [44] But the story had started much earlier at Cambridge.

5. *Logic, Algebra and Universal Algebra at Cambridge*

Whitehead's *Treatise on Universal Algebra, with applications* [38] has played a role more important than generally thought. Before embarking with Russell on the joint enterprise of writing *Principia Mathematica*, the reference for logic of the next century, the *Treatise* can be seen as a last 19^{th} century attempt at the edification of the "mathesis universalis". And it certainly has its place in the history of logic and algebra.

5.1. *The Forefathers*

One way to consider universal algebra is to see it originating first in W. R. Hamilton's investigations of $\sqrt{-1}$, and in the consequences drawn from there, the hypercomplex numbers. [32] In particular, it first appeared and developed in the algebraic investigation of these numbers by J. J. Sylvester on the basis of Cayley's matrices, and in Whitehead's application of Grassmann's algebra in his *Treatise*.

$\sqrt{-1}$ also motivated the development of ordinary algebra in the works of Peacock, Gregory, De Morgan and Boole. Algebra, that had been considered as "universal arithmetic" since Newton's time, was transforming into symbolical algebra. For example, with A. De Morgan, algebra is the science of uninterpreted symbols and their laws of combination by the mechanical processes of a calculus. [10]

Shortly after Hamilton's *Quaternions* and Grassmann's *Ausdehnungslehre*, both published in 1844, G. Boole's *Mathematical Analysis of Symbolic Logic, being an Essay toward a Calculus of Deductive Reasoning* appeared. [6]

Boole is mainly known for his *Investigation of the Laws of Thought on which are Founded the Mathematical Theories of Logic and Probabilities*, [8] the complete exposition of his theory. But his earlier book, published the year that saw the publication of De Morgan's *Formal Logic: or, the Calculus of Inference, Necessary and Probable*, [10] is no less important. This book, the "*Mathematical Analysis*", will give rise to what will later be called Boolean algebra and — not forgetting Leibniz's contributions — it can be considered as the beginning of modern algebra as well as of algebraic and mathematical logic.

From this point on, logic will be conceived of as a calculus whose object, the processes of thought, is submitted to mathematical operations on logical symbols. It is the difficult issue defended by Boole of the laws of logic being the laws of thought that attracted the attention and prompted a large debate. The aim of Boole in [6], [8], [9], was to use the symbolical method to investigate, in the logical calculus, the operations and laws of the mind by which reasoning is performed. This investigation would not only elucidate what thought is, the laws of thought corresponding to the laws of operating with logical symbols, but it would also give a foundation to logic. If there is a calculus of logic or if logic is developed under the form of a calculus, it is because there exists a formal analogy between the processes of logic and the operations of mathematics. Moreover, the processes of symbolical reasoning are independent of their interpretation and of the symbolic representation and use of symbols and belong to the relations of thought and language.

Boole applies the symbolical method to mathematics and discusses its value in [9]. He notes the importance of considering that the method makes visible the connexion of language with thought. Indeed, in this method, the operations are separated from their objects by a mental abstraction and "*are expressed by symbols in whose laws the laws of the operations themselves are represented.*" (381) Boole proceeds to show this on the example of applying this symbolic method to the operations involved in solving differential equations. The method reveals a formal analogy, or a similitude of relations, between the differential equations and the algebraic expressions subjected to various laws that determine their forms.
The laws of symbols are determined from the corresponding operations performed in thought. But, while the formal rules of two systems of symbols may agree, their interpretation may differ, or only one of them may represent real operations of thought. Nevertheless, Boole maintains that the processes of symbolical reasoning are independent of the conditions of their interpretation. And, this shows that the principle of symbolic representation and use of symbols, whether a priori or acquired by experience, is not a mathematical principle but "*claims a place among the* general *relations of Thought and Language.*" [9] (399)

5.2. *A. N. Whitehead's Universal Algebra*

Whitehead writes that his *Treatise of Universal Algebra*, is "*a thorough investigation of the various systems of Symbolic Reasoning allied to ordinary Algebra.*" [38] (v)
In the same way as Benjamin Peirce who had initiated the comparison of the symbolic structure of the algebraic systems in [25], Whitehead intends to

study and to compare the systems of William Rowan Hamilton's Quaternions Theory, Hermann Grassmann's Calculus of Extension and George Boole's Symbolic Logic.

