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HYPERREAL EXPECTED UTILITIES AND PASCAL'S WAGER

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Abstract

This paper re-examines two major concerns about the validity of Pascal's Wager: (1) The classical von Neumann-Morgenstern Theorem seems to contradict the rationality of maximising expected utility when the utility function's range contains infinite numbers (McClennen 1994). (2) Apparently, the utility of salvation cannot be reflexive under addition by real numbers (which some interpretations of *Pensées* §233 demand) and strictly irreflexive under multiplication by scalars < 1 at the same time (Hájek 2003).

Robinsonian nonstandard analysis is used to establish a hyperreal version of the von Neumann-Morgenstern Theorem: an affine utility representation theorem for internal, complete, transitive, independent and infinitesimally continuous preference orderings on lotteries with hyperreal probabilities. (Herein, a preference relation $\preceq$ on lotteries is called *infinitesimally continuous* if and only if for all $x \prec y \prec z$, there exist hyperreal, possibly infinitesimal, numbers p, q such that the "perturbed preference ordering" $px + (1-p)z \prec y \prec qx + (1-q)z$ holds. Infinitesimal Continuity is hence a much weaker condition than continuity.) This Hyperreal von Neumann-Morgenstern Theorem yields a hyperreal version of the Expected Utility Theorem — affirming a conjecture by Sobel (1996). This responds to objection (1).

To address objection (2), a convex linearly ordered superset S of the reals whose maximum is both reflexive under addition by finite numbers and strictly irreflexive under multiplication by scalars < 1 is constructed.

If the Wagerer is indifferent among the pure outcomes except salvation (an orthodox soteriological position) and her preference ordering satisfies certain rationality axioms, then this preference ordering can be represented through an S-valued (not just hyperreal-valued) utility function. This result responds to objections (1) and (2) simultaneously.

Behold, I set before you the way of life and the way of death.

Jeremiah 21,8
(King James Version)

But your happiness? Let us weigh the gain and the loss in wagering that God is. Let us estimate these two chances. If you gain, you gain all; if you lose, you lose nothing. Wager, then, without hesitation that He is.

Blaise Pascal, *Pensées*, §233
(Trotter translation)

1. *Introduction*

1.1. *The context of Pascal's Wager*

Pascal's Wager [*Pensées* §233] is a Christian apologetic argument.[1] It is, however, meant to address individuals who already hold certain beliefs about the supernatural (cf. Rescher 1985), which explains the strength of some of the argument's premises (see Subsection 1.2):

First, the subjective probability for the existence of the Christian God is assumed to be positive and non-infinitesimal. (Otherwise the argument would no longer be valid, cf. Oppy 1990 and Hájek 2003, see also Footnote 5.)

Secondly, the audience is assumed to consider the Christian faith the only viable alternative to atheism for themselves. (Otherwise they might as well become attracted to any religion that promises paradise to its followers. Several variants of this so-called many-gods objection have been studied systematically by Bartha 2007.)

From a theological perspective, it is important to note that Pascal did not expect that anyone who is convinced of the conclusion of the argument could earn their salvation themselves — let alone by merely accepting the rationality of some gambling strategy.[2] To the contrary, Pascal's (Jansenist) theology places great emphasis on grace and predestination.

The purpose of this apologetic argument is, therefore, simply to "incite to the search after God" [*Pensées* §181].

[1] The context of *Pensées* §233, in particular *Pensées* §181, implies that the Wager is an argument of Christian apologetics, not specifically Jansenist or Catholic apologetics. (Jansenism was the Roman Catholic, later considered heterodox, sect to which Pascal belonged.) It is not addressing Christian believers without an attachment to Jansenism or Roman Catholicism, but rather individuals who lack interest in a personal faith (hence Section III of the *Pensées* is entitled "Of the necessity of the Wager") or lean towards agnosticism or atheism.

[2] Cf. Pascal in *Pensées* §240.

1.2. *The structure of the Wager. Mixed strategies*

Pascal's argument — directed at someone who is choosing between either Christianity or atheism and, in addition, assigns positive, non-infinitesimal probability to the existence of the Christian God — can be formalised as follows:

(1) Premise: One has to wager for or against God, and the payoff of the wager is as follows:

	Christian God exists (with some probability $p \gg 0$)	Christian God does not exist
Wager for God	I	f_2
Wager against God	f_3	f_4

Herein, I denotes an infinitely large number[3], f_2, f_3, f_4 are finite[4], and $p \gg 0$ means that p is non-infinitesimal[5].

(2) Premise: Reason demands to maximise expected utility.

(3) Conclusion: Reason demands to wager for God.

Formalisations of the Wager have to identify I mathematically in some proper superset[6] $S \supsetneq \mathbb{R}$ of the field of the reals.

Now, in order to clarify Premise 2, one needs to define what kind of choices the Wagerer is allowed to make. In this paper, we allow the Wagerer to base his/her decision to wager for or against God on a random event of some probability q. (For instance, by tossing a coin to determine what to wager for.) Such a strategy is called a *mixed* strategy of chance q. In his prized critique of Pascal's Wager, Hájek (2003) has demanded that mixed strategies should also be taken into account, because if mixed strategies yielded the same expected utility as wagering directly for God's existence, then the conclusion of Pascal's Wager (that reason demands to wager for God) would no longer hold. However, we will find that mixed strategies yield strictly

[3] I.e., $I > n$ holds for every $n \in \mathbb{N}$.

[4] I.e. $|f_2|, |f_3|, |f_4| \leq n$ for some $n \in \mathbb{N}$.

[5] I.e., there exists some $n \in \mathbb{N}$ such that $p > \frac{1}{n}$. Zero probabilities would make the argument invalid straightaway, and infinitesimal probabilities would require a sufficiently high utility of salvation in order to preserve the validity of the argument, cf. Oppy (1990) and Hájek (2003).

[6] For arbitrary sets R, S, we say that S is a superset of R if and only if R is a subset of S.

lesser expected utility than outright wagering for God's existence in suitable formalisations of S, thus meeting Hájek's (2003) challenge.

If one were to exclude the possibility of mixed strategies, the decision of the Pascalian Wagerer amounts to the choice of one of two continuum-size sets of lotteries — for each value for the probability p of God's existence, she has to choose between two possible lotteries (wagering for or against God).

As we do allow for mixed strategies, the Wagerer has to choose one continuum-size set of lotteries among a continuum of continuum-size set of lotteries. For, there is a continuum of possible lotteries — one for each value for the chance that he wagers for God — for each value for the probability p of God's existence.

Thus, Premise 2 can now be phrased as follows: Let $\langle \bar{p}, \bar{q} \rangle$ denote the lottery where the probability that the Christian God exists is $\bar{p}$ and the probability that the Wagerer actually wagers for Him is $\bar{q}$. Then, for every $\bar{p} \in (0, 1]$ (excluding infinitesimal probabilities $\bar{p}$, see Premise 1), a rational Wagerer must strictly prefer $\langle \bar{p}, 1 \rangle$ over $\langle \bar{p}, \bar{q} \rangle$ for any $\bar{q} < 1$.

This statement is exactly what the Pascalian must prove (in some formal setting) in order to justify Premise 2.

1.3. *Two concerns about Pascal's Wager*

Pascal's Wager faces at least two major challenges: (1) McClennen's decision-theoretic objection, and (2) Hájek's dilemma.

(1) McClennen (1994) points out that Premise 2, the rationality of maximising expected utility, lacks a decision-theoretic justification (such as the von Neumann-Morgenstern Theorem) since the Wagerer's utility function is allowed to take infinite values: For, the classical von Neumann-Morgenstern Theorem only says that a preference ordering on lotteries can be represented by a real-valued expected utility function if and only if the preference ordering has certain properties, among them continuity. Now, on the one hand, continuous preference-orderings are inconsistent with infinite utilities (which Pascal's Wager entails), and on the other hand, the Pascalian wants to allow for infinite (not just real-valued) expected utility functions. Hence, the Pascalian cannot justify Premise 2 through classical utility theory. Instead, a new expected utility theorem is needed in order to defend Premise 2.

(2) Hájek (2003) contends that there is a dilemma for any conceivable mathematical (re)formulation of the Wager: On the one hand, a historically faithful reading of Pascal's *Pensées* §233 demands that the

utility of salvation be reflexive under addition by real numbers.[7] On the other hand, the utility of salvation must be (strictly) irreflexive under multiplication by probabilities > 0,[8] in order to ensure that one can distinguish between the expected utility of outright wagering for God and mixed strategies (where the Wagerer only ends up wagering for God with some probability $p > 0$, cf. Duff 1986). Hence, one must find a convex linearly ordered set which contains the reals and has a maximum that is both reflexive under addition by reals and strictly irreflexive under multiplication by positive scalars < 1. However, Hájek thought that this is impossible: "[There are] no prospects for characterizing a notion of the utility of salvation that is reflexive under addition without being reflexive under multiplication by positive, finite probabilities" (Hájek 2003 [p. 49]).

1.4. *Outline of the argument*

In a recent paper, Bartha (2007) proposed a new formalisation of Pascal's Wager, based on generalised utility ratios, which addresses both McClennen's objection and Hájek's dilemma. The aim of this article is to demonstrate how McClennen's objection and Hájek's dilemma can also be addressed by means of *one-place* hyperreal[9]-valued utility functions; if one drops the requirement of reflexivity under addition, this approach can be simplified even further.

In particular, we shall prove:

(1) There is an expected-utility representation theorem for hyperreal utility functions: Every standard-definable, complete, transitive, independent and infinitesimally continuous preference relation can be represented by a hyperreal-valued affine utility function. (See Section 2, in particular the Hyperreal Expected Utility Theorem 3.)
(2) There are two candidates for a mathematical model of the Wagerer's utility function where the maximal utility is both reflexive under addition and irreflexive under multiplication by positive probabilities.

[7] This means that $x + I = I$ for all real numbers x. In Appendix B, we shall reexamine this claim and see that, in fact, irreflexivity under addition may be more in accordance with Pascal's theology.

[8] This means $qI < I$ for all $q \in [0, 1)$.

[9] The hyperreals — in the sense of Robinsonian (1966, 1996) nonstandard analysis — form a real-ordered, non-Archimedean field which is usually constructed as the ultrapower of the reals with respect to a non-principal ultrafilter. The non-Archimedean property entails that the hyperreals includes infinitesimals and infinitely large numbers — as well as all real numbers.

(See Subsection 3.3 and Appendix C.) In particular, there exists a superset $S_{\mathsf{RA-IM}}$ of the reals — motivated by the hyperreals — on which both convex combinations as well as a linear order can be defined in a mutually consistent manner, and the maximum of $S_{\mathsf{RA-IM}}$ is reflexive under addition without being reflexive under multiplication by positive probabilities. We shall also prove an expected-utility representation theorem, under fairly restrictive conditions, for $S_{\mathsf{RA-IM}}$-valued utility functions.

Hence, each of the challenges by McClennen and Hájek can be addressed separately. The combination of Hájek's dilemma and McClennen's critique is potentially troublesome for the Pascalian. However, we shall prove that under additional hypotheses on the Wagerer's metaphysical stance, the range of this utility function can even be chosen as the linearly-ordered convex set $S_{\mathsf{RA-IM}}$ whose maximum is both reflexive under addition and strictly irreflexive under multiplication, thus answering at the same time McClennen's objection and Hájek's dilemma. Besides, in an appendix to this paper, we shall argue that despite its philosophical merits and faithfulness to Pascal's statement in *Pensées* §233, the reflexivity under addition of the utility of salvation is in tension with other aspects of Pascalian theology.

We close the Introduction with three remarks about our use of hyperreal expected utilities.

In order to apply the Hyperreal Expected Utility Theorem to Pascal's Wager, we must assume that the Pascalian Wagerer has a completely defined preference relation over lotteries with arbitrary hyperreal chances. This does, of course, by no means entail that the Wagerer is assumed to assign infinitesimal probability to the existence of God — which would be inconsistent with Premise 1. It only means that the Wagerer is able to compare those lotteries where the probability for the event that he wagers for God while God does not exist is hyperreal (e.g. infinitesimal) with other lotteries.

