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NEGATION IN METACOMPLETE RELEVANT LOGICS*

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1. *Introduction*

An often-asked question in the context of relevant logics is "What does negation mean?". This is usually asked when it is explained that the Boolean negation of classical logics no longer applies in relevant logics. Boolean negation gives a concept of negation, based on change of truth-value. This is in turn based on the assumption that the truth values, truth and falsity, are mutually exclusive and exhaustive. Indeed, it is better to capture Boolean negation in terms of the mutual exclusiveness and exhaustiveness of this negation rather than dealing with truth and falsity as these concepts are just following the negation, i.e. A is false iff ∼A is true. Mutual exclusiveness is formally captured by the Disjunctive Syllogism (∼A, A ∨ B ⇒ B) (DS), whose disjunctive form is interderivable with that of Ex Falso Quodlibet (A, ∼A ⇒ B) (EFQ), with minimal logical assumptions.[1] Mutual exhaustion, on the other hand, is formally captured by the Law of Excluded Middle (A ∨ ∼A) (LEM). Thus, the LEM and the DS are defining properties of Boolean negation. Indeed, in [UL] (see p. 42), they are used to axiomatize classical formulae, as distinct from general formulae, and, using the logic DJ^d of the book [UL], are sufficient to derive all tautologies.

We will attempt to explicate the meaning of negation in the context of the universal logic DJ^d of [UL] and of some surrounding relevant logics.

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[1] The disjunctive form of a rule $A_1, \ldots, A_n \Rightarrow B$ is $C \vee A_1, \ldots, C \vee A_n \Rightarrow C \vee B$, as introduced in Brady [1984]. It is a slight strengthening of the rule, enabling the '⊃' form of the rule to be derived, provided the LEM holds for each of the premises. The derivation from the disjunctive form of the DS to that of the EFQ uses disjunction introduction, and the converse also uses disjunction introduction with the C of EFQ being replaced by C ∨ B.

These surrounding logics will all be metacomplete relevant logics, in either of Slaney's M1 or M2 varieties. (See his [1984] and [1987] for details of this. We will introduce these metacomplete logics in section 6.) None of these logics have the LEM as a theorem, as they are metacomplete, and hence satisfy the Priming Property: if $\vdash$ A $\vee$ B then $\vdash$ A or $\vdash$ B. Also, none have the DS as a derived rule, as they are paraconsistent, due to their being contained in the logic RM3, used in Brady and Routley [1989] to prove the nontriviality of Extensional Dialectical Set Theory, which contains explicit inconsistencies. (The point here is that if the DS is a derived rule then so is EFQ, which would make RM3-based Extensional Dialectical Set Theory trivial.)

The reason metacomplete logics are important is that they bear a close relationship with those logics which solve the set-theoretic and semantic paradoxes. Neither the paradox-producing A $\vee$ $\sim$A nor A $\&$ $(\text{A} \rightarrow \text{B}) \rightarrow$ B can occur in a metacomplete logic, the former producing Russell's Paradox and the latter producing Curry's Paradox. Further, the main M1-metacomplete logics are from B through to TJ (see section 6 for details), for which simple consistency has been proved in my [UL] for naive set theory and higher-order predicate logic. For the M2-metacomplete logics EW and RW, White in [1979] showed that the comprehension axiom of naive set theory is simply consistent. Further, metacomplete logics are entailment-focussed in that theorems of such a logic are simply built from their entailment theorems using $\sim$, $\&$ and $\vee$. It is due to this that the above Priming Property holds. A prime example of a metacomplete logic is the logic DJ^{d} of meaning containment, called MC in later work. This logic, which reflects the set-theoretic containment properties, is developed in [UL] and has a content semantics with these properties (set out in Brady [1996] and also in [UL]).

In the process of explicating the meaning of negation, we will determine a common metavaluational structure for the theorems of each of these logics and show from these structures that "negations essentially come in pairs". Thus, negation is interrelated to itself in a circular kind of way, though this is a useful rather than a vicious circularity. This raises the question of what negation means in a single rather than this double sense. This is then seen to be just classical Boolean negation, which, as has been argued elsewhere (see [UL], [4BLI] and [WWCDA]), holds in a restricted domain. This gives a bifurcated meaning to negation, with the key classical negation properties split up in these two types, one of which is necessary and intensional, whilst the other is contingent and extensional. This, I believe, is one of the reasons why the concept of negation is so difficult to capture.

Our natural language 'not' can thus be of either type, depending on the use to which it is put. The intensional 'not' is more suited to theoretical contexts where the meanings of concepts are primary, and the extensional 'not' is more suited to observation and more generally to perception, where it is clear that a negated statement and its corresponding unnegated statement are mutually exclusive and exhaustive.