5.2.1. *Generalised algebra*

According to V. Lowe, *Generalised Algebra* was the title originally chosen for this *Treatise* but Whitehead eventually preferred to borrow a more appropriate title, *Universal Algebra*, from J. J. Sylvester. [22] (191)

Whitehead writes that his goal was to expose a generalized conception of space whose properties and operations would lead to a uniform method of interpretation of the various algebras that would thus appear "*as systems of symbolism, and also as engines for the investigation of the possibilities of thought and reasoning connected with the abstract general idea of space.*" [38] (v)

Volume one of the *Treatise* is divided into seven Books. The first book which relies heavily on Grassmann's ideas exposes the principles of algebraic symbolism, the nature of a calculus, the manifolds, and the principles of universal algebra. The second Book is devoted to the algebra of symbolic logic, and the rest of the volume is concerned exclusively with the study and applications of Grassmann's Calculus.

The collaboration with B. Russell imposed to postpone the edition of the second volume. It was eventually planned as Volume IV of the *Principia Mathematica*, but it never appeared. [22]

This volume should have contained the comparison of Hamilton's theory of Quaternions with Benjamin Peirce's *Linear Associative Algebra* of 1870, later published with notes and additions by his son, Charles Saunders. [26] It is important to notice, a referee remarked, that Benjamin's work is exclusively an algebra, whereas under Charles' notes and appendix, it is shown how to classify algebras in terms of the logic of relations. That is, on the basis of his father's work [25], Charles [28] demonstrated that linear and multilinear algebras could all be reduced to interpretations of the logic of relations.

Whitehead considered ordinary algebra of his time as a "*set of propositions, inter-related by deductive reasoning, and based upon conventional definitions which are generalizations of fundamental conceptions.*" [38] (viii) In introducing his new field of universal algebra, he knew that it could be considered as uninteresting and useless as investigation tool in the same way as symbolic logic had appeared to logicians, a simple branch of mathematics, and conversely. Nevertheless, he wanted to show that it is a branch of Mathematics as serious as any other because, according to him, and repeating Benjamin Peirce first sentence of [25], "*Mathematics in its widest*

signification is the development of all types of formal, necessary, deductive reasoning." (vi)
Mathematics is formal because the meaning of its propositions is not taken into account. It is necessary because "*the business of mathematics is simply to follow the rule.*" (vi) And the reasoning is deductive because it is based on definitions that have only to be consistent and complete. Definitions in mathematics are existential or conventional. They refer to existing things in the world or they are obtained by abstraction from a set of interrelated things.

Traditionally, the object of mathematics had always been number, quantity, and ordinary space; but the discovery of complex numbers — defined conventionally — and the introduction of complex quantities in ordinary algebra extended the ordinary quantities to generalized entities. This implied, not only the development of new algebras, but also the creation of a new science that "*has relations to almost every event, phenomenal or intellectual which can occur.*" Indeed, Whitehead dreamed of mathematics constructing a calculus adapted to reasoning with respect to each domain of thought or external experience. This new extended ordinary algebra was a first step, and it concerns universal algebra provided that "*the newly invented algebras* [...] *exemplify in their symbolism or* [...] *represent in their interpretation interesting generalizations of important systems of ideas, and* [be] *useful engines of investigation.*" (viii)
The goal of Whitehead in his *Treatise* is thus to investigate these new algebras; among them, the algebra of symbolic logic.

5.2.2. *Calculus and manifolds*

A calculus is defined as "*The art of the manipulation of substitutive signs according to fixed rules, and of the deduction therefrom of true propositions*". (4)
Coming back to the definitions, conventional definitions are abstracted from a set of objects in various consistent and defined relations by an act of imagination. Although they are conventional, they must keep some connections with existing things if they have to be used as basis for founding mathematics. Indeed, the language of mathematics uses substitutive signs; these signs are manipulated by rules, and the application of these rules must be such that in the resulting state "*when the signs are interpreted in terms of the things for which they are substituted, a proposition true of the things that are represented*" (4) results. This constitutes a calculus. Its signs are symbols and in its use, the calculus is interpreted.
In a calculus, the propositions used in the deductions have the form of equivalence assertions. And, in agreement with Lotze and Bradley, equivalence is not identity, '=' is not the same as 'is'. A calculus based on propositions asserting identities would only result in identities while equivalence implies

non-identity. In the equation $2+3=3+2$, $2+3$ is not identical to $3+2$ because "*the order of the symbols is different in the two combinations, and this difference of order directs different processes of thought.*" Thus, the equation asserts that "*these different processes of thought are identical as far as the total number of things thought of is concerned.*" (6) In order to distinguish the two members of the equation, Whitehead names one the 'truism' and the other the 'paradox': they are identical with respect to the number 5, but they are also different objects. In assertions of equivalence in a calculus, only the second aspect matters, i.e., that two things which are different are equivalent. And it is the process of derivation from events, either phenomenal or mental, of one thing, for example, 5, from other things, 2 and 3, that allows to judge the equivalence because this process which manifests the operations of the mind is in the domain of application of a calculus.