Also, it has been argued that there is some "arbitrariness" in modelling subjective utility of salvation of a given human individual by some particular infinite hyperreal (cf. Hájek 2003 and Bartha 2007). The Hyperreal Expected Utility Theorem clarifies that this degree of freedom simply reflects an ubiquitous phenomenon in decision theory with cardinal preferences: Von Neumann-Morgenstern utility functions are only unique up to a positive factor and a shift by an additive scalar.

Finally, one might argue that modelling the Wagerer's maximal utility by an infinite hyperreal amounts to cutting human utility at a somewhat arbitrary level and thus represents a philosophical problem. This would then be another reason, in addition to the reflexivity under addition, why the Wagerer's utility function should be modelled as $S_{\mathsf{RA-IM}}$-valued rather than hyperreal-valued. However, whilst the Wagerer's personal salvation cannot be topped in utility by anything earthly, one should think that even a saved Wagerer's utility can be increased through the salvation of her loved ones. Therefore, there are good reasons to model the Wagerer's maximal utility through an infinite value which can be naturally improved upon, such as an infinite hyperreal. This argument also casts some additional doubt on the assumption of reflexivity under addition — in addition to the concerns expressed in Appendix B.

2. *The Hyperreal Expected Utility Theorem*

In order to justify Premise 2, i.e. the rationality of maximising expected utility whilst permitting infinite values in the utility function's range, a decision-theoretic argument is required. The classical decision-theoretic justification of the rationality of maximising expected utility invokes the expected-utility representation theorem of von Neumann and Morgenstern (1980) who "[re-construed] the utility concept" (Jeffrey 1992 [p. 171]). The theorem asserts that a certain set of rationality axioms (transitivity, completeness, continuity, independence) on an individual's preference ordering among lotteries is equivalent to the existence of an affine real-valued utility function which represents the preference ordering — whilst every affine function on the set of lotteries can be viewed as an expectation operator on the set of random utility values corresponding to the lotteries. In other words, a preference ordering among lotteries satisfies certain rationality axioms if and only if it is derived from some real-valued expected utility function. Unfortunately for the Pascalian, infinite utility values are outside the scope of von Neumann and Morgenstern's theorem.

However, as explained by Bartha (2007) [p. 10], reflexivity under multiplication (i.e. the axiom that $x \cdot I = I$ for all $x > 0$) is responsible for the inconsistency of infinite utilities with the hypotheses of the von Neumann-Morgenstern Theorem. Thus, since both infinite hyperreals and infinite surreal numbers[10] are (strictly) irreflexive under multiplication, there may be some hope to counter the first objection if I is a hyperreal or a surreal number.

To the present author, it is unclear how to develop a von Neumann-Morgenstern Theorem for *surreal* utilities and thus to respond to McClennen's objection through a surreal formalisation of the Wager.

[10] The surreal numbers — in the sense of Conway's (1966) *On numbers and games* — form a real-ordered, non-Archimedean field, which is essentially constructed as the set of Dedekind cuts of (signed) ordinals. This construction is parallel to the classical construction of the real numbers via signed Dedekind cuts except that one starts from the semi-ring of ordinals rather than from the semi-ring of natural numbers.

By means of Robinson's (1966, 1996) nonstandard analysis, we shall prove a von Neumann-Morgenstern Theorem for *hyperreal*-valued utility functions (*Hyperreal von Neumann-Morgenstern Theorem*, Theorem 1) and derive from there a rigorous decision-theoretic justification for the rationality of maximising hyperreal expected utility (*Hyperreal Expected Utility Theorem*, Theorem 3). Another non-Archimedean extension of the von Neumann-Morgenstern Theorem was established by Herzberg (2009).

Hence, if the utility of salvation in Pascal's Wager is modeled by positive infinite hyperreals, then nonstandard analysis provides a response to McClennen's objection.

The Hyperreal von Neumann-Morgenstern Theorem (Theorem 1) follows easily from the classical Expected Utility Theorem of von Neumann and Morgenstern. Indeed, if * denotes a nonstandard embedding[11] from the standard universe into the nonstandard universe, then one can prove the following equivalence theorem, which is an easy consequence of applying the so-called Transfer Principle to the standard Expected Utility Theorem of von Neumann and Morgenstern in Jensen's (1967) formulation. The Transfer Principle asserts that any formula $\varphi\left[a_1,\ldots,a_n\right]$ with bounded quantifiers and parameters $a_1,\ldots,a_n$ from the original standard universe is true if and only if the formula $\varphi\left[{}^*a_1,\ldots,{}^*a_n\right]$ is true. As an example, the Transfer Principle, when applied to the ordered field axioms for $(\mathbb{R},+,-,\times,\div,0,1,<)$, proves that $({}^*\mathbb{R},{}^*+,{}^*-,{}^*\times,{}^*\div,{}^*0,{}^*1,{}^*<)$ is an ordered field, too.

In the statement of the following Theorem 1 (Hyperreal von Neumann-Morgenstern Theorem), we employ the notion of internality in the sense of nonstandard analysis. *Internal* means to be an element of the *-image of a standard set — and this is also equivalent to being definable by a formula

[11] This is an embedding of the superstructure over the reals into the superstructure of a non-Archimedean model of the ordered field of the reals — usually obtained via an ultrafilter construction — which satisfies the *Transfer Principle*, the *Countable Saturation Principle* and the *Internal Definition Principle*. (The superstructure $V(M)$ over some set M is defined via $V_0 = M$, $V_{n+1}(M) = V_n(M) \cup \mathcal{P}(V_n(M))$ for all $n \in \mathbb{N}_0$ and $V(M) = \bigcup_{n=0}^{\infty} V_n(M)$.) The Transfer Principle states the following: Any first-order proposition $\phi[a_1,\ldots,a_n]$ of set theory that treats the reals as atoms and has only bounded quantifiers (and parameters $a_1,\ldots,a_n$ from the superstructure over the reals), holds if and only if the proposition $\phi\left[{}^*a_1,\ldots,{}^*a_n\right]$, sometimes also referred to as ${}^*\phi[a_1,\ldots,a_n]$ (the *-image of the formula $\phi[a_1,\ldots,a_n]$), holds in the nonstandard universe. The Countable Saturation Principle states that any decreasing countable chain of nonempty *internal* sets, i.e. sets that are elements of *-images of (standard) sets, must have a nonempty intersection. The Internal Definition Principle says that any subset of an internal set that is defined via a set-theoretic formula with internal parameters is itself internal. There are even definable (over Zermelo-Fraenkel set theory with the Axiom of Choice) nonstandard extensions of the superstructure over the reals, cf. Herzberg (2008a, 200b).

of set theory which treats the reals as atoms and has internal, e.g. standard, parameters.

A *-*linear space* is an internal linear space over the field $^*\mathbb{R}$. Furthermore, an internal subset X of a *-linear space is *-*convex* if and only if $px + (1 - p)y \in X$ for all $x, y \in X$ and $p \in {}^*[0, 1]$. Finally, an internal function $U : X \to {}^*\mathbb{R}$, defined on some *-convex set X is called *-*affine* if and only if $U(px + (1-p)y) = pU(x) + (1-p)U(y)$ for all $x, y \in X$ and $p \in {}^*[0, 1]$. Note that these definitions are consistent with the terminology regarding *-images of formulae in Footnote 11, when applied to the formal definitions of being a linear space or a convex set or an affine function; this consistency is crucial for the proof of the Theorem which relies on the use of the Transfer Principle.

Also, $^*(0, 1]$ and $^*(0, 1)$ denote the sets of hyperreals x satisfying $0 < x \leq 1$ and $0 < x < 1$, respectively. This definition, again, is consistent with the Transfer Principle outlined in Footnote 11.

Theorem 1: (Hyperreal von Neumann-Morgenstern Theorem) *Let X be an internal*[12] **-convex subset of a *-linear space, and let $\preceq$ be an internal binary relation $\subseteq X \times X$. There exists a *-affine function $U : X \to {}^*\mathbb{R}$ such that*

$$U(x) \leq U(y) \iff x \preceq y$$

holds for all $x, y \in X$ if and only if $\preceq$ possesses all of the following properties:

(1) Completeness. *For all $x, y \in X$, either $x \preceq y$ or $y \preceq x$.*

(2) Transitivity. *For all $x, y, z \in X$ with $x \preceq y$ and $y \preceq z$, one has $x \preceq z$.*

(3) Infinitesimal Continuity. *For all $x, y, z \in X$ with $x \prec y \prec z$,*[13] *there exist hyperreals $p, q \in {}^*(0, 1)$ such that*

$$px + (1-p)z \prec y \prec qx + (1-q)z.$$

(4) Independence. *For all $x, y, z \in X$ and every $p \in {}^*(0, 1]$, the relation $x \preceq y$ is equivalent to $px + (1-p)z \preceq py + (1-p)z$.*

[12] Since all models in applications of nonstandard analysis are (standard parts) of internal objects (otherwise, it is impossible to obtain any information on these objects via the Transfer Principle), the requirement of internality is even in general not a relevant restriction. (Cf. also Herzberg 2007.) But, as we shall see later on, we can even circumvent the notion of internality for the purposes of this article.

[13] For any $x, y \in X$, we write $x \prec y$ if $x \preceq y$ but $y \not\preceq x$. If $\preceq$ is complete, then $x \prec y$ if and only if $y \not\preceq x$.

Proof. See Appendix A. □

Herein, the interpretation of $x \preceq y$ should be read as 'x is not preferred over y' or 'either y is preferred over x or they are equivalent'.

The first two properties are just the weak order axioms[14].

When we compare Infinitesimal Continuity with ordinary continuity[15] of binary relations on convex spaces, we find that $(0, 1)$ has been replaced by ${}^*(0, 1)$. In particular, we may choose p infinitesimally close to 1 and q infinitesimally close to 0 in the definition of Infinitesimal Continuity. This corresponds to an *infinitesimal perturbation* $x' = px + (1 - p)z$ of x and an infinitesimal perturbation $z' = qx + (1 - q)z$ of z. In other words, Infinitesimal Continuity asserts the existence of a hyperreal (possibly infinitesimal) perturbation, while ordinary continuity asserts the existence of a real, non-infinitesimal perturbation. Hence, Infinitesimal Continuity is a much weaker condition than the ordinary continuity axiom in the sense of Jensen (1967).

Our Independence axiom says that a preference relation $x \preceq y$ is preserved if x and y are both mixed with another lottery and the same, possibly hyperreal, probability p. Whilst this is a stronger axiom than ordinary independence in the sense of Jensen (for, it replaces $(0, 1]$ by the larger set ${}^*(0, 1]$ in the definition of independence for binary relations on convex spaces), it is clearly the natural extension of the ordinary independence axiom to lotteries with hyperreal chances.

Theorem 2: (Internal Expected Utility Theorem) *Let W be an internal finite-dimensional linear space over the field ${}^*\mathbb{R}$ of the hyperreals, let $x_1, \ldots, x_m \in W$, and consider $Y = \{\sum_{i=1}^m p_i x_i \;:\; p_1, \ldots, p_m \in {}^*[0, 1], \quad \sum_{i=1}^m p_i = 1\}$. Let $\preceq$ be an internal binary relation $\subseteq Y \times Y$. If the relation $\preceq$ satisfies all the axioms of (1) Completeness, (2) Transitivity, (3) Infinitesimal Continuity and (4) Independence, then there exist hyperreals $u_1, \ldots, u_m$ such that*

$$\sum_{i=1}^{m} p_i x_i \preceq \sum_{i=1}^{m} q_i x_i \iff \sum_{i=1}^{m} p_i u_i \leq \sum_{i=1}^{m} q_i u_i$$

whenever $p_1, q_1, \ldots, p_m, q_m \in {}^*[0, 1]$ with $\sum_{i=1}^m p_i = 1$ and $\sum_{i=1}^m q_i = 1$.

Proof. See Appendix A. □

[14] In the terminology of Jensen 1967, the first two axioms characterise *complete preorderings*.

[15] The Continuity axiom is also known as the Archimedean property (cf. e.g. Jensen 1967).