2. *Why Deductive Logics Ought to be Four-Valued*

We start the process at the beginning by examining the derivation of conclusions from the premises of an argument. For such a derivation in either classical or a non-classical logic, it can be seen that not only can a formula A be proved or ∼A be proved, but also both of these can be proved as could neither of them. Whether the logic is classical or not, a logical theory could still be inconsistent, as can be seen with Cantor's naive set theory. Similarly, a logical theory can be ∼-incomplete, even for sentences with no free variables, as can be seen by Godel's First Theorem for Peano Arithmetic.

Indeed, classical logic cannot ensure all of the basic tenets of Boolean negation despite the presence of both the LEM and the DS in its logic. (See my [2007] for this point and also [WWCDA] for some discussion.) So, there is a separation between the formal mechanism used to capture Boolean negation, i.e. the LEM and the DS, from which all tautologies can be derived, and the Boolean negation itself as determined by change of truth-value. And, logical theories are essential to bring out possible ∼- incompleteness, as one must deal with sentences, i.e. formulae without free variables.

As mentioned in [WWCDA], a semantics of a logic ought to reflect the deductive system it is meant to provide a meaning for, and deductive systems are what deductive logic is all about. So, deductive systems, by their very nature should be 4-valued, each value representing a way in which a formula and its negation can be derived, and these four values should be appropriately embedded in such a semantics. A proof-theoretic version of such a semantics can be found in Schroeder-Heister [2006], who expresses the meaning of the logical connectives by their introduction and elimination rules in a natural deduction system. However, we leave the notion of values more open at this stage.

3. *De Morgan Negation*

All this said, we still need to determine the general meaning of negation. Such an attempt was made in Brady [1996] and in [UL], where negation was introduced for the purposes of building up a suitable content semantics. There, the negation of p was introduced as what might be called a "big disjunction" of situations alternative to those satisfying p. On the face of it, this definition seems circular in that negation is defined in terms of being alternative, which is negative in itself, but nevertheless, as shown in [1996] and

[UL], this produces a De Morgan negation in a content semantics. Such a negation satisfies both double negation laws and all four forms of contraposition. All the De Morgan properties deductively follow from these two principles, using basic lattice properties of conjunction and disjunction. This De Morgan negation applies specifically to the physical world, as well as generally. Indeed, we will use this in the next section to analyse the application of negation to the physical world.

The lattice structure of a De Morgan negation gives a mirror image picture of negation, as pointed out by Routley. This picture enables one to see that what is happening on the positive side of the mirror, as it were, has a reflection on the negative side, the components of which can be derived using negation from the positive side. Vice versa, the negative side relates back to the positive side. The fact that there is a single reflection of a point, which then reflects back to itself, represents the double negation law. Joins, representing disjunctions in the lattice, are reflected to meets, representing conjunction, and vice versa. This then represents the De Morgan Laws. An entailment on one side reflects to an entailment on the other, but the direction is the same: towards the mirror from the positive side and away from the mirror on the negative side; it is not reversed. This represents the Contraposition Laws. Statements can be on the line of the mirror, that is, on the line of symmetry of the lattice. These are what are usually called paradoxical statements and take the form $p \leftrightarrow \sim p$, as both the statement and its negation are represented by the same point. Examples of these paradoxes are the Liar sentence 'This very sentence is false', Russell's class is a member of itself, etc. Some of the paradoxes are what are called pseudo-paradoxes, where an assumption is made upon which $p \leftrightarrow \sim p$ is derived. An example of this is the Barber Paradox (see [UL], Ch. 8). As argued in [WWCDA], the derivation of Cantor's Theorem is also an example. Without the LEM or $A \rightarrow \sim A \Rightarrow \sim A$ in the logic, the contradiction $p \,\&\, \sim p$ may not be realized.[2] This is generally the case for the logics we will be introducing.

4. *Negation in the Classical Domain*

Getting back to the classical domain, classical logic was said to apply to the physical world and its abstractions and idealizations, in [UL]. However, as argued in [4BLI] and in [WWCDA], this is too broad, as can be seen from the two examples in the first paragraph of section 2. (Here, naive set theory

[2] Regarding the LEM, note that from $p \leftrightarrow \sim p$, $p \vee \sim p \rightarrow p \,\&\, \sim p$ easily follows. Also note, for intuitionist logic, $p \rightarrow \sim p \rightarrow \sim p$ holds, enabling $\sim p$ to be derived, which then yields p by applying Modus Ponens to $\sim p \rightarrow p$.

and Peano arithmetic would be reasonably considered to be about idealizations.) So, in these two papers, we restricted it further to the large part of the physical world, and also to what can be mapped into it. In [4BLI], we argued that there are $\sim$-incompletenesses due to Heisenberg's Uncertainty Principle in quantum mechanics and due to our inability to determine events as they are happening now in distant space. (A similar point can be made for much of our statements about the past.) We also argued that the classical domain should also include fictional situations, where a one-one mapping into corresponding real classical situations is possible. Further, in [WWCDA], we extended this to include all the finite numerical statements in Peano Arithmetic, as the natural numbers could not be reasonably truncated at some particular finite point. This leaves the classical domain as being quite fuzzy around the edges, with the full finite being used for arithmetic and similarly for set theory, whilst the physical world is truncated in its outer reaches, both with respect to past and future time and the large and small in space.

