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A PURELY COMBINATORIAL APPROACH TO DEONTIC LOGIC

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Abstract
In this paper I revisit the foundations of deontic logic. This results in a truth-table way of looking at the deontic modality. Going back to Frege, I define a semantics for deontic logic on the basis of four primitive terms — (the) True, (the) False, (the) Permitted, (the) Forbidden — without introducing an accessibility relation.

1. *Introduction*

Modern deontic logic began with Mally's book *Basic Laws of Ought: Elements of the Logic of Willing* (1926). His approach was axiomatic and inspired by Russell and Whitehead's *Principia Mathematica*. It led to counterintuitive results.

In 1951 von Wright published an epoch-making paper ("Deontic Logic") which is considered to be the first viable system of deontic logic. In von Wright's system, the validity of deontic formulas can be decided by a procedure which bears many resemblances with Wittgenstein's truth-table method. There are however differences. As we shall see, von Wright's decision procedure remains partly syntactic.

In this paper I shall try to give a combinatorial account of deontic logic which draws on von Wright's insights but is free of axioms.

2. *Von Wright's decision procedure is not purely recursive*

At first blush, the deontic realm abides by the same principle as the alethic realm: deontic units are independent of each other. A closer examination, however, reveals that this can be questioned.

The independence of deontic units would lead us, von Wright observes, to admit as possible that an act and its negation are both forbidden. By the definition of obligatory, this means that the act itself is both obligatory and forbidden.

But von Wright is reluctant to accept this consequence and settles the issue by an appeal to ordinary language: "At this point an appeal to ordinary language will, I think, be decisive. We seem prepared to reject a use of the words, according to which one and the same act could be truly called both obligatory and forbidden". However, if we reject this use, we must restrict the independence of deontic units and we are forced to accept a principle which can be laid down as a principle of permission (any act is itself permitted or its negation is permitted). This principle can also be phrased as: if an act is obligatory, then it is permitted (known as the axiom D of deontic logic).

This is unduly restrictive. There are, after all, situations in which the same act is both obligatory and forbidden. A striking example is provided by Raymond of Peñafort. The casuist imagines a householder who is asked by people bent on taking the life of someone hiding in the house whether he or she is in. Replying "Yes" is obligatory by virtue of the duty to tell the truth, but at the same time it is forbidden by virtue of the duty to save a life.

More fundamentally, von Wright designed a hybrid procedure for validity. A preliminary syntactic step takes place before the analysis itself. It consists in gradually transforming the given deontic formula into an equivalent formula, which is an alethic compound of deontic constituents. The validity of the latter is then recursively decided.

3. *A purely recursive procedure for validity*

In the *Tractatus Logico-Philosophicus* (1922), Wittgenstein succeeded in formulating a non-axiomatic logic in which tautologies and contradictions are defined as truth-functions of a certain kind. A *tautology* (resp. a *contradiction*) is defined by Wittgenstein as a proposition which is true (resp. false) for all the truth-possibilities of the elementary propositions it involves (TLP 4.46). Next Wittgenstein calls *truth-grounds* of a proposition those truth-possibilities of its arguments which make it true and states that the truth of a proposition p *follows from* that of a proposition q if all the truth-grounds of the latter are truth-grounds of the former (TLP 5.101 and 5.12). A major advantage of such an approach lies in the fact that tautological formulas can be identified by means of a truth-table.

One may surmise that von Wright wanted to extend Wittgenstein's analysis to deontic propositions, but he did not succeed in providing a purely recursive procedure for validity.

The first goal of this paper is to take a fresh start and to try to solve the problem without introducing extraneous elements of any kind.

As a second goal I try to accompany the solution I propose with a completeness result.

4. *The key concept of alethico-deontic configuration*

Sticking to the conception of logic upheld by Frege in "[Notes for Ludwig Darmstaedter]" (1919), I develop it in the following way.

I start with two primitive terms: the *alethic values*, (*the*) *True* and (*the*) *False*. These primitives serve to define the notion of proposition. A *proposition* is what is either true or false.

Definition 1. An *alethic configuration* of the set N (a finite set of propositions) is any function which maps each element of the set N to an alethic value.

Given a set N whose cardinality is n, we can form as many alethic configurations as there are subsets of the set N, namely 2^n.