The hypothesis of internality of the relation $\preceq$ may be replaced by a stronger, but conceptually more accessible assumption: standard-definability under a basis choice. $\preceq$, a relation on some subset Y of an n-dimensional linear space W over ${}^*\mathbb{R}$ ($n \in \mathbb{N}$) is said to be *standard-definable under a basis choice* if there exist

- an isomorphism $\psi : W \simeq {}^*\mathbb{R}^n$ (a bijective map that commutes both with addition and with multiplication by *hyperreals*) and
- a first-order formula $\varphi(x_1, \dots, x_n, y_1, \dots, y_n)$ in which the canonical extensions (*-images) of maps from $\mathbb{R}^M$ to $\mathbb{R}^N$ (for any $M, N \in \mathbb{N}$), as well as equality '$=$' and the order relation '$<$' may occur, with free variables $x_1, \dots, x_n, y_1, \dots, y_n$ and constants from ${}^*\mathbb{R}$

such that

$$\forall v, w \in Y \qquad (x \preceq y \Longleftrightarrow \varphi[\psi(v), \psi(w)])$$

(in other words: $\preceq = \left\{\langle v, w\rangle \in Y^2 \;:\; \varphi[\psi(v), \psi(w)]\right\}$).

In particular, $\varphi(x_1, \dots, x_n, y_1, \dots, y_n)$ may be any formula from the language of ordered rings[16]. Since, however, the theory of real-ordered fields admits quantifier elimination (which can, for instance, be proved via the so-called Tarski-Seidenberg Principle, cf. e.g. Bochnak, Coste, Roy 1998 [Proposition 5.5.2] or Marker 2002 [Theorem 3.3.15]), this would simply mean that there are polynomials $f_{i,j}$ ($i \leq M$, $j \leq N$) in the variables $X_1, \dots, X_n, Y_1, \dots, Y_n$ with coefficients from ${}^*\mathbb{R}$ such that

$$\preceq = \bigcup_{i=1}^{M} \bigcap_{j=1}^{N} \left\{\langle v, w\rangle \in Y^2 \;:\; f_{i,j}(\psi(v), \psi(w)) \geq 0\right\}.$$

Note that whenever $\chi : {}^*\mathbb{R}^n \simeq {}^*\mathbb{R}^n$ is an automorphism of the linear space ${}^*\mathbb{R}^n$ over ${}^*\mathbb{R}$ and $f : {}^*\mathbb{R}^{2n} \to {}^*\mathbb{R}$ is defined via canonical extensions of maps from $\mathbb{R}^M$ to $\mathbb{R}^N$ as well as constants from ${}^*\mathbb{R}$, then $f \circ \chi$ can also be defined that way. (The reason is that χ itself is definable, since it is a linear map from a finite-dimensional linear space onto itself.) Hence, the choice of ψ is irrelevant: It can be replaced by $\chi \circ \psi$ and thus by an arbitrary other isomorphism between W and ${}^*\mathbb{R}^n$ (as linear spaces over ${}^*\mathbb{R}$).

Every relation on an internal linear space that is standard-definable under a basis choice is internal; the converse, however, is not true.

This new concept of standard-definability under a basis choice allows to us state the following Theorem, which does no longer invoke the concept of

[16] The operations in the language of ordered rings are addition, subtraction and multiplication; the relations in this language are equality '$=$' and the order relation '$<$'. Cf. e.g. Marker 2002.

internality, as an immediate corollary to Theorem 2. This theorem affirms a conjecture by Sobel (1996).

Theorem 3: (Hyperreal Expected Utility Theorem) *Let W be a finite-dimensional linear space over the field ${}^*\mathbb{R}$ of the hyperreals, let $x_1, \ldots, x_m \in W$ and suppose $Y = \{\sum_{i=1}^m p_i x_i \,:\, p_1, \ldots, p_m \in {}^*[0,1], \quad \sum_{i=1}^m p_i = 1\}$ (the convex hull of $x_1, \ldots, x_m$ over ${}^*\mathbb{R}$). Let $\preceq$ be a binary relation $\subseteq Y \times Y$ and assume $\preceq$ to be standard-definable under a basis choice. If the relation $\preceq$ on Y satisfies all the axioms of (1) Completeness, (2) Transitivity, (3) Infinitesimal Continuity and (4) Independence, then there exist hyperreals $u_1, \ldots, u_m$ such that*

$$\sum_{i=1}^m p_i x_i \preceq \sum_{i=1}^m q_i x_i \Longleftrightarrow \sum_{i=1}^m p_i u_i \leq \sum_{i=1}^m q_i u_i$$

whenever $p_1, q_1, \ldots, p_m, q_m \in {}^*[0,1]$ with $\sum_{i=1}^m p_i = \sum_{i=1}^m q_i = 1$.

Proof. See Appendix A. □

Note that the statement of the Hyperreal Expected Utility Theorem (Theorem 3) does not involve the notion of an internal set any longer — in contrast to, e.g. the Hyperreal von Neumann-Morgenstern Theorem (Theorem 1).

The Hyperreal Expected Utility Theorem shows that expected hyperreal-valued utility functions represent preference orderings among lotteries based on a finite set of pure outcomes and nonstandard probabilities — provided that we impose certain natural conditions which are, apart from definability or internality, the direct analogues (the *-images) of the original von Neumann-Morgenstern conditions.

Hyperreal-valued utility functions have also been studied by Skala (1974), Kannai (1992) and Lehmann (2001), in chronological order. Lehmann's (2001) article is also concerned with nonstandard von Neumann-Morgenstern utility functions, but only allows for standard probabilities, which leads to a different representation theorem. Kannai (1992) shows that every convex preference ordering admits a concave utility function, provided one chooses an appropriate nonstandard extension of the reals as the range of the utility function. Skala's (1974) results are, to the author's knowledge, the most relevant in the literature to the subject of this article. Skala, in refuting Fishburn's (1971) impression that game theory with non-Archimedean utilities is "rather barren", constructs utility functions that represent *mean groupoids*. A mean groupoid is a generalisation of a convex set on which a complete transitive order is defined. This more general approach leads, however, when applied to our setting, to a significantly weaker result than our Hyperreal von Neumann-Morgenstern Theorem. In particular, Skala's (1974) representation theorem [Theorem 9] only works in one direction. More importantly, general weighted sums of pure outcomes as considered in the Hyperreal Expected Utility Theorem, are even undefined in the mean groupoid setting.

3. *Resolving Hájek's dilemma*

3.1. *Reflexivity under Addition and Pascal's soteriology*

In Subsection 1.3, we mentioned that Hájek (2003) reads Pascal as assuming that the reward of salvation, I, is reflexive under addition, i.e. it does not change when a positive utility is added onto it:

$$\forall x \in \mathbb{R} \qquad x + I = I$$

(which may also be read as a definition of addition of reals onto I).

As an example, consider the most simple contemporary formalisation of Pascal's Wager — where the Wagerer's utility function takes values in the set of the extended real numbers $\mathbb{R} \cup \{\pm\infty\}$ with their natural ordering. The utility of salvation is $I = +\infty$. Recalling the convention that $x + \infty = +\infty$ for every $x \in \mathbb{R}$, the condition of Reflexivity under Addition is clearly satisfied.

It should be noted at this point that there are good theological reasons not to interpret Pascal's *Pensées* §233 as stating that I should be reflexive under addition.[17] Furthermore, we have already argued in the introduction that even a saved Wagerer's utility could be improved upon through the salvation of others. This would then particularly favour a model of the wager where I is a hyperreal or surreal number.

Another argument which Hájek (2003) gave against modelling the utility of salvation by a particular positive infinite hyperreal I was that the choice of I would be somewhat "arbitrary". However, the "arbitrariness" in the

[17] For, reflexivity under addition directly contradicts a major and widely-held thesis in Biblical soteriology (in particular in Roman Catholicism, but by far not limited to it), viz. the belief that there is some hierarchy in Heaven: Not all of those who are saved will *a priori* receive the same reward on Judgement Day. In particular, it might be possible that Pascal himself shared that opinion: In Appendix B.2 we shall argue that Pascal seems to accept the soteriological claims of the New Testament in their literal meaning. These speak plainly about a hierarchy in Heaven, and hence of a non-trivial ordering of the utility associated with salvation. (Moreover, a distinctive of Jansenist doctrine of justification is salvation by grace alone and, at the same time, a hidden judgement.) Prior to these deliberations, in Appendix B.1, we shall reconsider Hájek's argument that Pascal viewed the utility of the saved as reflexive under addition and discuss some of the questions that it raises.

choice of I only reflects the "arbitrariness" in the choice of the hyperreal expected utility function U, which is only unique up to S-continuous monotone *-affine transformations, as a closer look at the Hyperreal Expected Utility Theorem reveals. In other words, the "arbitrariness" in the choice of I is no different from the "arbitrariness" of the classical expected-utility functions of von Neumann and Morgenstern, which are also only unique up to monotone affine transformations. (One could even go beyond this and argue that *all* cardinal utility functions are somewhat arbitrary, as they always allow for scaling.) Hence, the arbitrariness of the choice of I as some positive infinite hyperreal is conceptually as innocent as the "arbitrariness" of choosing a classical von Neumann-Morgenstern expected utility function. (In some sense it is simply a matter of scaling the range of a cardinal utility function.) Moreover, one should observe that in the setting of Pascal's wager, such a kind of arbitrariness may even be desirable for philosophical reasons, since in the setting of Pascal's wager it is God who determines the exact utility level that a given individual believer receives. All that the Wagerer can deduce from the Biblical revelation (and Pascal's interpretation thereof) is that this utility level will be infinite, but certainly not its exact size.

Despite these points, Hájek's interpretation of *Pensées* §233 (as demanding reflexivity under addition) has some appeal. Therefore, we shall accept the tension between this interpretation and other aspects of Pascal's theology for a moment and provide a solution to Hájek's dilemma.

3.2. *Irreflexivity under Multiplication*

In our presentation of Pascal's Wager (see Subsection 1.2), the Wagerer is allowed to adopt mixed strategies: The Wagerer may base his decision to wager for or against God on some random event of non-infinitesimal probability $q > 0$ (e.g. through dicing, tossing coins etc.). Such a strategy will be called *mixed strategy of chance* q.

In order to apply Premise 2 in that setting, the Pascalian must prove that the expected utility of any mixed strategy of chance $q < 1$ is less than the the expected utility of outright wagering for God (the "mixed strategy" of chance 1). As we will see, this is only possible if one has

$$\forall q \in [0,1) \qquad qI < I,$$

an axiom called (Strict) Irreflexivity under Multiplication.

For, suppose there existed some $q < 1$ such that $qI = I$, and consider a mixed strategy of chance q. Then the conditional expected utility, conditioned with respect to God's existence, of the mixed strategy would be $qI + (1-q)f_3 = I + (1-q)f_3$. If f_3 is non-negative, this would be at least as much as I, the conditional expected utility associated with outright wagering for God. Hence, if there was some $q < 1$ such that $qI = I$, then the expected utility of the mixed strategy of chance q would always be at least as much as expected utility of outright wagering for God. (In other words, if such a q exists, then the corresponding mixed strategy would be co-optimal. Put more bluntly: Flipping some suitably biased coin about whether to wager for or against God maximises utility equally well as faith proper.)

This reasoning in favour of (Strict) Irreflexivity under Multiplication is due to Duff (1986) and has been reiterated by Hájek (2003) as well as Bartha (2007); it must be taken into account by every formalisation of the Wager which allows the Wagerer to adopt a mixed strategy.