Nevertheless, to obtain some measure of definiteness, we could use human possibility as the yardstick for determination of the classical domain, in line with deduction itself with its insistence on recursively enumerable proofs and inductive specification of formulae. Indeed, this would then suggest strengthening the classical domain to include recursive sets, where both the membership and the non-membership can be recursively determined, the latter allowing Boolean negation to apply.

What is it about the physical world that engenders Boolean negation? Boolean negation seems to have as its core the placement of matter in Euclidean space-time, representing the essence of the physical world of immediate perception. We examine two principles of objects in space-time and show how the LEM and the DS are derived from them.

The first principle is that it is not possible for a material object to be in two distinct places at the same time. We first show that this is really an instance of the Law of Non-Contradiction.[3] We formalize this as follows:

$$\mathrm{p}_1 \neq \mathrm{p}_2 \rightarrow \sim(\mathrm{Amp}_1\mathrm{t} \,\&\, \mathrm{Amp}_2\mathrm{t}),$$

where p_1 and p_2 are the two distinct places, m is the material object, t is the time, and A is the 3-place relation of something being at a certain place at a certain time. The modality 'not possible' is incorporated into the entailment, which follows as a matter of necessity, given the assumption of Euclidean space-time.

We contrapose to obtain: $(\mathrm{Amp}_1\mathrm{t} \,\&\, \mathrm{Amp}_2\mathrm{t}) \rightarrow p_1 = p_2$. (*)

[3] The fact that this principle is a form of the Law of Non-Contradiction was suggested to me by Penelope Davies.

To establish the Law of Non-Contradiction, we assume:

$$\text{Ampt} \ \& \ {\sim}\text{Ampt}.$$

By the "big disjunction" meaning of negation:

$$\text{Ampt} \ \& \ (\text{Amp}_2\text{t} \vee \text{Amp}_3\text{t} \vee \ldots), \text{ with } \text{p}_\text{i} \neq \text{p}, \text{ for all i} \geq 2,$$

with these Amp_it's representing the alternative situations.
Distributing, we get: $(\text{Ampt} \ \& \ \text{Amp}_2\text{t}) \vee (\text{Ampt} \ \& \ \text{Amp}_3\text{t}) \vee \ldots$
By repeated applications of (*), $\text{p} = \text{p}_2 \vee \text{p} = \text{p}_3 \vee \ldots$, each one of which contradicts our distinctness of place assumption. And, the distinctness of place is a more fundamental assumption than the one concerning the placement of matter in space-time, given the assumption of Euclidean space-time. As such, the original contradiction cannot be so and we have the Law of Non-Contradiction, applied to the placement of matter in space-time. If such a Law holds, then the DS in the form: ${\sim}$Ampt, Ampt $\vee$ B $\Rightarrow$ B, is justified as Ampt is then consistent.

The second principle is that a material object must be somewhere at a given time. We show that this is really an instance of the Law of Excluded Middle. We formalize it as follows: $\exists$pAmpt.
This can be re-expressed as:

$$\text{Ampt} \vee \text{Amp}_2\text{t} \vee \text{Amp}_3\text{t} \vee \ldots, \text{ with } \text{p}_\text{i} \neq \text{p}, \text{ for all i} \geq 2.$$

By the "big disjunction" meaning of negation:

$$\text{Ampt} \vee {\sim}\text{Ampt}.$$

So, we have the LEM.

Thus, these two laws are derived from properties concerning the occupation of Euclidean space-time, but such properties are surely nonlogical properties expressing how matter in space-time happens to be, at least at the level of immediate perception. All that is required is for there to be a certain spatio-temporal otherness and somewhereness that matter can occupy, which provides the core of Boolean negation.[4,5]

[4] There is a parallel universe or multiverse theory of quantum mechanics which allows a particle to be at two distinct places at the same time, but in distinct universes. Also, Heisenberg's Uncertainty Principle can leave a particle without position. However, these are well outside immediate perception.