The notion of a manifold has a geometrical origin in the works of H. Grassmann and B. Riemann. It is on this notion that Whitehead builds up his abstract and generalised notion of space. It is generalised in the sense that, given its definition, a manifold can be constituted of the musical notes as well as of the colors of the spectrum, or the points of ordinary space.

Indeed, various things may share a common property; and they can all have that property in different modes. Call each separate mode of possessing that property an element. Then, the set of these elements is the manifold of the property.

Various relations exist between the various modes of a property, that is, there are relations between objects which possess a same property in different modes; these relations define how they possess the property, that is, how the objects differ. The axioms from which all relations between all elements of a manifold can be logically deduced are called the characteristics of the manifold. To give an example of a manifold, Whitehead considers the empty space with respect to a system of coordinates: a point is a mode of the property 'spatiality' and the axioms of geometry constitute its characteristics.

"*It is the logical deductions from the characteristics of a manifold which are investigated by means of a calculus.*" (14)

5.3. *The Algebra of Symbolic logic*

"*Universal Algebra is the name applied to that calculus which symbolizes general operations* [...] *called addition and multiplication.*" (18) This is the short characterization of the principles of universal algebra given by Whitehead.

These principles depend on general definitions of the processes of addition and multiplication which hold for all branches of universal algebra. There are also special definitions of special kinds of addition and multiplication whose investigation constitutes special branches of universal algebra. Each of these branches is a special algebraic calculus or a special algebra. Otherwise, except for these special operations, algebra is ordinary algebra and its manifolds, whose elements can be added and multiplied, i.e., whose elements comply with the laws defining these operations, are called algebraic manifolds.

These operations of addition and multiplication defined by their usual properties (commutativity, associativity, etc.) do not vary with the different algebras. But a difference appears when a term is added to itself (multiplied by itself): while ordinary algebra distinguishes a and $a+a$ (a and aa), the algebra of symbolic logic, the algebra of Boole's *Investigations*, identifies the two. The reason is simple and is found in the interpretation of the calculus.

When developing the symbolism of a calculus, Whitehead suggests to keep its interpretation simple and to develop it concurrently with the algebra. In order to keep the interpretation of the algebra of symbolic logic simple, he considers only intersection or non-intersection of regions of space.

Thus, similarly to the intuitive interpretation of the postulates in 4.2, consider the elements of the algebraic manifold of the algebra of symbolic logic as regions of space. Let the terms representing the elements now represent the mental act of apprehending the regions they represent. Then the operation of addition corresponds to the act of apprehending in the mind the region represented by all terms added. A term added to itself, $a + a$, is presented twice to the mind for apprehension, but it cannot be duplicated since the region represented by $a+a$ remains always the same, i.e., a. Hence, $a+a = a$.

While, as usual, commutativity and associativity are required in defining addition, distributivity and absorption are also required for multiplication. Indeed, multiplying terms results in a term representing the entire region common to all terms multiplied. If abc is such a term, it represents the region contained at once in a, b, and c. Apprehending first region ab imposes to apprehend region a, then region b, and then the region which is their interesection. Obviously, this process satisfies distributivity and absorption, $a + ab = a$.

We end up with two kinds of addition and multiplication, the numerical and the non-numerical addition and multiplication; thus, with two kinds of algebras, the numerical kind (*genus*) of algebra characterized by the equations $a + a = 2a$ and $aa = a^2$, and the non-numerical algebra characterized by $a + a = a$, and $aa = a$.

"*The Algebra of Symbolic logic is the simplest possible species of its genus and has accordingly the simplest interpretation in the field of deductive logic.*" (29)

This algebra of symbolic logic is not all of the field of deductive logic but it is the only non-numerical algebra that has been developed. It is amenable to algebraic treatment because it is an algebra of extension, that of concepts and

propositions conceived as multiplicities and, as we have seen, its operations of addition and multiplication respect the various properties of commutativity, associativity, distributivity of multiplication over addition, idempotence and absorption.
In addition, and according to the intuitive interpretation, the null element, 0, represents the non-existence of a region; and the universe, i, is the entire space. Obviously, they are such that $a + 0 = a$ and $ai = a$. A special element is added, symbolized by $'$; it is such that a' is the complement of a, that is, the b such that $a + b = i$ and $ab = 0$. And also, $a + a' = i$ and $0a' = 0$. Moreover, the duality of the operations $+$ and $\cdot$ discovered independently by C. S. Peirce and Schröder, the duality of the null and universal classes of elements, and Boole's translation of any algebraic function into polynomials and normal forming of logical equations are all respected.
Note that Whitehead considers the other duality of the operators, that corresponding to division and substraction which is sometimes useful. As expected, associativity is a problem for substraction, and that keeps these operators out of considerations in this algebra.
Finally, this algebra of logic is essentially, as Whitehead writes, the "*algebra in all essential particulars ... invented and perfected by Boole*" in his "*Laws of Thought*".