Definition 2. The *alethic space* $E_0(N)$ is the totality of alethic configurations of the set N.

Definition 3. A *truth-function on the alethic space* $E_0(N)$ is any function which maps each element of the alethic space $E_0(N)$ to an alethic value.

Next I introduce two more primitive terms: the *deontic values*, (*the*) *Permitted* and (*the*) *Forbidden*, on a par with (the) True and (the) False. These primitives serve to define the notion of action. An *action* is what is either permitted or forbidden.

Then I go on to postulate that any action is an alethic configuration of a finite set of propositions. This postulate is of crucial importance. It makes it possible to give a purely combinatorial account of deontic logic.

Definition 4. A *deontic configuration* of the set N (a finite set of propositions) is any function which maps each element of the alethic space $E_0(N)$ to a deontic value.

Given a set N whose cardinality is n, we can form $2^{(2^n)}$ deontic configurations.

Definition 5. An *alethico-deontic configuration* of the set N (a finite set of propositions) is an ordered pair whose first component is an alethic configuration and whose second component is a deontic configuration.

Given a set N whose cardinality is n, the number of alethico-deontic configurations is thus equal to the number of alethic configurations times the number of deontic configurations, namely $2^n \times 2^{(2^n)}$.

Definition 6. The *alethico-deontic space* $E_1(N)$ is the totality of alethico-deontic configurations of the set N, that is the Cartesian product of the set of alethic configurations with the set of deontic configurations.

Definition 7. A *truth-function on the alethico-deontic space* $E_1(N)$ is any function which maps each element of the alethico-deontic space $E_1(N)$ to an alethic value.

Before showing that the purely combinatorial approach outlined here makes it possible to decide by means of a truth-table whether or not a formula is tautological, let me give a concrete illustration of the concepts of alethic, deontic and alethico-deontic configurations applied to the simplest case, i.e. to the case where $n = 1$.

5. *An example of alethic, deontic and alethico-deontic configurations*

Let us take as our working example the single proposition p:

The householder replies "*Yes*".

Two alethic configurations can be constructed with "True" and "False" applied to the above proposition.

(1) *It is the case that the householder replies "Yes".*
(2) *It is not the case that the householder replies "Yes".*

Four deontic configurations can be constructed with "Permitted" (Per) and "Forbidden" (For) applied to the two alethic configurations.

(1) *None of the two alethic configurations are forbidden.*
(2) *The second of the two alethic configurations is forbidden.*
(3) *The first of the two alethic configurations is forbidden.*
(4) *Both alethic configurations are forbidden.*

Eight alethico-deontic configurations — exclusive and exhaustive — emerge.

The first one is:
(1) *None of the two alethic configurations are forbidden and the householder chooses the first.*

The last two ones are:
(7) *Both alethic configurations are forbidden and the householder chooses the first.*
(8) *Both alethic configurations are forbidden and the householder chooses the second.*

Notice that the last two illustrate the dilemma of the agent in Raymond of Peñafort's example who has to choose between two evils: sending someone to death or telling a lie, a case which is by no means ethically impossible — Hartmann (1926) takes it to be a case of unavoidable culpability —, but which is entirely left out of the picture if we adopt von Wright's approach. Both (7) and (8) can be used as counterexamples to the controversial axiom D of deontic logic. I shall use a truth-table to prove that obligation does not entail permission in the deontic logic presented here (see Section 7 below).

6. *An alternative to Kripke semantics*

The first goal of the paper is achieved by spelling out a truth-table semantics for deontic logic. By this I mean a semantics which recursively assigns to each formula a truth-function. The language I use is made up out of n propositional letters $p_1, ..., p_n$ denoting respectively the n propositions of the set N. It is standard in that it allows nesting of modalities.

Definition 8. Let (α, Δ) be an alethico-deontic configuration of the set N. The *semantics* $S_1(N)$ has the following truth-conditions:

(1) (α, Δ) satisfies p_i iff α maps the proposition denoted by p_i to True.

(2) (α, Δ) satisfies $\top$.

(3) (α, Δ) does not satisfy $\bot$.

(4) (α, Δ) satisfies $\neg\varphi$ iff (α, Δ) does not satisfy φ.

(5) (α, Δ) satisfies $\varphi \wedge \psi$ iff (α, Δ) satisfies both φ and ψ.