[5] It has been suggested to me by Lloyd Humberstone that set partitions would do a similar job in the set-theoretic context. There, each member is in one partition or other and cannot be

However, let us consider the effect of Zeno's paradoxes, since they are set in Euclidean space-time and go beyond immediate perception. As they have similar characteristics, we just consider the familiar Paradox of Achilles and the Tortoise. Also, to avoid vagueness which goes beyond the scope of this paper, we will precisify the Paradox. Referring to Clark [2002], pp. 1–2, let Achilles move at twice the speed of the tortoise, giving the tortoise a half-mile start. By the time Achilles has made up this head start, the tortoise has gone a quarter-mile further. By the time Achilles has made up this quarter-mile, the tortoise is an eight of a mile ahead, and so on. There are infinitely many such stages; the Paradox then raises the question of whether Achilles ever reaches the tortoise. However, by taking the mathematical limit of this sequence, Achilles should reach the tortoise exactly after one mile, a finite distance. This illustrates a recursive sequence, of the sort that was mentioned at the beginning of this section, and such recursive sequences fall within the classical domain. We do take it that full Peano Arithmetic is not built into the specification of this sequence as this would have to include Godel's Theorem, which we have already discussed as yielding a $\sim$-incompleteness.

5. *The Propagation View of Negation in some Relevant Logics*

There is the rough view that the weak relevant logics, based on De Morgan negation, enable one to derive one negated formula from another, but without producing an initial single negated formula. As stated in section 3, the basic De Morgan properties are double negation and contraposition, from which the De Morgan's Laws are derivable, using the standard lattice properties of conjunction and disjunction. These all involve two negations, either with one (or more) on each side of the main '$\rightarrow$' or with two (or more) negations on one side of the '$\rightarrow$' and none on the other. So, it seems that one cannot derive a single negation property from no negation and that negations tend to propagate from one to another. We aim to capture this roughly stated view in some formal way, thereby giving it some substance.

However, in stronger relevant logics such as R, E and T (see their axiomatizations below, in section 6), the additional negation property, $A \rightarrow \sim A \rightarrow \sim A$, is added, from which $A \rightarrow B \rightarrow \sim A \vee B$ is easily proved (by substituting $A \mathbin{\&} \sim B$ for A), and hence also $A \vee \sim A$. The latter two properties consist of a single negation property entailed by a non-negation and of a single negation property. The sort of requirement on relevant logics that will separate such logics with $A \rightarrow \sim A \rightarrow \sim A$ from the weaker ones with just

in two partitions at the one time. Further, as suggested in [4BLI], one-one correspondences can be set up between such set-theoretic elements and matter in space-time, ensuring the maintenance of Boolean negation.

the De Morgan negation properties is that of metacompleteness, which we introduce in the following section.

6. *The Key Metacomplete Relevant Logics*

We start by defining the key metacomplete relevant logics, as set out below. For simplicity, we will restrict ourselves to sentential logics. We consider both the M1 and M2 types of metacomplete logics, as distinguished by Slaney in his [1984] and [1987] papers. We also set up a wider range of relevant logics, so we can see how they all relate to each other.

Primitives.
$\sim$, &, $\vee$, $\rightarrow$ (connectives).
p, q, r, ... (sentential variables).
Axioms.
1. $A \rightarrow A$.
2. $A \,\&\, B \rightarrow A$.
3. $A \,\&\, B \rightarrow B$.
4. $(A \rightarrow B) \,\&\, (A \rightarrow C) \rightarrow .A \rightarrow B \,\&\, C$.
5. $A \rightarrow A \vee B$.
6. $B \rightarrow A \vee B$.
7. $(A \rightarrow C) \,\&\, (B \rightarrow C) \rightarrow .A \vee B \rightarrow C$.
8. $A \,\&\, (B \vee C) \rightarrow (A \,\&\, B) \vee (A \,\&\, C)$.
9. $\sim\sim A \rightarrow A$.
10. $A \rightarrow \sim B \rightarrow .B \rightarrow \sim A$.
11. $(A \rightarrow B) \,\&\, (B \rightarrow C) \rightarrow .A \rightarrow C$.
12. $A \rightarrow B \rightarrow .B \rightarrow C \rightarrow .A \rightarrow C$.
13. $A \rightarrow B \rightarrow .C \rightarrow A \rightarrow .C \rightarrow B$.
14. $A \rightarrow .A \rightarrow B \rightarrow B$.
15. $(A \rightarrow .A \rightarrow B) \rightarrow .A \rightarrow B$.
16. $A \rightarrow \sim A \rightarrow \sim A$.