As Hájek (2003) and Bartha (2007) noted, any formalisation of the Wager where I is a hyperreal or surreal number automatically satisfies Strict Irreflexivity under Multiplication (and hence is not susceptible to the reasoning above). We shall not replicate Hájek's argument here, since it appears to tacitly assume that $f_2 > f_4$. Instead, we give a new proof. Recall, for this sake, that whenever J is a surreal or hyperreal number the implication

(1) $$J \text{ infinite} \Rightarrow \forall r \gg 0 \quad rJ \text{ infinite}$$

(wherein, as before, $a \ll b$ means that $b-a$ is positive and non-infinitesimal) holds for all J. From here, we can readily deduce that regardless of the exact values for f_2, f_3, f_4 (provided they are finite) mixed strategies always carry an infinitely lesser reward than outright wagering for God. Indeed, observe that choosing to wager for God yields expected utility $pI + (1-p)f_2$, whilst choosing to Wager for God with some probability q yields expected utility $p\,(qI + (1-q)f_3) + (1-p)\,(qf_2 + (1-q)f_4)$. The difference between the former and the latter value is

(2) $$p(1-q)\,(I - f_3) + (1-p)(1-q)\,(f_2 - f_4)\,.$$

Now, whenever f_2, f_3, f_4 are finite and $p \gg 0$ as well as $q \ll 1$, the first addend is always positive infinite (due to implication (1) applied to $J = I - f_3$ and $r = p(1-q)$) whilst the second addend is finite. Hence the difference in expected utility between outright wagering for God and a mixed strategy is always positive, even infinite. Therefore, mixed strategies where the probability of wagering against God is non-infinitesimal always carry a lesser reward if I is some positive infinite hyperreal or surreal utility. Hence, Strict Irreflexivity under Multiplication holds whenever I is an infinite hyperreal or surreal number. (Hájek's original argument tacitly assumed that $f_2 > f_4$ and proved that the difference in expected utility between outright wagering for God and a mixed strategy of chance q is strictly decreasing in q and zero for $q = 1$, which proves that mixed strategies are suboptimal.)

However, neither hyperreals nor surreals are reflexive under addition in ${}^*\mathbb{R}$ or No (the field of surreal numbers, i.e. the Dedekind completion of the field generated by the semi-ring of ordinals), respectively. Thus, we have yet to show that there exists a set S of utilities of the Wagerer in which the utility of salvation satisfies both Reflexivity under Addition and Strict Irreflexivity under Multiplication.

This will be accomplished in Subsection 3.3: We will construct a convex linearly ordered set $S := S_{\mathsf{RA-IM}}$ containing the reals which does satisfy both Reflexivity under Addition and Strict Irreflexivity under Multiplication.[18] Moreover, even Premise 2 can be defended for $S_{\mathsf{RA-IM}}$-valued utility functions (as we saw in the discussion of Corollary 6) under additional hypotheses on the Wagerer's soteriological presuppositions. Hence, formalising Pascal's Wager through an $S_{\mathsf{RA-IM}}$-valued utility function allows to respond to McClennen's (1994) decision-theoretic objection and Hájek's dilemma at the same time.

3.3. *A model for S with Strict Irreflexivity under Multiplication and Reflexivity under Addition for all infinite utilities*

We shall construct a linearly-ordered set $S \supsetneq \mathbb{R}$ such that the maximum $I \in S$, the utility of salvation, has the property of Irreflexivity under Multiplication and Reflexivity under Addition. Furthermore, taking convex combinations of elements of S will be defined in a way that is consistent with the linear order on S. Taking convex combinations with 0 will implicitly define an operation of multiplication by elements of $[0,1]$, furthermore it will define an operation of addition for some pairs of elements of S. These operations are, as we will see, associative as well as commutative and satisfy the law of distributivity. Hence the set S defined in this Subsection is a rather well-behaved model for the set of possible utilities of a Pascalian Wagerer.

Irreflexivity under Multiplication and Reflexivity under Addition for I imply, via the law of distributivity (in the form $q(x+y) = qx+qy$ for $q \in (0,1]$ and $x, y \in S$), Reflexivity under Addition for qI. Indeed,

$$\forall x \in \mathbb{R} \quad \forall q \in (0,1] \qquad qx + qI = q(x+I) = qI,$$

hence (inserting $x = y/q$): $y + qI = qI$ for all $y \in \mathbb{R}$. Thus, $qI \in S$ must be reflexive under addition for all $q \in (0,1]$.

[18] In Appendix C, we construct another linearly ordered superset of the reals which satisfies Reflexivity under Addition and Strict Irreflexivity under Multiplication, but the ordering is inconsistent with mixing the utilities of mixed strategies. Also, it is not clear how to defend Premise 2 when S is the set constructed in Appendix C.

Hence a natural candidate for S is

$$S := S_{\mathsf{RA-IM}} := \mathbb{R} \cup \{qI \ : \ q \in (0,1]\}\,. \tag{3}$$

So, in addition to the maximal utility $I := 1I$, there is a continuum of other infinite utilities, each denoted by qI for some $q \in (0,1)$. (In all of this subsection, $\mathbb{R}$ and its subintervals may be replaced by ${}^*\mathbb{R}$. This would allow to consider nonstandard probabilities as well.)

In order to develop decision theory under risk with this set of utilities, we need to be able to form convex combinations of elements of S. For $x, y \in \mathbb{R} \subset S$, convex combinations shall be defined in the ordinary way. For $x \in \mathbb{R}$, $q \in (0,1]$ and $r \in [0,1]$, we define

$$(1-r)x + r(qI) = (rq)I$$

(with the convention — typical for probability theory — that $0I = 0$), in line with associativity of multiplication and Reflexivity under Addition for $(rq)I$. Finally, for $q, q' \in (0,1]$ and $r \in [0,1]$ we set

$$(1-r)qI + r(q'I) = \left((1-r)q + rq'\right) I.$$

This implicitly defines addition for some pairs of elements in S (viz. for those $\langle x, y\rangle \in S^2$ where $x \in \mathbb{R}$ or $y \in \mathbb{R}$ or $\langle x, y\rangle = \langle qI, q'I\rangle$ wherein $q + q' \in [0,1]$ with $q + q' = 1$) is not closed under addition, e.g. $I + I$ is undefined), and it also defines multiplication by elements of $[0,1]$ (simply take $y = 0$ as the second element of a convex combination). It is an easy exercise to check that the law of distributivity holds, and that both multiplication by elements of $[0,1]$ and addition are associative as well as commutative.

Moreover, one can extend the linear order $<$ on the reals to S by setting

$$\forall q \in (0,1] \quad \forall x \in \mathbb{R} \qquad qI > x$$

(thus making each qI infinite and hence S non-Archimedean) and

$$qI < rI \Leftrightarrow q < r$$

for all $q, r \in (0,1]$.

The strict ordering $<$ is preserved by multiplication by elements of $(0,1]$, i.e.

$$\forall x, y \in S \quad \forall r \in (0,1] \qquad x < y \Rightarrow rx < ry$$

and the weak ordering $\leq$ is preserved by addition:

$$\forall x, y, z \in S \qquad x < y \Rightarrow x + z \leq y + z.$$

(Note that if $x < y$ are reals, then $x + qI = qI = y + qI$, hence addition by qI, for any $q \in (0,1]$ does not preserve the strict ordering.) The strict ordering $<$ is preserved, however, by adding a real.

These observations yield that forming convex combinations is consistent with the strict ordering $<$:

$$\forall x, y \in S \quad \forall r \in (0,1) \qquad x < y \Rightarrow x < rx + (1-r)y < y.$$

(One can easily prove this directly as well: The right-hand side obviously holds whenever $x, y \in \mathbb{R}$ or both $x = qI$ and $y = q'I$ for some $q, q' \in (0,1]$. It also holds whenever $x \in \mathbb{R}$ and $y = qI$ for some $q, q' \in (0,1]$, since then $rx + (1-r)y = (1-r)qI$ is infinite, but dominated by $y = qI$.)

Finally, mixed strategies do not yield optimal utility in this setting: The expected utility of wagering for God with probability $q \in (0,1)$ equals $p\,(qI + (1-q)f_3) + (1-p)\,(qf_2 + (1-q)f_4)$ whilst the expected utility of outright wagering for God is $pI + (1-p)f_2$, and, as we see from expression (2), the difference between the two expected utilities is

$$(1-q)\,(p\,(I - f_3) + (1-p)\,(f_2 - f_4))\,.$$

Recalling that f_2, f_3, f_4 are finite and in light of the Reflexivity under Addition for I and pI, we obtain that the expected utility of wagering for God with probability $q \in (0,1)$ is $(1-q)pI$, which is strictly less than I.

Now we shall provide a decision-theoretic foundation for the use of $S_{\mathsf{RA-IM}}$-valued utility functions. In the following theorem, Y is a linear space, $B \subseteq Y$ a subset, $x_1 \in B$, $B_0 := B \setminus \{x_1\}$, whilst X and X_0 denote the convex hulls of B and B_0, respectively.

Theorem 4: *Let $\preceq$ be a binary relation on X, and suppose $\preceq$ is complete, transitive, continuous and independent. Assume $x_1 \succ x_0 \sim y_0$ for all $x_0, y_0 \in B$. Then, there exists a function $\bar{U} : X \to S_{\mathsf{RA-IM}}$ such that*

$$\forall p \in [0,1] \quad \forall x_0 \in X_0 \qquad \bar{U}\,(px_1 + (1-p)x_0) = 1 + pI$$

(thus $\bar{U}\,(x_0) = 1$ and $\bar{U}\,(px_1) = pI$ for all $p \in (0,1]$, $x_0 \in X_0$) and

$$\forall x, y \in X \qquad x \preceq y \Longleftrightarrow \bar{U}(x) \leq \bar{U}(y).$$

The following lemma is used in the proof of Theorem 4.

Lemma 5: *Let $\preceq$ be a binary relation on X_0 that is complete, transitive and independent. Suppose $x \sim y$ for all $x, y \in B_0$. Then $x \sim y$ for all $x, y \in X_0$.*

(The proofs can again be found in Appendix A.)

One of the advantages of choosing $S = S_{\mathsf{RA-IM}}$ is that the infinite value I is not specified any further. It is simply an entity outside $\mathbb{R}$ for which convex combinations with reals, multiplication with scalars between 0 and 1 and also an order relation can be naturally defined (as outlined above). For this reason, no charges of "arbitrariness" can be brought against using $S = S_{\mathsf{RA-IM}}$, while the axioms used to characterise the relations among $S = S_{\mathsf{RA-IM}}$ (in particular the relation of I to $\mathbb{R}$, e.g. the transitivity rule, the axioms of irreflexivity under multiplication and reflexivity under addition etc.) are still mathematically natural or philosophically necessary.

4. *Application to Pascal's Wager*

In the previous two sections we have refuted McClennen's point about decision theory with infinite utilities and (for special cases) solved Hájek's dilemma in general terms. We shall now apply this to the Pascal's Wager. We start with a corollary to the Hyperreal Expected Utility Theorem. All proofs can be found in Appendix A.

If $x_1, \ldots, x_m \in {}^*\mathbb{R}^n$, then the *convex hull of* $x_1, \ldots, x_m$ *over* ${}^*\mathbb{R}$ is the set

$$\left\{ \sum_{i=1}^{m} p_i x_i \;:\; p_1, \ldots, p_m \in {}^*[0,1], \quad \sum_{i=1}^{m} p_i = 1 \right\},$$

and the convex hull of $\{x_1, \ldots, x_m\}$ over $\mathbb{R}$ is defined as

$$\left\{ \sum_{i=1}^{m} p_i x_i \;:\; \sum_{i=1}^{m} p_i = 1, \quad p_1, \ldots, p_m \in [0,1] \right\}.$$

Corollary 6: *Let $x_1, \ldots, x_m \in {}^*\mathbb{R}^n$, let V and Y be the convex hulls of $x_1, \ldots, x_m$ over $\mathbb{R}$ and ${}^*\mathbb{R}$, respectively, and let $\preceq$ be an internal binary relation $\subseteq Y \times Y$ that satisfies the axioms of Completeness, Transitivity, Infinitesimal Continuity and Independence (cf. Theorem 1). Then, the restriction of $\preceq$ to V is transitive, complete and independent. Furthermore,* *there are* $f_1, \ldots, f_m \in {}^*\mathbb{R}$ *such that*

$$\sum_{i=1}^{m} p_i x_i \preceq \sum_{i=1}^{m} q_i x_i \Longleftrightarrow \sum_{i=1}^{m} p_i f_i \leq \sum_{i=1}^{m} q_i f_i \tag{4}$$

whenever $p_1, q_1, \ldots, p_m, q_m \in {}^*[0,1]$ *with* $\sum_{i=1}^{m} p_i = 1$ *and* $\sum_{i=1}^{m} q_i = 1$.