In the intuitive interpretation of the algebra, elementary and complex propositions are interpreted in terms of regions and relations between these regions, that is, in terms of the algebraic manifold and its submanifolds. Since the symbolic treatment of these relations has analogies with the theory of inequalities of ordinary algebra while having also some properties of algebraic equations, Whitehead introduces two symbols to express these relations: one for incidence (subset), $x \mathrel{\,=\!\!\!\!\!\!<\,} y$, and one for inclusion (the reverse), $y \mathrel{\,=\!\!\!\!\!\!>\,} x$ which are defined in terms of equality: if $x \mathrel{\,=\!\!\!\!\!\!<\,} y$ then $x = yx$, and conversely. (See the table in the next subsection). Actually, he borrows from Schröder the symbol of subsumption and he remarks that it is from the relation of containment, $\mathrel{=\!\!\!\!\!\!>}$, that C. S. Peirce deduced his theory of symbolic logic, i.e., his algebra of logic. (C. S. Peirce used the symbol of inference $A \prec C$ to express the primary mode of relation between two propositions; it means that every state of things in which a proposition of the antecedent class A is true is a state of things in which the corresponding propositions of the consequent class C are true). [27], [30]

Whitehead develops in two chapters the methods of construction of symbolic equations from symbolic terms and the method of solution of algebraic equations which the analogy between logical and algebraic equations requires.
The first chapter is concerned with the methods to solve equations of the algebra of logic in one and several unknowns. One may note here the existence of fields of equations, i.e. domains of value for the evaluation of variables or

unknown of the equations.
In the next chapter, existential expressions are introduced. These are expressions containing symbols, j or ω, that assert the existence, or that limit the extension of the regions, that is, symbols such that $x.j$ represents “x *exists*” (i.e., is not 0) and $x + \omega$ represents “x is not all of i”. Then, the symbol $\equiv$ is used in place of $=$ to mean that the regions on either sides of the symbol are not only the same but, also, that the existential information of the right-hand side can be deduced from the left-hand side.
This existential notation can be extended further with the help of umbral or shadow letters, i.e. Greek characters attached to their corresponding Roman characters, called regional letters, and indicating where the adjoined symbols of the algebra are assumed to exist. For example, $x\alpha$ means that “regions x and a overlap”, i.e., $x\alpha$ implies $xa.j$ but it only denotes the region x.

5.4. *Application of the Algebra to Logic*

The second Book of the *Treatise* concludes with two chapters devoted to the application of the algebra to “*Formal Logic conceived as the Art of Deductive Reasoning.*” (99)

First, as Boole had done in his *Investigation*, [8] the algebraic calculus is applied to the classical theory of syllogisms. While providing some authoritative ancestry to his structural perspective, Whitehead’s formal position is supported by similar convictions of Leibniz in his *New Essays*: “*I consider the invention of the form of syllogisms one of the most beautiful, and also one of the most important, made by the human mind. It is a species of universal mathematics* [...] *Now you must know that by arguments in form, I mean not merely this scholastic mode of argument used in colleges, but all reasoning which concludes by the force of the form* ”. [20] (559)
This application to syllogisms is also an appropriate example of the notion of implication as relationships between propositional forms that was mentionned earlier in section 3.

The algebra of logic applies easily to the syllogisms because a syllogism can be represented as $xy \quad \notin \quad z$, that is, from x and y, conclude y. Nevertheless, according to Whitehead, this form, as such, does not represent the process of thought at work in a syllogism.

One may remember that the theory of syllogism is based on combinations of the various forms of propositions or judgements traditionally labelled A, E, I, O to abbreviate the four traditional forms: “all a is a b”; “no a is a b”; “some a is a b” and “some a is not a b”. Given what was said earlier about the algebra and the existential expressions, it is easy to see in the table below how these syllogistic forms translate into a form of the algebra of symbolic logic. Of course, given the formal laws of this algebra, these algebraic forms are not unique and have each other equivalent forms.