Rules.
1. $A, A \rightarrow B \Rightarrow B$.
2. $A, B \Rightarrow A \,\&\, B$.
3. $A \rightarrow B, C \rightarrow D \Rightarrow B \rightarrow C \rightarrow .A \rightarrow D$.
4. $A \rightarrow \sim B \Rightarrow B \rightarrow \sim A$.
5. $A \Rightarrow A \rightarrow B \rightarrow B$.
6. $A \Rightarrow \sim(A \rightarrow \sim A)$.

Meta-Rule.
1. If $A \Rightarrow B$ then $C \vee A \Rightarrow C \vee B$.

Logics.
B = A1–9, R1–4.

DW = A1–10, R1–3. [R4 is derivable.]
DJ = A1–11, R1–3.
TW = A1–10,12–13, R1–2. [R3 is derivable.]
TJ = A1–13, R1–2.
EW = A1–10,12–13, R1–2,5.
RW = A1–10,12–14, R1–2. [R5 is derivable.]
T = A1–10,12–13,15–16, R1–2.
E = A1–10,12–13,15–16, R1–2,5.
R = A1–10,12,14–15, R1–2. [A13, A16 and R5 are derivable.]
The logics X^d are obtained by adding MR1 to a logic X. DJ^d is the logic developed in Brady [UL].

The logics B, DW, DJ, TW and TJ, together with their '^{d}' extensions are all M1-metacomplete. The logics EW and RW, which, together with their '^{d}' extensions, are M2-metacomplete. Due to these metacompleteness results, each of the '^{d}' extensions do not add any further theorems to the logics, and thus are only added to enhance the strength of the logical rules. Further, any of B, DW and TW, with the addition of R6, is M2-metacomplete.

To introduce the properties of M1- and M2-metacomplete logics, we start with Slaney [1984], p. 162, where two parallel metavaluations M and M* are introduced, as follows, with modified notation:
(I) $M(p) = F$, for all sentential variables p;
$M^*(p) = T$, for all sentential variables p.
(II) $M(A \& B) = T$ iff $M(A) = T$ and $M(B) = T$;
$M^*(A \& B) = T$ iff $M^*(A) = T$ and $M^*(B) = T$.
(III) $M(A \vee B) = T$ iff $M(A) = T$ or $M(B) = T$;
$M^*(A \vee B) = T$ iff $M^*(A) = T$ or $M^*(B) = T$.
(IV) $M({\sim}A) = T$ iff $\vdash {\sim}A$ and $M^*(A) = F$;
$M^*({\sim}A) = T$ iff $M(A) = F$.
(V) $M(A \to B) = T$ iff $\vdash A \to B$ and (if $M(A) = T$ then $M(B) = T$) and (if $M^*(A) = T$ then $M^*(B) = T$);
$M^*(A \to B) = T$ (for M1-logics) *or* $M^*(A \to B) = T$ iff, if $M(A) = T$ then $M^*(B) = T$ (for M2-logics). (This is the only difference between M1- and M2-logics.)

Slaney goes on in [1984] to prove the following:
Lemma 1: $\vdash A$ iff $M(A) = T$.
Lemma 2: $\vdash {\sim}A$ iff $M^*(A) = F$.

Using Lemma 2, we simplify (IV) to:
$M({\sim}A) = T$ iff $M^*(A) = F$;
$M^*({\sim}A) = T$ iff $M(A) = F$.
Using these lemmas, the following key properties follow from the above

metavaluation conditions:
Property 1 : $\vdash$ A $\vee$ B iff $\vdash$ A or $\vdash$ B. (Priming Property)
Property 2 : Not-$\vdash$ $\sim$(A $\rightarrow$ B) (for M1-logics), or
$\vdash$ $\sim$(A $\rightarrow$ B) iff $\vdash$ A and $\vdash$ $\sim$B (for M2-logics)
(Negated Entailment Property)

7. *The Structure of Negation in the Theorems of Metacomplete Relevant Logics*

Our aim is to provide some subformula structure for the theorems of metacomplete relevant logics of either type, in order to support the view that negations essentially come in pairs. We will then contrast this with those non-theorems of these logics which have so-called single negations. We also aim to extend this to rules in order to support the propagation view of negation for these logics.

We could start by examining simple formation trees of the theorems of these logics, which would include all the subformulae in the composition of the theorem. However, these would include subformulae that may not play a key role in the proof of the theorem. For example, B does not play a key role in the proof of A & B $\rightarrow$ A or A $\rightarrow$ A $\vee$ B. Whatever negations occur in B in such cases, they do not matter. It is the negations occurring in the A's that matter and these will nicely balance each other off. So, we need to look into subformula trees that focus on negations that play a role in proof.