If, moreover, $x_1 \succ x_2 \sim \cdots \sim x_m$ [19]*, then* $f_2 = \cdots = f_m$ *can be any hyperreal and* f_1 *can be any hyperreal* $> f_2$ *(e.g.* f_1 *positive infinite,* $f_2 = \cdots = f_m = 1$*).*

Let us now apply the Corollary 6 to Pascal's Wager. $n = 2$ and $m = 4$. Recall that $\langle \bar{p}, \bar{q} \rangle$ is the lottery described in Subsection 1.2, where $\bar{p}$ is the subjective probability for the existence of the Christian God and $\bar{q}$ is the probability that the Wagerer chooses to wager for Him. Hence, for the rest of this section:

(1) $x_1 = \langle 1, 1 \rangle$ represents the pure outcome where the Christian God exists and is wagered for.
(2) $x_2 = \langle 0, 1 \rangle$ represents the pure outcome where the Christian God does not exist, but is nevertheless wagered for.
(3) $x_3 = \langle 1, 0 \rangle$ represents the pure outcome where the Christian God exists, but is wagered against.
(4) $x_4 = \langle 0, 0 \rangle$ represents the pure outcome where the Christian God does not exist and is wagered against.

We now rephrase Corollary 6 in non-technical terms: Whenever the Wagerer's preference relation is

- transitive
- complete (on the space of lotteries with hyperreal chances),
- unaffected by infinitesimal perturbations (Infinitesimal Continuity),
- unaffected by mixing with other lotteries (Independence), and
- internal, e.g. definable through standard functions with hyperreal parameters,

there are cardinal utilities $\{f_1, \ldots, f_4\}$ associated with the four pure outcomes $x_1, \ldots, x_4$, and for any two lotteries $\sum_{i=1}^{4} p_i x_i$ and $\sum_{i=1}^{4} q_i x_i$, the first lottery is not preferred over the first if and only if the expected utility from the first lottery ($\sum_{i=1}^{4} p_i f_i$) is less than or equal to the expected utility from the second lottery ($\sum_{i=1}^{4} q_i f_i$). If we interpret the assumptions on the Wagerer's preference relation as rationality axioms, then we obtain indeed that reason demands the maximisation of expected utility.

[19] We write $x \sim y$ if both $x \preceq y$ and $y \preceq x$.

In particular, outright wagering for God is strictly preferable to any mixed strategy of chance $\bar{q} \ll 1$:

Corollary 7: *Assume the hypotheses of Corollary 6. Suppose, in addition, that $n = 2$, $m = 4$, that f_1 is a positive infinite hyperreal, and that f_2, f_3, f_4 are finite. Then for all $\bar{p}, \bar{q} \in {}^*[0, 1]$ such that both $\bar{p}$ and $1 - \bar{q}$ are non-infinitesimal,*

$$\langle \bar{p}, \bar{q} \rangle \prec \langle \bar{p}, 1 \rangle.$$

Before we consider the special case where $f_2 = f_3 = f_4$, let us note the following points about the use of hyperreals in formalising Pascal's Wager:

- The axiom of Completeness requires the preference relation to be *defined* between lotteries with hyperreal (including infinitesimal) chances for each of the pure outcomes (e.g. the event that the Wagerer wagers for God and the existence of God). This has no consequences whatsoever for the Wagerer's subjective probability for the existence of God; it does by no means imply that the Wagerer assigns a non-real or even infinitesimal probability to the existence of the Christian God (which would contradict Premise 1). In applying the Hyperreal Expected Utility Theorem to the Pascalian Wagerer, we merely require him to have a preference realation that is defined over lotteries with hyperreal — including infinitesimal — subjective probabilities for the existence of God and for the event that the Wagerer actually wagers for Him.
- Corollary 6 is a consequence of the Hyperreal Expected Utility Theorem and hence ultimately of the classical Expected Utility Theorem of von Neumann and Morgenstern. Therefore, as was already discussed in Subsection 3.1, the "arbitrariness" of modelling $f_1 = I$, the subjective utility of salvation of a given human individual, by some particular infinite hyperreal (an objection of Hájek 2003 and Bartha 2007 against the use of hyperreals as values for I) is due to the fact that von Neumann-Morgenstern utility functions are only unique up to a positive factor (and a shift by an additive scalar). Hence, this "arbitrariness" merely reflects a typical property of decision-theoretic cardinal utility functions, and moreover there are sound theological reasons for accepting such an "arbitrariness" in order to save the transcendence of God's judgement.

  However, a Wagerer might nevertheless not be prepared to impose such an arbitrary cut-off on her utility in her model. Therefore, we will now construct, under additional assumptions on the Wagerer's preferences, an expected-utility representation of the Wagerer's preference ordering where the Wager's utility function takes values in the set $S = \mathbb{R} \cup \{rI \,:\, r \in (0,1]\}$, wherein I is conceived as representing an infinite number. (This superset of the reals can be closed under convex combinations in a natural way, and in can be linearly ordered in a manner that is consistent with convex combinations.)

If $f_2 = f_3 = f_4 = 1$ and f_1 is a positive infinite hyperreal I and the Wagerer's preference ordering $\preceq$ satisfies the hypotheses of Corollary 6, then the following Corollary 8 shows that $\preceq$ can be represented through an $S_{\mathsf{RA-IM}}$-valued utility function. We recall that $S_{\mathsf{RA-IM}}$ is the set $\mathbb{R} \cup \{rI \,:\, r \in (0,1]\}$, on which a linear order as well as convex combinations which respect the linear ordering can be defined naturally (see Equation (3) in Subsection 3.3 and the discussion surrounding it).

In fact, the following Corollary 8 overlaps with a simpler theorem — Theorem 9 below — about the representation of preference orderings by means of $S_{\mathsf{RA-IM}}$-valued utility functions.

Corollary 8: *Assume the hypotheses of Corollary 6, suppose that* $n = 2$ *and* $m = 4$ *as well as* $x_1 \succ x_2 \sim x_3 \sim x_4$. *We may put* $f_1 = f_2 = f_3 = 1$ *and* $f_4 = I$ *for some infinite hyperreal I. Then*

$$\sum_{i=1}^{4} p_i x_i \preceq \sum_{i=1}^{4} q_i x_i \iff \sum_{i=1}^{4} p_i f_i \leq \sum_{i=1}^{4} q_i f_i \;\; \textit{in } S_{\mathsf{RA-IM}}$$

whenever $p_1, q_1, \ldots, p_4, q_4 \in [0,1]$ *with* $\sum_{i=1}^{4} p_i = \sum_{i=1}^{4} q_i = 1$.

Also, for all $\bar{p}, \bar{q} \in [0,1]$,

$$\langle \bar{p}, \bar{q} \rangle \prec \langle \bar{p}, 1 \rangle.$$

The hypotheses of Corollary 6 (and hence of Corollary 8) entail that $\preceq$ can even compare lotteries with arbitrary hyperreal (e.g. infinitesimal) chances. Corollary 8 provides a $S_{\mathsf{RA-IM}}$-valued expected-utility representation of the *restriction* of this preference relation $\preceq$ to lotteries with standard, real chances.

The following theorem, a special case of Theorem 4, is similar to Corollary 8. It can be proved without any reference to nonstandard analysis or the hyperreals. Whilst it does not require the preference ordering to be defined for hyperreal chances (let alone being internal), it does assume that the preference ordering is continuous. This assumption simply reflects that there is a continuum of infinite utilities in $S_{\mathsf{RA-IM}}$ (one for each lottery with chance $q > 0$) and thus is not problematic.

Recall that $x_1 := \langle 1, 1 \rangle$, $x_2 := \langle 0, 1 \rangle$, $x_3 := \langle 1, 0 \rangle$, and $x_4 := \langle 0, 0 \rangle$.

Theorem 9: *Let $X = [0,1]^2$ and let $\preceq$ be a binary relation on X. Suppose $\preceq$ is complete, transitive, continuous and independent. Assume $x_1 \succ x_2 \sim x_3 \sim x_4$. Denote the convex hull of $\{x_2, x_3, x_4\}$ by X_0, and consider the function $\bar{U} : X \to S_{\mathsf{RA-IM}}$ given by*

$$\forall p \in [0,1] \quad \forall x_0 \in X_0 \qquad \bar{U}\,(px_1 + (1-p)x_0) = 1 + pI,$$

thus $\bar{U}\,(x_0) = 1$ and $\bar{U}\,(px_1) = pI$ for all $p \in (0,1]$, $x_0 \in X_0$. Then

$$\forall x, y \in X \qquad x \preceq y \Longleftrightarrow \bar{U}(x) \leq \bar{U}(y).$$

Proof of Theorem 9. Apply Theorem 4 to $B = \{x_1, \ldots, x_4\}$. □

This theorem again yields (even under simpler assumptions than those of Corollary 8), that outright wagering for God is strictly preferable to any mixed strategy with chance $\bar{q} < 1$:

Corollary 10: *Under the assumptions of Theorem 9,*

$$\forall \bar{p} \in (0,1] \quad \bar{q} \in [0,1) \qquad \langle \bar{p}, \bar{q} \rangle \prec \langle \bar{p}, 1 \rangle.$$

Note that the maximum I of $S_{\mathsf{RA-IM}}$ is both reflexive under addition and strictly irreflexive under multiplication. Hence, if $f_2 = f_3 = f_4$, one can respond, based on Corollary 8 and the more general Theorem 9, to both McClennen's objection and Hájek's dilemma by using a single-valued utility function, viz. a utility function with values in $S_{\mathsf{RA-IM}}$. The $S_{\mathsf{RA-IM}}$-valued utility representation of the Wagerer's preferences seems philosophically much more adequate than the ${}^*\mathbb{R}$-valued utility representation, since $S_{\mathsf{RA-IM}}$ has a maximum (viz. I, the utility of salvation), whereas ${}^*\mathbb{R}$ has none. Hence, the maximal utility of the Wagerer, when measured through an $S_{\mathsf{RA-IM}}$-valued utility function, is just the maximum of $S_{\mathsf{RA-IM}}$, whilst a ${}^*\mathbb{R}$-valued utility function would cut off the Wagerer's utility at a somewhat arbitrary value in ${}^*\mathbb{R}$ (as was argued above).

The philosophical interpretation of the equality $f_2 = f_3 = f_4$ is, of course, that the Wagerer is indifferent among all pure outcomes except salvation. It can be decomposed into the following two orthodox theological propositions. The first proposition is Reformed theology's view of hell; the second proposition is supported by Jesus' promises in Matthew 6,33 and Mark 10,30:

(1) *Separation from God as judgement for non-believers:* $f_3 = f_4$ holds if the Wagerer believes that the Christian God would not punish those who choose not to have fellowship with Him, but simply "leaves

them alone", i.e. as "well"-off (viz. f_3) as they were if there was no God (f_4).

(2) *Utility-neutral sanctification:* $f_2 = f_4$ holds if the Wagerer assumes that he does not have to make any sacrifices for his faith on earth that would reduce his overall utility (through mental peace, joy, fellowship with other believers etc.). Any sacrifices that he makes will, at least in the long run, result in an offsetting increase in utility, even without special divine intervention. If God does not exist and he wagers for Him nevertheless, he is just as well-off (viz. f_2) as he would be if he decides otherwise (f_4).

In summary, if the Wagerer's view of the Christian gospel entails these two propositions, she will be indifferent among all pure outcomes except salvation, and therefore, by Corollary 8 and Theorem 9, reason demands that she wagers for God and that her preference ordering can be represented through an $S_{\mathsf{RA-IM}}$-valued utility function. There are two sets of rationality axioms on the Wagerer's preference ordering which yield this result: Either

- the Wagerer's preference ordering is complete for lotteries with real chances, is transitive, is unaffected by real perturbations (Continuity), and is unaffected by mixing with other real-chance lotteries (Independence), or
- the Wagerer's preference ordering is complete for lotteries with hyperreal chances, is transitive, is unaffected by infinitesimal perturbations (Infinitesimal Continuity), is unaffected by mixing with other lotteries (Independence), and is internal (e.g. definable through standard functions with hyperreal parameters).

If $f_2 = f_3 = f_4$, one can also prove the validity of Pascal's Wager via the (game-theoretic) principle of stochastic dominance, based on either an $S_{\mathsf{RA-IM}}$-valued or a hyperreal-valued utility function.