A	All a is b	:	region a is included in b	$\equiv$	$a \ \overline{\in}\ b$
E	No a is b	:	no regions overlap	$\equiv$	$ab = 0$
I	Some a is b	:	regions a and b overlap	$\equiv$	$ab.j$
O	Some a is *not* b	:	regions a and b' overlap	$\equiv$	$ab'.j$

In order to exclude meaningless forms, that is, forms such as in A, for example, where in $a \ \overline{\in}\ b$, one would have $a = 0$ or $b = i$ (and even both), the propositional form is transformed into an existential one: $aj \ \overline{\in}\ bj$ or $aj \equiv bj$ in the first case, and into $a{+}\omega \ \overline{\in}\ b{+}\omega$ or $(a{+}\omega) \equiv (a{+}\omega)(b{+}\omega)$ in the second case. From these transformed forms, the other equivalent forms can then be deduced by symbolic reasoning.

The traditional moods of syllogisms, the nineteen combinations of forms considered as valid forms of reasoning, and whose conclusions can be reached from premises by purely algebraic methods are thus investigated. Of these, five forms having too strong premises are excluded and Whitehead considers the symbolic equivalents of the fourteen moods retained.
Since the treatment of syllogisms amounts to the elimination of the middle term, the symbolic methods developed in the preceding chapters apply. To give an example, one mood in its simplest algebraic expression is that which has the universal form A in the premises and in the conclusion. Represented as [AAA], it is expressed as "$b \ \overline{\in}\ c$, $a \ \overline{\in}\ b$, therefore $a \ \overline{\in}\ c$" or, in one of its equivalent expression, as "$b = bc$, $a = ab$, therefore $a = ab = abc = ac$". Applying the algebraic methods, the elimination of b results in the symbolic equivalent $ac' = 0$. And similarly, a symbolic equivalent is obtained for each mood of the four figures.

Since the symbolic methods of the algebra permit to reach the conclusion of syllogisms from their premises, the conclusion of any reasoning valid by virtue of deductive logic can also be obtained by the same algebraic and symbolic methods. On this ground, Whitehead makes the suggestion that the processes at work in solving systems of equations of logical expressions are a generalization of the processes of syllogism and, hence, he suggests to generalize these processes to the whole of ordinary logic.

Doing so, Whitehead explicitly follows Boole's *Investigation* program. Indeed, with respect to equations, Boole's "general problem" was to find a rule such that "*Given any equation connecting the symbols* $x, y, \ldots, w, z, \ldots$", it determines "*the logical expression of any class expressed in any way by the symbols* $x, y, \ldots$ *in terms of the remaining symbols* $w, z, \ldots$ *&c.*" [8] (140)
With respect to the systems of traditional categorical forms (*A, E, I, O*), Boole saw "*The processes of Formal logic* [...] *described as of two kinds, viz., "Conversion" and "Syllogism."*" [8] (227) Conversion is the process of converting any of these propositions into an equivalent form, an application of a very general process in logic, "*the determination of any element in any proposition,* [...] *as a logical function of the remaining elements.*" (230) Syllogism is the usual process of deducing from two such propositions having

a term in common, a third proposition. Boole's goal was to show that these processes, often seen as universal types of reasoning, could be conducted "*upon the principles of the present treatise, and, viewing them thus in relation to a system of Logic*" whose foundations "*have been laid in the ultimate laws of thought*." (228)

Thus, in Whitehead, these two processes applied to universal propositions symbolised by equations and to particular propositions symbolised by existential propositions, are performed according to the rules of the algebra of symbolic logic. These are rules of transformation and reduction of a general form of a system of universal propositions involving one (or several) unknown element(s), the other elements being known. That form of a system is a set of n equations $a_i x + b_i x' = c_i x + d_i x'$ $(i \leq i \leq n)$. In order to determine the unknown x, the required information is obtained in solving the system of equations according to rules which do not concern us here. Actually, this process essentially amounts to form two regions, A and B, out of the regions involved in the system of propositions, and this amounts to reorganise the original knowledge to express the new information conveyed in the system. Formally, this amounts to a selection of certain regions defined by inter-relations but, in practice, this process may add knowledge to the definition of x.