Trees are currently used in logic for various purposes. The most commonly used ones are semantically based, with the tree rules following the truth conditions of the connectives in the semantics. However, we wish to consider trees based on proof considerations and we can do this, making use of the above metavaluations for the M1- and M2-logics that we introduced. The fact that metavaluations represent theoremhood can be seen from Lemmas 1 and 2 above. We can make use of the M* valuations to demarcate the single uses of negation and, since M functions as the M** valuation, any doubling up of negation returns us back to the M valuation. That is, single negation occurs when M is changed to M* or when M* is changed to M. Doubling up of negation occurs when M becomes M*, which then reverts to M or when M* becomes M, which then reverts to M*. In this latter case, since formulae are evaluated at M initially, there would still be a single negation present in the M*. As we shall see, these metavaluations will separate the De Morgan negation properties, which will contain only negations which double up, from the core Boolean properties, the LEM (A $\vee$ $\sim$A) and the DS ($\sim$A, A $\vee$ B $\Rightarrow$ B), whose dislayed negations are single negations.

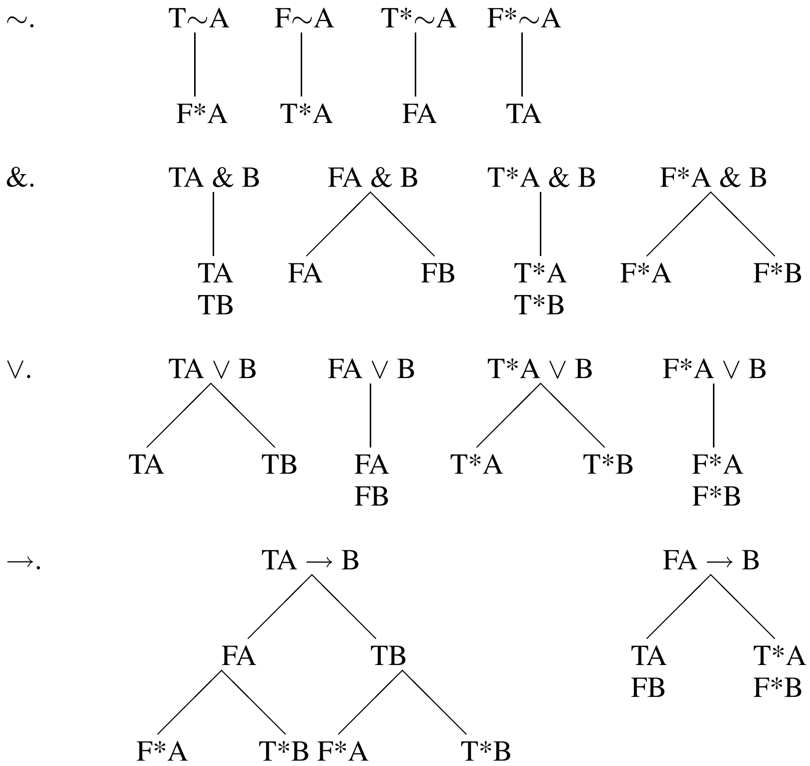
However, the initial problem in using metavaluations for this purpose is the occurrence of '$\vdash$ A $\rightarrow$ B' in the M valuation for A $\rightarrow$ B. All the rest of the metavaluational conditions are clear-cut in that they either reduce to metavaluations of subformulae or terminate, as they do for $M^*(A \to B)$ (for M1- logics), $M(p)$ and $M^*(p)$. So, let us see what the clause '$\vdash A \to B$' does in showing the metavaluational truth of theorems of the logics, as in the proof of the $L \to R$ direction for Lemma 1. We refer to Slaney's proof of his corresponding Lemma 3 on pp. 162–165 of his [1984]. In his proof of $A \to B \to .B \to C \to .A \to C$, the rule forms, $A \to B \Rightarrow B \to C \to .A \to C$ and $A \to B, B \to C \Rightarrow A \to C$ are also used. What we see here, and for the other axioms and rules, is that the inclusion of the clause '$\vdash A \to B$' helps to ensure that $M(A \to B) = T$ holds for theorems of form $A \to B$. Its corresponding rule-form $A \Rightarrow B$ can also be used to show that if $M(A) = T$ then $M(B) = T$, if both A and B are of $\to$-form or if B is of $\to$-form. Similarly, further rule-forms do a similar job, as in the above proof. If we are dealing with theorems in the first place then, given Modus Ponens for '$\to$', this is already known. The use of $\vdash A \to B$ in showing $M(A \to B) = T$ can be separated off from the use of the M and M*-conditions. So, what we will do is simply delete this clause from the metavaluation M for '$\to$'. This will still leave the key metavaluational structure of the proof of the truth of the metavaluation M. This will suffice for our purposes, as it will still highlight the difference between single and double usage of negation in the theorems in which we are interested. Double negation is catered for by putting $M^{**} = M$ and contraposition is catered for by the clause 'if $M^*(A) = T$ then $M^*(B) = T$', reading as 'if $\vdash \sim B$ then $\vdash \sim A$' (using Lemma 2), in the evaluation of $M(A \to B) = T$. For the special case of M2-logics, Property 2 is catered for by the clause 'if $M(A) = T$ then $M^*(B) = T$' in the evaluation of $M^*(A \to B) = T$.