At the end of this Section, we should recall that we only studied the special case $f_2 = f_3 = f_4 = 1$ after we had accepted that the use of hyperreal-valued utility functions in formalising Pascal's wager is controversial (on the grounds of infinite hyperreals' irreflexivity under addition and their "arbitrariness") and looked for other ways to meet both McClennen's decision-theoretic objection and Hájek's dilemmas. The present author thinks nevertheless that the objections against the use of hyperreal-valued expected utility functions can be answered for mathematical and theologico-philosophical reasons and that on the contrary there are excellent reasons to employ hyperreal-valued expected utility functions in formalising Pascal's wager (see Subsection 3.1). Well, and in a formalisation based on hyperreal expected utility functions, the validity of Pascal's wager can be maintained in general, not just in the special case $f_2 = f_3 = f_4 = 1$.

5. *Conclusion*

We have shown that the concept of hyperreal expected utility has a sound decision-theoretic basis: Under some natural conditions on the preference orderings, expected hyperreal-valued utility functions on convex sets represent preference orderings among lotteries based on a finite set of pure outcomes and hyperreal probabilities.

This is good news for the Pascalian since Pascal's Wager — and most of its generalisations, such as the many-gods wagers studied by Bartha (2007) —, only allow a finite number of pure outcomes. In the original Wager, there are just four pure outcomes: the Wagerer believes in God and God exists, or he does not wager for Him, although He exists, or he does wager for Him, whilst He does not exist, or he does not wager for Him, nor does He exist.

Therefore, a formalisation of Pascal's argument by means of hyperreals is consistent with a decision theory that incorporates nonstandard probabilities, whilst every internal (nonstandard) probability measure canonically induces a standard real-valued probability measure. (In particular, if one composes a nonstandard probability measure on a finite set with the standard part map[20], one obtains a standard probability measure on that finite set[21]).

The Hyperreal Expected Utility Theorem (Theorem 3) provides a general decision-theoretic justification of hyperreal-valued expected utility functions. Thus, one may now consider, in the spririt of Bartha's (2007) conclusion ("Beyond Pascal's Wager" [pp. 39-41]), the use of hyperreal stochastic utilities in other situations where either an infinite good is at stake, or where intolerable outcomes should be avoided, or where both Kantian and utilitarian deliberations seem to have their point.

In addition, we have constructed — motivated by the hyperreals — a convex linearly ordered superset $S = S_{\text{RAIM}}$ of the reals which has a maximum that is both reflexive under addition by finite numbers and strictly irreflexive under multiplication by scalars < 1, thereby proposing a way out of Hájek's dilemma.

Moreover, if one assumes that the Wagerer is indifferent among all pure outcomes except salvation ($f_2 = f_3 = f_4$ in the notation of the Wagerer's

[20] Let r be a hyperreal number such that r is S-bounded, i.e. there exists some standard natural number $N \in \mathbb{N}$ with $-N \leq r \leq N$. Then, due to the Hausdorff property of the order topology on $\mathbb{R}$, there exists a unique real number s, which minimises $|r - s|$ among all $s \in \mathbb{R}$. This s is then denoted $^\circ r$ and referred to as the *standard part* of r. The function $\mathrm{st} : r \mapsto {}^\circ r$ is called *the standard part map*.

[21] This is just a special case of a general construction: Any internal probability function can be extended to a (σ-additive) probability measure with standard values, as was shown by Loeb (1975). Almost all contributions to probability theory using nonstandard methods rely on this basic result of Loeb (1975).

payoff matrix), which e.g. follows from all those theological systems (*nota bene*, on the Wagerer's part) where judgement just means separation from God and where sanctification is utility-neutral, then Theorem 9 shows that the preference ordering can be represented by a utility function whose range is contained in the aforementioned convex linearly ordered set S.

Summing up, we have determined under which hypotheses one can simultaneously refute two major arguments against Pascal's Wager, viz. McClennen's decision-theoretic objection and Hájek's dilemma, through a formalisation with a single-valued utility function whose range is a certain superset S reals: Aside from rationality axioms on the preference ordering, one has to impose the assumption of $f_2 = f_3 = f_4$ (in the notation of the Wagerer's payoff matrix above), which can be upheld if the Wagerer views sanctification as utility-neutral and believes that the Christian God will judge nonbelievers by "mere" separation from Him. However, the most elegant — and for mathematical, decision-theoretic, philosophical and theological reasons perhaps most suitable — formalisation of Pascal's wager is probably the one based on hyperreal expected utility functions.

Appendix A. Proofs

Proof of the Hyperreal von Neumann-Morgenstern Theorem (Theorem 1). The (standard) Expected Utility Theorem of von Neumann and Morgenstern about preference relations on convex sets says the following: A binary relation on any convex subset X of a linear space satisfies the axioms of completeness, transitivity, continuity and independence if and only if there exists some affine function $U : X \to \mathbb{R}$ such that the estimate $U(x) \leq U(y)$ is equivalent to $x \preceq y$ (for all $x, y \in X$).

We shall apply the Transfer Principle to this Expected Utility Theorem. Note, for this purpose, that the Transfer Principle also yields that the *-image of the set of binary relations on convex subsets of linear spaces is just the set of internal binary relations on *-convex subsets of *-linear spaces.

Hence, applying the Transfer Principle to the standard Expected Utility Theorem leads to the following result: Any internal binary relation on a *-convex subset of a *-linear space is *-complete, *-transitive, *-continuous and *-independent if and only if there exists a *-affine function $U : X \to {^*\mathbb{R}}$ satisfying

$$\forall x, y \in X \qquad U(x) \leq U(y) \Leftrightarrow x \preceq y.$$

But, *-transitivity is the same as transitivity, and *-completeness is the same as completeness. Furthermore, if we apply the Transfer Principle to the definitions of continuity and independence (for standard binary relations on convex sets), we obtain that *-continuity is the same as Infinitesimal Continuity, and *-independence is the same as Independence. This completes the proof of the Theorem.

□

Proof of the Internal Expected Utility Theorem (Theorem 2). By the Hyperreal von Neumann-Morgenstern Theorem (Theorem 1), there exists a *-affine function $U : Y \to {}^*\mathbb{R}$ which represents $\preceq$. Since $U\left(\sum_{i=1}^{m} p_i x_i\right) = \sum_{i=1}^{m} p_i U\left(x_i\right)$ holds for all $p_1, \ldots, p_m, \in {}^*[0,1]$ with $\sum_{i=1}^{m} p_i = 1$, we may simply put $u_i := U\left(x_i\right)$ for every $i \in \{1, \ldots, m\}$. □

Proof of the Hyperreal Expected Utility Theorem (Theorem 3). First, suppose $Y = \{\sum_{i=1}^{m} p_i x_i \;:\; p_1, \ldots, p_m \in {}^*[0,1], \quad \sum_{i=1}^{m} p_i = 1\}$ and $\preceq$ is an internal binary relation $\subseteq Y \times Y$. We have to show that the Hyperreal von Neumann-Morgenstern Theorem (Theorem 1) may be applied in the setting of the Hyperreal Expected Utility Theorem (Theorem 3).

For this sake, note that if $n = \dim_{^*\mathbb{R}} W$ is the dimension of W as a linear space over the field $^*\mathbb{R}$, then W is isomorphic (over $^*\mathbb{R}$) to $^*\mathbb{R}^n$. Let us denote this isomorphism by $\psi : W \simeq {}^*\mathbb{R}^n$.

If $y_1, \ldots, y_m$ are elements of an arbitrary *-linear space Z, then the set

$$C\left(y_1, \ldots, y_m\right) :=$$

$$\left\{ z \in Z \;:\; \exists p_1, \ldots, p_m \in {}^*[0,1] \left(\sum_{i=1}^{m} p_i = 1, \quad z = \sum_{i=1}^{m} p_i y_i \right) \right\}$$

is internally defined and therefore — according to the Internal Definition Principle[22] — itself an internal set. Moreover, it is closed under convex combinations with weights from $^*\mathbb{R}$. Hence it is a *-convex set (the *-convex hull of $y_1, \ldots, y_m$).

Since ψ is an isomorphism, we find that

$$(5) \quad \psi\left(Y\right) =$$

$$\left\{ z \in {}^*\mathbb{R}^n \;:\; \exists p_1, \ldots, p_m \in {}^*[0,1] \left(\sum_{i=1}^{m} p_i = 1, \quad z = \sum_{i=1}^{m} p_i \psi\left(x_i\right) \right) \right\}.$$

[22] See Footnote 11.

Therefore, the observation of the previous paragraph may be applied to $C(\psi(x_1),\ldots,\psi(x_m)) = \psi(Y)$. Hence $X := \psi(Y)$ is a *-convex subset of the *-linear space $^*\mathbb{R}^n$.

Now, the internality of $\preceq$ on Y ensures that the relation $\preceq_X$, defined by

$$\xi_1 \preceq_X \xi_2 :\Leftrightarrow \psi^{-1}(\xi_1) \preceq \psi^{-1}(\xi_2),$$

is also internal (and if $\preceq$ is standard-definable under a basis choice, $\preceq_X$ will have that property, too: $\preceq_X = \left\{\langle x,y\rangle \in X^2 \;:\; \varphi(x,y)\right\}$). Since the formula φ only involves the canonical extensions (*-images) of standard maps as well as equality and the order relation, $\preceq_X$ is internally defined and thus, according to the Internal Definition Principle, internal.

Thus, we may apply the Hyperreal von Neumann-Morgenstern Theorem (Theorem 1) to the set $X = \psi(Y)$ and the relation $\preceq_X$ on X. Observe that $\preceq_X$ satisfies the axioms of Completeness, Transitivity, Infinitesimal Continuity and Independence if and only if the relation $\preceq$ on Y satisfies them (because ψ is an isomorphism and thus commutes with *-convex combinations, i.e. convex combinations with weights from $^*\mathbb{R}$). Furthermore, the equivalence assertion

$$\forall \xi_1,\xi_1 \in X \qquad \xi_1 \preceq_X \xi_2 \Leftrightarrow U(\xi_1) \leq U(\xi_2)$$

is true if and only if

$$\forall y_1,y_2 \in Y \qquad y_1 \preceq y_2 \Leftrightarrow U(\psi(y_1)) \leq U(\psi(y_2)).$$

Hence, after we have applied the Hyperreal von Neumann-Morgenstern Theorem to X and $\preceq_X$, we actually obtain the following statement: The relation $\preceq$ on Y satisfies the axioms of Completeness, Transitivity, Infinitesimal Continuity and Independence if and only if there is some *-affine function $U : \psi(Y) \to {}^*\mathbb{R}$ with

$$\forall y_1,y_2 \in Y \qquad y_1 \preceq y_2 \Leftrightarrow U(\psi(y_1)) \leq U(\psi(y_2)).$$

Finally, it is easy to see that the existence of a *-affine function $U : \psi(Y) \to {}^*\mathbb{R}$ with $y_1 \preceq y_2 \Leftrightarrow U(\psi(y_1)) \leq U(\psi(y_2))$ for all $y_1,y_2 \in Y$ implies the existence of the hyperreals $u_1,\ldots,u_m$ as in the statement of the Hyperreal Expected Utility Theorem: Given $U : \psi(Y) \to {}^*\mathbb{R}$, simply set

$$\forall i \in \{1,\ldots,m\} \qquad u_i := U(\psi(x_i)).$$

Since U is *-affine, this already entails that

$$U\left(\sum_{i=1}^{m} p_i \psi\left(x_i\right)\right) = \sum_{i=1}^{m} p_i u_i$$

for all $p_1, \ldots, p_m \in {}^*[0,1]$ with $\sum_{i=1}^{m} p_i = 1$.

□

Proof of Lemma 5. Fix any $x_0 \in X_0$. Note that $\sim$ is transitive as $\preceq$ is transitive (in fact, $\sim$ is an equivalence relation). Therefore, whenever $x, y \sim x_0$, we have $x \sim y$ and thus, by the independence of $\preceq$, even

$$px + (1-p)y \quad \preceq \quad py + (1-p)y \preceq px + (1-p)y,$$

hence $px + (1-p)y \sim py + (1-p)y = y \sim x_0$ for all $p \in [0,1]$. Therefore, the set $C_0 := \{x \in X_0 \ : \ x \sim x_0\}$ is convex. Moreover, $B_0 \subseteq C_0$, whence C_0 must even contain the convex hull of B_0. So $C_0 \supseteq X_0$ and thus $X_0 = C_0$. Hence, for all $x, y \in X_0$ we already have $x \sim x_0 \sim y$ and therefore $x \sim y$ (again by transitivity of $\sim$). □

Proof of Theorem 4. Since X_0 is convex, the restriction of $\preceq$ to X_0 is not only complete and transitive, but independent, too. Therefore, Lemma 5 may be applied to $\preceq$ and we obtain that $x \sim y$ for all $x, y \in X_0$.