Reference to undefined information or to what is known or not in reasoning introduces the last chapter devoted to the interpetation of propositions. There, Whitehead applies the calculus to classical logic: "*There is another possible mode of interpreting the Algebra of Symbolic logic which forms another application of the calculus to Logic.*" [38] (108)

Indeed, in the calculus, any symbol represents a categorical proposition or a complex proposition. A simple proposition is an assertion of a fact and two propositions x, y are equivalent if assenting to one entails assent to the other. A complex proposition tells that two or more simple propositions are conjunctively true or that at least one of the propositions of the complex is true. In the first case, it is a conjunctive complex represented as $(abc\ldots)$; in the second case, a disjunctive one, $(a + b + c + \ldots)$. A proof is then required that the operations of additions and multiplication of propositions can be identified with the operations of the algebra of symbolic logic.

This is easily done by showing that the complexes follow the rules of the algebra. As expected, a product is interpreted as a conjunction, and a sum as a disjunction of propositions. With respect to the elements 0 and i, the element 0 of the algebra corresponds to rejection of motives for assent to a proposition; thus it corresponds to the rejection of the validity of the proposition: $x = 0$ means that x does not enter the process of reasoning or the act of assertion. The class of elements equal to 0 is the class of those elements inconsistent with the propositions equal to i. These last are the propositions that have absoluteness of assent, conventionally or naturally. These propositions can be the "Laws of Thought of the Logic"; they are self-evident propositions. Therefore, the elements of the class equal to 0 are the self-contradictory or, as Whitehead calls them, the self-condemned propositions. Then, obviously, for any proposition x and its negation, x', $x + i = i$ and $xi = x$ as well as $xx' = 0$ and $x + x' = i$.

The first interpretation of the algebra of logic was restricted to classes of propositions in relation of inclusion and exclusion. This second interpretation of logic assumes the existence of some domain of knowledge from which all consequences of some categoric proposition or set of such propositions in either conjunctive, disjunctive or hypothetical relations to each other can be deduced. It is essentially a modification of the system of Boole, and Whitehead claims that it can be taken as the appropriate system of symbolic logic. Indeed, he shows that this interpretation includes the first one as particular case.

Similarly to Boole's system, it is built on the fundamental principles of identity and non-contradiction symbolised in $x^2 = x$ and in $xx' = 0$ (since $x' = i - x$). Although this second interpretation cannot exhibit the process of thought at work in a syllogism, there is a way out devised by McColl. It requires some precise analysis of predication and consists in analysing propositions of any traditional form into a relation between other propositions. Taking the example of the traditional form A, "*All A is B*", is analysed as saying that the validity of what is called a primitive predication "*It is A*" is equivalent to that of the conjunctive complex "*It is A and It is B*", which is thus represented as $a = ab$.

5.5. *A Memoir on the Algebra of Symbolic logic*

Shortly after the publication of his *Treatise*, Whitehead published "*a purely mathematical investigation concerning the Algebra of Symbolic Logic.*", [40] (139) the algebra originating in Boole's *Laws of Thought* and perfected by Peirce and Schröder that he recommends as "*the first object of mathematical study.*" because it is concerned with inclusion and exclusion of classes and it is also the simplest of all algebraic systems. (139)

This memoir was written in order to show that many interesting mathematical properties of the algebra of symbolic logic had not been worked out because attention was concentrated on its application to the operations of logic. As examples, Whitehead mentions Venn's *Symbolic Logic* (1881) that considered the interpretation of the algebraic symbolism in logic, and Peano's school which used that symbolism "*as a practical means for the exact expression of deductive reasoning*". (140)

Whitehead thus continued his own investigation of the theory of Boolean equations and the translation of Boolean functions into polynomials. Among

other topics, he studied the conditions under which the transformations of a function constitute a group.
Earlier, in 1899, in a communication at the Royal Society, [39] only published as abstract, Whitehead had related the theory of finite groups to an algebra, the algebra of groups of finite order that he compared to the Boolean algebra of symbolic logic of his *Treatise*. Both algebras are non-numerical algebras and they share several properties.

At the second International Congress of Mathematicians, coupled to the first International Congress of Philosophy held in Paris in 1900, Whitehead and Russell discovered Peano and his school of mathematics. This encounter was decisive in the development of symbolic logic.
Soon after, in a paper written in 1901, Whitehead generalized his theory of Boolean equations in terms of Peano's notation and of Russell's theory of relations which he saw as "*indispensable for the development of the theory of Cardinal numbers...*[they] *form an epoch in mathematical reasoning.*" [41] (367) In order to deal with infinitely many variables, he applied his theory of transfinite cardinal numbers. [41], [42] But this is another story.
Noteworthy in this last paper, and witness of Whitehead's originality, his definition of a '*multiplicative class*' and the related theorem. Stated in a readable form in Russell [33], it goes as follows: "*Let* k *be a class of classes, no two of which have any term in common. Form what is called the multiplicative class of* k*, i.e., the class each of whose terms is a class formed by choosing one and only one term from each of the classes belonging to* k*. Then the number of terms in the multiplicative class of* k *is the product of all the numbers of the various classes composing* k." (119) This principle would be soon famous as an axiom, but not under Whitehead's name.