What we wish to do is to set up some easier way of representing the metavaluational structures, instead of setting out the full inductive procedures for each formula or rule under test. The ideal medium for this is a tree structure, whose branches will close when negations double up and remain open when the negations are single. We will see this in the examples to follow. So, what we will do is set up tree structures based on the M and M* metavaluations, for each of the theorems of a metacomplete logic L. We will call these *M-trees*. These trees will work in much the same way as trees for classical logic, except that we distinguish M and M* valuations, essentially making the logic 4-valued. We place the symbol T, T*, F or F* in front of each subformula A under consideration, according to whether $M(A) = T$, $M^*(A) = T$, $M(A) = F$ or $M^*(A) = F$, respectively.

Much as for classical logic, we use a reductio argument and assume that a formula A under consideration takes the M valuation to F, with the aim of showing that each branch *closes* by having a T and F or a T* and F* of the same formula within each branch. In this case, we will say that the M-tree *closes*. Since we are basically analysing theorems of a metacomplete logic

L, this should be the case. Further, when a negation occurs in a branch a T* or F* will be introduced, which will ultimately have to be matched up with a F* or T*, respectively, to close the branch, or alternatively move to a T** or F**, which are T and F, respectively. Each of these subsequent steps will involve introducing a second negation. And, we will also analyse some non-theorems of these logics to show some examples of single negations.

We set up the following connective rules for M-trees following the inductive specifications for M and M* metavaluations, with the given difference between M1- and M2-logics.



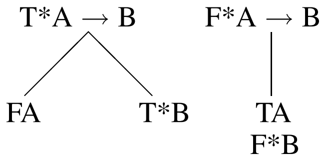
(Often, for TA → B, only one of the unstarred or starred branches will be used, as is appropriate to show closure.)

For M1-logics:

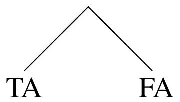
T*A → B
Redundant, since true.

F*A → B
x
Closure, since false.

For M2-logics:



For the sake of the rules and meta-rule of the logics, we add the following tree rule, which enables a 'T' to be placed in front of the premises of each of the logical rules:

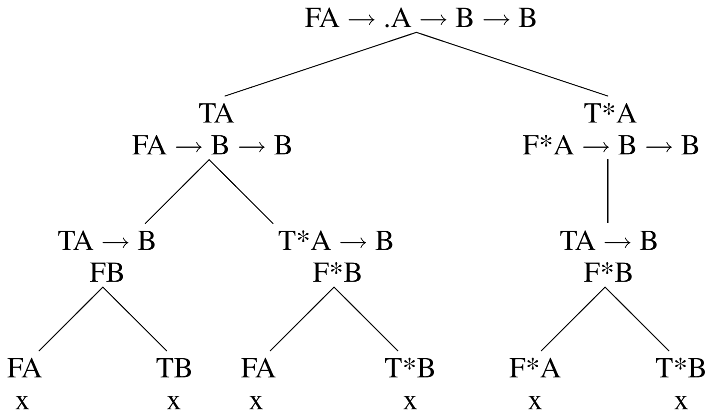


Theorem 1.
The M-trees of the theorems of the M1-logic TJ (and hence all weaker logics) are all closed.
Proof. As in Lemma 3 of Slaney [1984], modified as for his Theorem 3, if A is a theorem of any of these M1-logics then $M(A) = T$. As stated above, we can drop the clause '$\vdash A \to B$' from the metavaluation $M(A \to B) = T$ and the M and M* part of the soundness arguments still persist. We can show this by examining each of the axioms, rules and the meta-rule of the above logics. By classical meta-logical considerations, the M-trees are set up in such a way that they will close for all these theorems. Note that the metavaluations T and F are classically related for both M and M*. We give some examples of M-trees in section (viii) below that will verify some of the axioms and rules of these logics.