(6) (α, Δ) satisfies $\varphi \vee \psi$ iff (α, Δ) satisfies either φ or ψ, or both.

(7) (α, Δ) satisfies $\varphi \rightarrow \psi$ iff if (α, Δ) satisfies φ then (α, Δ) satisfies ψ.

(8) (α, Δ) satisfies $\varphi \leftrightarrow \psi$ iff (α, Δ) satisfies φ if and only if (α, Δ) satisfies ψ.

(9) (α, Δ) satisfies $O\varphi$ iff for all β that Δ maps to Per, (β, Δ) satisfies φ.

(10) (α, Δ) satisfies $P\varphi$ iff for some β that Δ maps to Per, (β, Δ) satisfies φ.

If I succeed in showing that the tautological formulas are exactly those derivable from the axioms of the modal system K45, I shall have, by the same token, provided an alternative to Kripke semantics which is commonly used in interpreting these systems. This completeness result, the second goal of the paper, can be achieved by extending Kalmár's method to K45. Essentially it is to prove the following two lemmas.[1]

Lemma 1 (can be proved by induction on the complexity of φ). *If* (α, Δ) *satisfies* φ*, then* $p^{\alpha}_{\Delta} \to \varphi$ *is derivable in K45, else* $p^{\alpha}_{\Delta} \to \neg\varphi$ *is derivable in K45. In particular, if* φ *is tautological, for all* (α, Δ)*,* $p^{\alpha}_{\Delta} \to \varphi$ *is derivable in K45.*

The symbol "p^{α}_{Δ}" denotes $p^{\alpha} \wedge p_{\Delta}$, with:

(1) "p^{α}" for $(\bigwedge_{\underline{\alpha}} p_i) \wedge (\bigwedge_{\bar{\alpha}} \neg p_i)$, $\underline{\alpha}$ (resp. $\bar{\alpha}$) being the set of p_i that α maps to True (resp. to False).

(2) "p_{Δ}" for $(\bigwedge_{\underline{\Delta}} Pp^{\beta}) \wedge (\bigwedge_{\bar{\Delta}} \neg Pp^{\beta})$, $\underline{\Delta}$ (resp. $\bar{\Delta}$) being the set of β that Δ maps to Per (resp. to For).

Lemma 2. *The disjunction of all* p^{α}_{Δ} *is derivable in K45.*

From Lemmas 1 and 2, it also follows that the canonical model for K45 (over n propositional letters) is structurally identical to the logical space $E_1(N)$ equipped with the semantics $S_1(N)$. There is however a fundamental difference. Since it builds on the notion of maximal consistency, the canonical model concept remains partly syntactic. On the contrary, the semantics proposed here is both complete and free of axioms.

[1] Kripke (1959) gives similar lemmas but for S5.

7. *Obligation does not entail permission*

Whether or not obligation entails permission in the deontic logic presented here can be decided by means of a truth-table. Such tables show everything logically important about a given formula.

Configurations	O	p	→	P	p
1. $(\alpha_1; \alpha_1-$Per, α_2-Per)	0	1	1	1	1
2. $(\alpha_2; \alpha_1-$Per, α_2-Per)	0	0	1	1	0
3. $(\alpha_1; \alpha_1-$Per, α_2-For)	1	1	1	1	1
4. $(\alpha_2; \alpha_1-$Per, α_2-For)	1	0	1	1	0
5. $(\alpha_1; \alpha_1-$For, α_2-Per)	0	1	1	0	1
6. $(\alpha_2; \alpha_1-$For, α_2-Per)	0	0	1	0	0
7. $(\alpha_1; \alpha_1-$For, α_2-For)	1	1	0	0	1
8. $(\alpha_2; \alpha_1-$For, α_2-For)	1	0	0	0	0

We see that the last two configurations, that in which both α_1 and α_2 are forbidden, satisfy Op but not Pp. This suffices to conclude that Op does not entail Pp.

8. *Conclusion*

In this paper the foundations of deontic logic have been revisited. This results in a truth-table way of looking at the deontic modality. The philosophical significance of this fresh approach can be summed up as follows:

- It makes it possible to define a semantics for deontic logic on the basis of four primitive terms — (the) True, (the) False, (the) Permitted, (the) Forbidden — without introducing an accessibility relation.
- It provides an alternative interpretation, truth-table based, of the modal system K45.

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