By the classical von Neumann-Morgenstern Theorem, there exists an affine function $U : X \to \mathbb{R}$ such that

$$\forall x, y \in X \qquad x \preceq y \Leftrightarrow U(x) \leq U(y).$$

For all $x_0, y_0 \in X_0$, we have $x_1 \succ x_0 \sim y_0$ by Lemma 5 and therefore $U\left(x_1\right) > U\left(x_0\right) = U\left(y_0\right)$. We may assume that $U\left(x_0\right) > 0$ for all $x_0 \in X_0$. This means that for all $p, q \in [0,1]$ and $x_0, y_0 \in X_0$, one has $U\left(px_1 + (1-p)x_0\right) > U\left(qx_1 + (1-q)y_0\right)$ if and only if $pU\left(x_1\right) + (1-p)U\left(x_0\right) > qU\left(x_1\right) + (1-q)U\left(y_0\right) = qU\left(x_1\right) + (1-q)U\left(x_0\right)$ which in turn holds if and only if $(p-q)U\left(x_1\right) > (p-q)U\left(x_0\right)$, and this inequality is equivalent to $p > q$ (since $U\left(x_1\right) > U\left(x_0\right)$).

Hence,

$$\forall p, q \in [0,1] \quad \forall x_0, y_0 \in X_0$$
$$U\left(px_1 + (1-p)x_0\right) > U\left(qx_1 + (1-q)y_0\right) \Leftrightarrow p > q.$$

This readily yields

$$\forall p \in [0,1] \quad \forall x_0, y_0 \in X_0 \quad U\left(px_1 + (1-p)x_0\right) = U\left(px_1 + (1-p)y_0\right).$$

Therefore, there exists an increasing function $v : [0,1] \to \mathbb{R}$ such that

$$\forall p \in [0,1] \quad \forall x_0 \in X_0 \qquad U\left(px_1 + (1-p)x_0\right) = v(p).$$

Since U is affine, v must be affine, too. Hence $v(p) = v(0) + (v(1) - v(0))\, p$ for all $p \in [0,1]$.

It is clear that $v(0) = \min v = \min U = U(x_0)$ and $v(1) = \max v = \max U = U(x_1)$, so $v(1) > v(0)$. Define now a function $\varphi : \mathbb{R} \to S$ via

$$\forall u \in \mathbb{R} \qquad \varphi(u) = 1 + \frac{u - v(0)}{v(1) - v(0)} I$$

and

$$\bar{U} := \varphi \circ U.$$

Then, for all $p \in [0,1]$ and $x_0 \in X_0$,

$$\begin{aligned} \bar{U}\left(px_1 + (1-p)x_0\right) &= \varphi \circ U\left(px_1 + (1-p)x_0\right) = \varphi\left(v(p)\right) \\ &= \varphi\left(v(0) + (v(1) - v(0))\, p\right) = 1 + pI \end{aligned}$$

Moreover, since φ is increasing and U is a von Neumann-Morgenstern utility function, we obtain

$$\forall x, y \in X \qquad x \preceq y \Leftrightarrow \bar{U}(x) \leq \bar{U}(y).$$

□

Proof of Corollary 6. By Theorem 3 there are $f_1 = u_1, \ldots, f_m = u_m \in {}^*\mathbb{R}$ such that the equivalence statement (4) holds. If $x_1 \succ x_2 \sim \cdots \sim x_m$, then $f_1 > f_2 = \cdots = f_m$. Since the vector $\langle u_1, \ldots, u_m \rangle$ in Theorem 3 is only unique up to affine transformations, $f_2 = \cdots = f_m$ can indeed be any given hyperreal and f_1 can be any hyperreal $> f_2$. □

Proof of Corollary 7. Observe that for all $\bar{p}, \bar{q} \in {}^*[0,1]$, one has

$$\begin{aligned} \langle \bar{p}, \bar{q} \rangle &= \langle \bar{p}, \bar{p}\bar{q} \rangle + \langle 0, \bar{q} - \bar{p}\bar{q} \rangle \\ &= \bar{p}\left(\bar{q}\langle 1,1 \rangle + (1-\bar{q})\langle 1,0 \rangle\right) + (1-\bar{p})\left(\bar{q}\langle 0,1 \rangle + (1-\bar{q})\langle 0,0 \rangle\right) \\ &= \bar{p}\left(\bar{q}x_1 + (1-\bar{q})\, x_3\right) + (1-\bar{p})\left(\bar{q}x_2 + (1-\bar{q})\, x_4\right). \end{aligned} \tag{6}$$

Therefore, the equivalence statement (4) of Corollary 6 teaches that $\langle \bar{p}, \bar{q} \rangle \prec \langle \bar{p}, 1 \rangle$ is equivalent to

$$(7) \quad \underbrace{\bar{p}\left(\bar{q} f_1 + (1-\bar{q}) f_3\right) + (1-\bar{p})\left(\bar{q} f_2 + (1-\bar{q}) f_4\right)}_{=\bar{p}\bar{q} f_1 + \bar{p}(1-\bar{q}) f_3 + (1-\bar{p})(\bar{q} f_2 + (1-\bar{q}) f_4)} < \bar{p} f_1 + (1-\bar{p}) f_2,$$

which in turn is equivalent to

$$(8) \quad \bar{p}(1-\bar{q}) f_3 + (1-\bar{p})\left(\bar{q} f_2 + (1-\bar{q}) f_4\right) - (1-\bar{p}) f_2 < \bar{p}\left(1-\bar{q}\right) f_1,$$

and this inequality holds obviously whenever f_1 is positive infinite, f_2, f_3, f_4 are finite and both $\bar{p} > 0$ and $1 - \bar{q} > 0$ are non-infinitesimal (since then the left-hand side in inequality (8) is finite, while the right-hand side is positive infinite). □

Proof of Corollary 8. $\langle \bar{p}, \bar{q} \rangle \prec \langle \bar{p}, 1 \rangle$ for all $\bar{p}, \bar{q} \in [0, 1]$ is an immediate consequence of Corollary 7 (which imposed weaker assumptions).

Let $p_1, q_1, \ldots, p_4, q_4 \in [0, 1]$ in ${}^*\mathbb{R}$ with $\sum_{i=1}^4 p_i = \sum_{i=1}^4 q_i = 1$. By the choice of $f_1, \ldots, f_4$, the estimate $\sum_{i=1}^4 p_i f_i \le \sum_{i=1}^4 q_i f_i$ is tantamount to $p_2 + p_3 + p_4 + (1 - p_2 - p_3 - p_4)\, I \le q_2 + q_3 + q_4 + (1 - q_2 - q_3 - q_4)\, I$ (both in ${}^*\mathbb{R}$ and in $S_{\mathsf{RA-IM}}$). Regardless whether we evaluate this latter estimate in ${}^*\mathbb{R}$ or in $S_{\mathsf{RA-IM}}$, it holds if $p_1 \le q_1$ and it fails if $p_1 > q_1$, and is thus equivalent to $p_1 \le q_1$.

Therefore, we obtain the following chain of equivalences:

$$\begin{aligned}\sum_{i=1}^4 p_i x_i \preceq \sum_{i=1}^4 q_i x_i \quad &\Leftrightarrow \quad \sum_{i=1}^4 p_i f_i \le \sum_{i=1}^4 q_i f_i \text{ in } {}^*\mathbb{R} \Leftrightarrow p_1 \le q_1 \\ &\Leftrightarrow \quad \sum_{i=1}^4 p_i f_i \le \sum_{i=1}^4 q_i f_i \text{ in } S_{\mathsf{RA-IM}}.\end{aligned}$$

□

Proof of Theorem 9. Apply Theorem 4 to $B = \{x_1, \ldots, x_4\}$. □

Proof of Corollary 10. For all $\bar{p}, \bar{q} \in [0,1]$, we have by Equation (6),

$$
\begin{aligned}
&\langle \bar{p}, \bar{q} \rangle \\
&= \bar{p}\left(\bar{q}x_1 + (1-\bar{q})\,x_3\right) + (1-\bar{p})\left(\bar{q}x_2 + (1-\bar{q})\,x_4\right) \\
&= \bar{p}\bar{q}x_1 + \bar{p}\,(1-\bar{q})\,x_3 + (1-\bar{p})\left(\bar{q}x_2 + (1-\bar{q})\,x_4\right) \\
&= \bar{p}\bar{q}x_1 + (1-\bar{p}\bar{q}) \underbrace{\left(\frac{\bar{p}\,(1-\bar{q})}{1-\bar{p}\bar{q}}x_3 + \frac{\bar{q}\,(1-\bar{p})}{1-\bar{p}\bar{q}}x_2 + \frac{(1-\bar{p})\,(1-\bar{q})}{1-\bar{p}\bar{q}}x_4\right)}_{\in X_0}
\end{aligned}
$$

(note that $\frac{\bar{p}(1-\bar{q})}{1-\bar{p}\bar{q}} + \frac{\bar{q}(1-\bar{p})}{1-\bar{p}\bar{q}} + \frac{(1-\bar{p})(1-\bar{q})}{1-\bar{p}\bar{q}} = 1$), hence

$$\forall \bar{p}, \bar{q} \in [0,1] \qquad \bar{U}\left(\langle \bar{p}, \bar{q} \rangle\right) = 1 + \bar{p}\bar{q}I.$$

Therefore, if $\bar{p} \in (0,1]$ and $\bar{q} \in [0,1)$, then

$$\bar{U}\left(\langle \bar{p}, \bar{q} \rangle\right) = 1 + \bar{p}\bar{q}I < 1 + \bar{p}I = \bar{U}\left(\langle \bar{p}, 1 \rangle\right),$$

hence $\langle \bar{p}, \bar{q} \rangle \not\succeq \langle \bar{p}, 1 \rangle$ by Theorem 9, so $\langle \bar{p}, \bar{q} \rangle \prec \langle \bar{p}, 1 \rangle$ due to the completeness of $\preceq$.

□

Appendix B. Reflexivity under Addition vs. Soterical Differentiation

B.1. Pascal on Reflexivity under Addition

As was mentioned in the Introduction, Hájek suggests that Pascal would have required I, the utility associated with salvation, to be *reflexive under addition* (and we have already indicated our disagreement with that statement):

(9) $$\forall x \in \mathbb{R}_{>0} \qquad x + I = I$$

(and thereby, by adding $y = -x$ to both sides of the equation, even $y + I = I$ for all $y \in \mathbb{R}_{<0}$, hence $y + I = I$ for every real y).

At first glance, Hájek's interpretation of Pascal as assuming the Reflexivity under Addition is convincing. First, Reflexivity under Addition seems to express that "Nothing could be better for you than your salvation" (Bartha 2007; note, however, even this sentence can be interpreted as being consistent with graded utilities of salvation). Moreover, Pascal writes in the preface to the Wager [*Pensées* §233]:

> Unity joined to infinity adds nothing to it, no more than one foot to an infinite measure. The finite is annihilated in the presence of the infinite, and becomes a pure nothing. [...]

The first sentence of this passage seems to support the axiom of Reflexitivity under Addition (9). The second sentence, however, explicates that a finite number is nothing *in the presence of* — in other words: *compared to* — an infinite value.

If we view this second sentence as an explanation of the first one, then we are not forced to adopt the assumption of Reflexitivity under Addition.

Rather, we could postulate that S, the set of possible utilities of the Pascalian Wagerer, is a convex subset of an ordered field[23] and satisfies for all I in some nonempty proper subset $\Sigma \subsetneq S$ the following estimate:

$$\forall x \in \mathbb{R}_{>0} \quad \forall n \in \mathbb{N} \qquad 0 < \frac{x}{I} < \frac{1}{n}. \tag{10}$$

In words, this means that for all $x \in \mathbb{R}_{>0}$ and $I \in \Sigma$, $\frac{x}{I}$ is a positive infinitesimal. We may assume that the set Σ has been chosen as the maximal subset of S with the property that all elements $I \in \Sigma$ satisfy estimate (10).