6. *Conclusions*

Christine Ladd, a student of J. J. Sylvester and C. S. Peirce at Johns Hopkins University in the 1880's, wrote: "*There are in existence five algebras of logic, — those of Boole, Jevons, Schröder, McColl, and Peirce, — of which the later ones are all modifications,* [...], *of that of Boole.*" [19] (17) Some years later, following the publication of his *Treatise*, the name of Whitehead could have been added to the list. Or indeed, the list shortened if we listen to Huntington who referred to "*The name Boolean algebra (or Boolean "algebras") for the calculus originated by Boole, extended by Schröder, and perfected by Whitehead*". [15] (278)

J. Venn [37] considered that Boole had made the natural mistake of regarding logic as a branch of mathematics, simply applying mathematical rules to logical problems. Given his interpretation of the algebra of symbolic logic in the context of his universal algebra, and the justification he gives to his

approach, it is obvious that Whitehead did not make the same mistake.
According to C. I. Lewis [21] who wrote an extended account of Boole-Schröder algebra of logic, Whitehead's *Treatise* had been a most notable addition and improvement of the methods while "*Jevons, in simplifying Boole's system, destroyed its mathematical form; Peirce, retaining the mathematical form, complicated* [...] *the original calculus.*" (118)

Lowe [22] writes that in 1905, a reviewer for the doctoral degree found that Whitehead's published work gave "*new life to the study of symbolic logic.*" (263) This would appear in full light shortly after with the publication of *Principia Mathematica*, the joint work with Russell that had started five years earlier.
Symbolic logic and algebraic logic will then continue to develop along their own ways, and Boolean algebra, no longer Boole's algebra of symbolic logic, will later become the algebra known today by this name.
Several of the avenues opened by Whitehead will be forgotten, in particular, universal algebra as he had devised it. His last paper on logic with which we started, was a reminder of some of his essential conceptions that he left for logicians to meditate on.

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REFERENCES

[1] B. A. Bernstein, *A simplification of the Whitehead-Huntington set of postulates for Boolean algebras*, Bull. Am. Math. Soc., 22 (1916), 458–459.
[2] B. A. Bernstein, *Sets of postulates for the logic of propositions*, Trans. Am. Math. Soc., 28 (1926), 472–478.
[3] Garrett Birkhoff, *On the structure of abstract algebra*, Proc. Camb. Phil. Soc., 31 (1935), 433–454. Reprinted in: *Selected Papers on Algebra and Topology by Garrett Birkhoff*, G.-C. Rota and J. S. Oliveira, (eds.), Basel, Birkhäuser, 1987.
[4] Garrett Birkhoff, *Universal Algebra*, in: *Proceedings of the First Canadian Mathematical Congress*, Toronto U. P., 1945, 310–326.
[5] Garrett Birkhoff, *Mathematics at Harvard, 1836–1944*, in: *A Century of Mathematics in America, Part II*, P. Duren et al., (eds.), Providence, RI, AMS, 1989, 3–58.
[6] G. Boole, *The Mathematical Analysis of Logic, being an Essay toward a Calculus of Deductive Reasoning*, London, G. Bell, 1847.

[7] G. Boole, *The Calculus of Logic*, Cambridge and Dublin Math. J., Vol. III (1848), 183–198.
[8] G. Boole, *An Investigation of the Laws of Thought on which are Founded the Mathematical Theories of Logic and Probabilities*, London, Walton and Maberly, 1854.
[9] G. Boole, *A Treatise on Differential Equations*, Cambridge, Macmillan & Co., 1859.
[10] A. De Morgan, *Formal Logic: or, The Calculus of Inference, Necessary and Probable*, London, Taylor and Walton, 1847.
[11] A. H. Diamond, *The complete existential theory of the Whitehead-Huntington set of postulates for the algebra of logic*, Trans. of the Am. Math. Soc., 35 (1933), 940–948.
[12] I. Grattan-Guinness, *The Search for Mathematical Roots, 1870–1940: Logics, Set Theories and the Foundations of Mathematics from Cantor through Russell through Gödel*, Princeton, Princeton U. P., 2000.
[13] T. Hailperin, *Boole's Algebra Isn't Boolean Algebra*, Math. Magazine 54 (1981), 172–184.
[14] E. V. Huntington, *Sets of independent postulates for the algebra of logic*, Trans. of the Am. Math. Soc., 5 (1904), 288–309.
[15] E. V. Huntington, *New sets of independent postulates for the algebra of logic, with special reference to Whitehead and Russell's Principia Mathematica*, Trans. of the Am. Math. Soc., 35 (1933), 274–304.
[16] E. V. Huntington, *The postulational method in mathematics*, The Am. Math. Monthly, 41 (1934), 84–92.
[17] E. V. Huntington, *The inter-deducibility of the new Hilbert-Bernays theory and Principia Mathematica*, Ann. of Math., 2nd Ser., 36 (1935), 313–324.
[18] E. V. Huntington, *The method of postulates*, Phil. of Sc., 4 (1937), 482–495.
[19] C. Ladd, *On the algebra of logic*, in: C. S. Peirce et al., *Studies in Logic by Members of Johns Hopkins University*, Boston, Little, Brown and Comp., 1883, 17–71.
[20] G. W. Leibniz, *New Essays Concerning Human Understanding*, transl. by A. G. Langley, New York, The Open Court Pub. Comp., 1916.
[21] C. I. Lewis, *A survey of Symbolic Logic*, Berkeley (Cal), Un. of Cal. Pr., 1918.
[22] V. Lowe, *Alfred North Whitehead, the Man and his Work*, Vol. I, Baltimore, The Johns Hopkins Un., 1985.
[23] E. H. Moore, *Introduction to a Form of General Analysis*, in: *The New Haven Colloquium Lectures*, E. H. Moore et al., eds., New Haven, Yale Un. Pr., 1910, 1–150.
[24] G. Peano, *Arithmetices Principia, Nova Methodo Exposita*, Roma, Frat. Bocca, 1889.

[25] B. Peirce, *Linear associative algebra*, Washington City, privately printed lithograph, 1870.
[26] B. Peirce, *Linear associative algebra*, Am. J. of Math., 4 (1881), 97–229.
[27] C. S. Peirce, *On the algebra of logic*, Am. J. of Math., 3 (1880), 15–57.
[28] C. S. Peirce, *On the logic of number*, Am. J. of Math., 4 (1881), 85–95.
[29] C. S. Peirce, *On the algebra of logic: a contribution to the philosophy of notation*, Am. J. of Math., 7 (1885), 180–196.
[30] C. S. Peirce, *On the algebra of logic [Continued]*, Am. J. of Math., 7 (1885), 197–202.
[31] W. V. O. Quine, *A System of Logistic*, Cambridge, Harvard U. P., 1934.
[32] J. Riche, *From Universal Algebra to Universal Logic*, in: J.-Y. Béziau, A. Costa-Leite (eds.), *Perspectives on Universal Logic*, Monza, Polimetrica, 2007, 3–39.
[33] B. Russell, *The Principles of Mathematics*, Cambridge, Cambridge U. P., 1903.
[34] E. Schröder, *Vorlesungen über die Algebra der Logik*, Leipzig, Teubner, 1890.
[35] H. M. Sheffer, *A set of five independent postulates for Boolean algebras, with appplications to logical constants*, Trans. of the Am. Math. Soc., 14 (1913), 481–488.
[36] P. B. Shields, *Peirce's Axiomatization of Arithmetic*, in: N. Houser et al., (eds.), *Studies in the Logic of Charles S. Peirce*, Bloomington, Indiana U. P., 1997), 43–52.
[37] J. Venn, *Boole's logical system*, Mind, 4 (1876), 479–491.
[38] A. N. Whitehead, *A Treatise on Universal Algebra*, Cambridge, Cambridge U. P., 1898.
[39] A. N. Whitehead, *Sets of operations in relation to groups of finite order*, Proc. Roy. Soc. London, 64 (1898–1899), 319–320.
[40] A. N. Whitehead, *Memoir on the algebra of symbolic logic*, Am. J. of Math., 23 (1901), 139–165, 297–316.
[41] A. N. Whitehead, *On cardinal numbers*, Am. J. of Math., 24 (1902), 367–394.
[42] A. N. Whitehead, *The logic of relations, logical substitution groups and cardinal numbers*, Am. J. of Math., 25 (1903), 157–178.
[43] A. N., Whitehead and B. Russell, *Principia Mathematica*, 3 Vols, Cambridge, Cambridge U. P., 1910–1913. Sec. ed., 1925.
[44] A. N., Whitehead, *Indication, Classes, Numbers, Validation*, Mind, New Ser., Vol. 43, No. 171 (July 1934), 281–297. *Corrigenda*, 543.