Note that in this formalisation, Hájek's (2003) axiom of I being an *Overriding Utility*, holds — provided we model the utilities of the saved by elements of Σ and the values f_2, f_3, f_4 (as defined in the Wagerer's payoff matrix in Premise 1) by $S \setminus \Sigma$. For, in order for someone to have an infinite level of utility in this model, she must wager for God.

In particular, the axiom (10) is satisfied if

- S is the ordered field of hyperreal numbers and Σ the subset of positive infinite hyperreals, or
- S is the ordered field of surreal numbers and Σ the subset of positive infinite surreal numbers.

In both of these ordered fields, 'finite' means being bounded by some $n \in \mathbb{N}$.

[23] An *ordered field* Q is a field on which a linear order is defined, in such a way that addition of any element of the field preserves the order relation between two elements, and so does multiplication by positive elements. A subset $A \subseteq Q$ is called *convex* if and only if for all $x, y \in A$ and every $p \in Q$ with $0_Q <_Q p <_Q 1_Q$, one has $p \times_Q x +_Q (1_Q -_Q p) \times_Q y \in A$. One is inclined to demand the convexity of the set of possible utilities of the Wagerer in order to allow for mixed strategies.

B.2. Pascal and Soterical Differentiation

Up to now, we have only argued that Pascal need not be read as supporting the axiom of Reflexivity under Addition. But can he be interpreted as subscribing to the principle that the Wagerer's utility should have multiple levels of infinite utility in its range (*Soterical Differentiation*)?

It is difficult to argue directly in favour of this. Given the apologetic nature of most of his writings, Pascal has written next to nothing on eschatology (not even in the Prophecies section, Section XI, of the *Pensées*), and his soteriological comments are mainly concerned with matters of justification and salvation, in particular the doctrine of grace and predestination. Also, neither Cornelius Jansen, the founder of the Roman Catholic sect Pascal belonged to — nor Augustine, the church father whose soteriology greatly influenced Pascal and Jansen, seem to have published anything that would either favour or contradict Soterical Differentiation.

It is important to note at this point that Pascal's and Jansenism's emphasis on a justification by grace through faith (quite as the Protestant *sola gratia*), as opposed to works, cannot be seen as an argument against Soterical Differentiation. With the same right that God proves Himself gracious to some and not to others, He may as well reward some of the saved more and some less.

Strictly speaking, we can therefore only speculate what Pascal's views regarding Soterical Differentiation might have been like. However, Pascal had a very high appreciation of Scripture, even of what he terms "obscure passages". For instance, in *Pensées* §575, he writes the following (which itself consists half of indirect Scripture quotations)

> All things work together for good to the elect [*cf. Romans 8,28*], even the obscurities of Scripture; for they honour them because of what is divinely clear. And all things work together for evil to the rest of the world, even what is clear; for they revile such, because of the obscurities which they do not understand [*cf. 2nd Peter 3,16*].

(Comments of the author in squared parentheses.) Similarly, in *Pensées* §568, §579 as well as §889 ("the true guardians of the Divine Word have preserved it unchangeably") he defends the divine inspiration of the whole of Scripture; he goes even further to claim, quoting Augustine, that "He who will give the meaning of Scripture, and does not take it from Scripture, is an enemy of Scripture" [*Pensées* §900].

However, the New Testament multiply mentions a hierarchy in Heaven. For instance, in one of the first paragraphs of the Sermon on the Mount after the Benedictions, Jesus says:

> Whosoever therefore shall break one of these least commandments, and shall teach men so, he shall be called the least in the kingdom of heaven: but whosoever shall do and teach them, the same shall be called great in the kingdom of heaven. [Matthew 5,19 (King James Version)]

In the same Gospel, Jesus teaches as follows:

> Verily I say unto you, Among them that are born of women there hath not risen a greater than John the Baptist: notwithstanding he that is least in the kingdom of heaven is greater than he. [Matthew 11,11 (King James Version)]

What is translated as "least" in the New International Version or the King James Version, would be *ἐλάξιστος* (the elative or superlative of *μικρός*) and *μικρότερος* (comparative of *μικρός*), respectively in the original Greek New Testament sources, clearly suggesting that there is a hierarchy in the 'kingdom of heaven' (a term which refers to God's dominion in the individual lives of the faithful and in the corporate life of the church) and presumably in Heaven, too. Outside the Gospel of Matthew, Soterical Differentiation can be found in the book of Revelation:

> And I saw thrones, and they sat upon them, and judgement was given unto them: and I saw the souls of them that were beheaded for the witness of Jesus, and for the word of God, and which had not worshipped the beast, neither his image, neither had received his mark upon their foreheads, or in their hands; and they lived and reigned with Christ a thousand years. [Revelation 20,4 (King James Version)]

The order in which those sharing power with Jesus Christ in the Millennium are mentioned reflects the faithfulness they had displayed for His sake and thus suggests a Millennial hierarchy and presumably a Heavenly hierarchy, too. Note that there is agreement on these verses among all major textual witnesses and early translations (cf. the critical apparatus of Nestle-Aland's *Novum Testamentum Graece*, 27th rev. ed.).

Given that central passages of the New Testament mention a hierarchy in Heaven, it is reasonable to assume that Pascal would have approved of the idea of Soterical Differentiation.

It should be noted that under the Jansenist or Augustinian idea of predestination, which Pascal subscribed to, "not only are the reasons for the judgement hidden (which the Calvinists admit), but the judgement itself is also" (cf. Miel 1969 [p. 105-106]) and therefore, the believers should "work out [their] salvation with fear and trembling" (Miel 1969 [p. 105] citing Philippians 2,12). Consequently (and this is even consistent with Calvinism), humans may not know how much exactly, compared to other saved ones, they will be rewarded in Heaven — all one can say is, that, in case of salvation, it has to be an infinite value compared with any earthly utility.

Appendix C. An alternative way of resolving Hájek's dilemma

In this Appendix, we construct a model for S where only the maximum satisfies Reflexivity under Addition. However, this model for S is not covered by the Corollary 6, whence it is susceptible to McClennen's objection.

Let S, the set of possible utilities of a Pascalian Wagerer, be defined as

$$S = \{\langle 1, 0\rangle\} \cup [0, 1) \times \mathbb{R},$$

i.e. the union of the singleton $\{\langle 1, 0\rangle\}$ with the set of all pairs of real numbers where the first entry is ≥ 0 and < 1. (Here and in the following, $\mathbb{R}$ and its various subintervals could again be replaced by any real-ordered field, thus allowing, for instance, for nonstandard probabilities.)

The first coordinate should be seen as representing 'heavenly utility' and the second coordinate 'earthly utility" . In the payoff matrix of the Wager, $f_2, f_3, f_4 \in \{0\} \times \mathbb{R}$ and $I = \langle 1, 0\rangle$. The utility $I = \langle 1, 0\rangle$ might be interpreted as the utility of someone who heeded Pascal's advice and "wager[ed for God] without hesitation", thus not considering mixed strategies.

Let the total order $<$ on S be just the lexicographic ordering:

$$\forall \langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in S$$
$$\langle x_1, y_1\rangle < \langle x_2, y_2\rangle \Longleftrightarrow x_1 < x_2 \vee (x_1 = x_2 \wedge y_1 < y_2).$$

Hence, $\langle 1, 0\rangle$ is the strict maximum of S:

$$\forall \langle x, y\rangle \in [0, 1) \times \mathbb{R} \qquad \langle x, y\rangle < \langle 1, 0\rangle.$$

Let the operation of addition in S be defined as follows:

$$\forall \langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in [0, 1) \times \mathbb{R}$$
$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle \max\{x_1, x_2\}, y_1 + y_2\rangle$$

and

$$\forall \langle x, y\rangle \in S \qquad \langle x, y\rangle + \langle 1, 0\rangle = \langle 1, 0\rangle + \langle x, y\rangle = \langle 1, 0\rangle.$$

In particular, $I = \langle 1, 0\rangle$ is reflexive under addition. Multiplication by elements of $[0, 1]$ will be defined as the ordinary multiplication on $\mathbb{R}^2$:

$$\forall \langle x, y\rangle \in S \qquad p\,\langle x, y\rangle = \langle px, py\rangle.$$

Hence, $I = \langle 1, 0\rangle$ meets the requirement of Irreflexivity under Multiplication

$$\forall p < 1 \qquad p\langle 1, 0\rangle = \langle 0, p\rangle < \langle 1, 0\rangle.$$

This entails that both S and the proper subset $[0, 1) \times \mathbb{R}$ of S are closed under addition as well as under multiplication by elements of $[0, 1]$.

Having constructed S and observed that in this setting, the utility of salvation, I, is both reflexive under addition and irreflexive under multiplication, we close with the following remarks:

(1) Albeit exhibiting some resemblances to the set of vector-valued utilities considered by Hájek [Subsection 4.2, pp. 39-41], it is different in that S contains just one value with maximal 'heavenly utility', viz. I, the maximum of S itself.

(2) Adding up the overall utility of one individual with the utility of another one, does not make much sense if salvation is at stake. This is reflected by the ordering on S being inconsistent with addition: For, if $x_3 \geq x_2 > x_1$ but $y_2 < y_1$, then both $\langle x_1, y_1\rangle < \langle x_2, y_2\rangle$ and

$$\langle x_1, y_1\rangle + \langle x_3, y_3\rangle > \langle x_2, y_2\rangle + \langle x_3, y_3\rangle\,.$$

For, the left hand side in the inequality then equals $\langle x_3, y_1 + y_3\rangle$ and the right hand side equals $\langle x_3, y_2 + y_3\rangle$. Also, the ordering is obviously inconsistent with mixing the utilities of mixed strategies, i.e. elements of $(0, 1) \times \mathbb{R}$: If $p \in (0, 1)$ and $\langle x_1, y_1\rangle < \langle x_2, y_2\rangle$ are elements of $(0, 1) \times \mathbb{R}$, then we will have

$$\langle x_1, y_1\rangle > p\,\langle x_1, y_1\rangle + (1 - p)\,\langle x_2, y_2\rangle$$

whenever $(1 - p)x_2 < x_1$. Hence, in order to estimate the utility of a strategy that uses multiple random experiments, one must first compute the overall probability that the Wagerer will, at the very end, Wager for God.

(3) This, however, will lead to no further inconsistencies: The ordering is consistent with mixing pure strategies: If $p \in (0, 1)$ and $\langle x_1, y_1\rangle <$

$\langle x_2, y_2\rangle$ are elements of $\{0\} \times \mathbb{R} \cup \{\langle 1,0\rangle\}$ (since $\{0\} \times \mathbb{R}$ is the set of utilities associated with wagering against God), then

$$\langle x_1, y_1\rangle < p\,\langle x_1, y_1\rangle + (1-p)\,\langle x_2, y_2\rangle\,.$$

(4) Mixed strategies no longer yield maximal utility: The utility of wagering for God with probability q is

$$p\,(q\langle 1,0\rangle + (1-q)\,\langle 0, y_{f_3}\rangle) + (1-p)\,(q\,\langle 0, y_{f_2}\rangle + (1-q)\,\langle 0, y_{f_4}\rangle)\,,$$

wherein $\langle 0, y_{f_i}\rangle = f_i$ for $i \in \{2,3,4\}$. This can be reduced to

$$\langle pq, p(1-q)y_{f_3}\rangle + \langle 0, (1-p)qy_{f_2} + (1-p)(1-q)y_{f_4}\rangle$$

which equals

$$\langle pq, p(1-q)\,(y_{f_3}) + (1-p)qy_{f_2} + (1-p)(1-q)y_{f_4}\rangle < \langle 1,0\rangle = I.$$

So, I is reflexive under addition, and mixed strategies are suboptimal.

There is no way to apply Corollary 6 to this set S. Therefore, it only provides a response to Hájek's dilemma, but not to McClennen's challenge. The only possibility to respond to both Hájek and McClennen simultaneously is via a $S_{\mathsf{RA-IM}}$-valued utility representation of the Wagerer's preferences, as stipulated in Corollary 8.

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