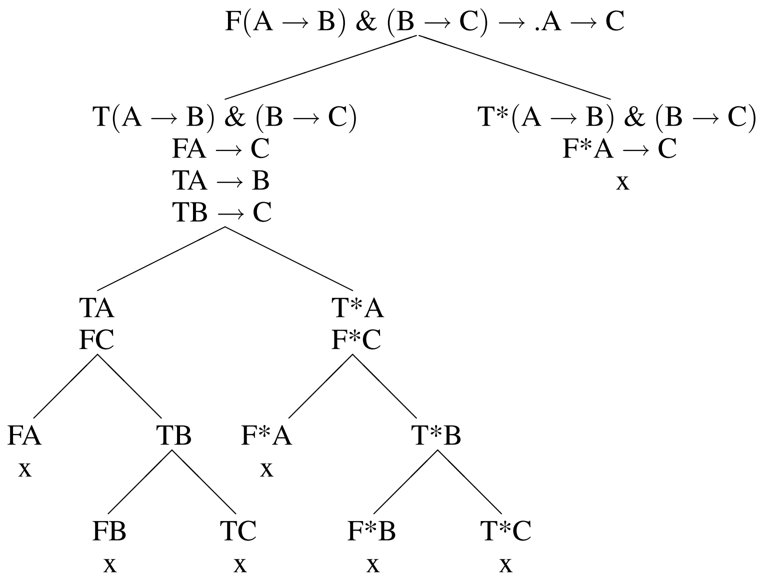
Theorem 2.
The M-trees of the theorems of the M2-logic RW (and hence all weaker logics) are all closed.
Proof. The proof follows as for Theorem 1, except that it concerns the logic RW and uses just Lemma 3 of Slaney [1984].

8. *Some examples of M-trees*

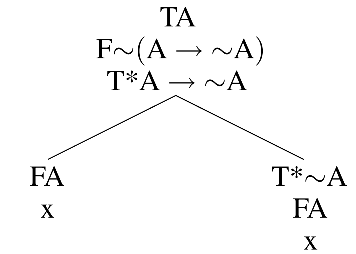
A14. (This axiom can only occur in an M2-logic.)



A11. (This axiom can only occur in an M1-logic.)

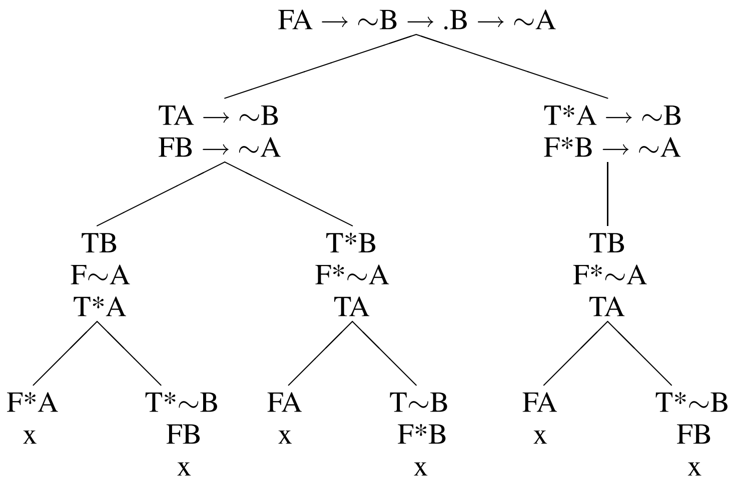


R6. (This rule can only occur in an M2-logic.)

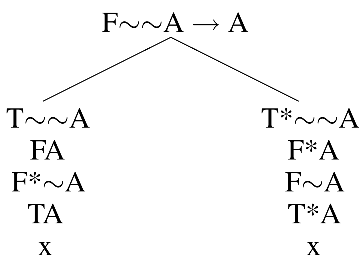


[Applying the additional rule to TA, we would assume FA to be closed.]

A10. (Another example of negation. For the sake of this example, we will assume an M2-logic, though M1 is also possible.)



A9. (In either an M1- or M2-logic.)



∼(A ∨ B) → ∼A & ∼B. (Again, in either an M1- or M2-logic.)

F∼(A ∨ B) → ∼A & ∼B

T∼(A ∨ B)		T*∼(A ∨ B)	
F∼A & ∼B		F*∼A & ∼B	
F*A ∨ B		FA ∨ B	
F*A		FA	
F*B		FB	
F∼A	F∼B	F*∼A	F*∼B
T*A	T*B	TA	TB
x	x	x	x

We will now examine some examples of non-theorems, to show up single negations in non-closing M-trees. In each case, we will determine whether a ∼-consistency or ∼-completeness assumption is needed to close the non-closing branches. A way of representing the ∼-consistency for the formula A in an M-tree is: TA ⇒ T*A. Similarly, a way of representing the ∼-completeness for the formula A in an M-tree is: T*A ⇒ TA. We will use these inferences to plug gaps in order to close a branch of an M-tree. We start with examples involving ∼-completeness and then move onto those involving ∼-consistency, though some involve both. This process enables us to show up Boolean influences in a variety of other formulae and rules.

A & (A → B) → B. (for an M2-logic)

FA & (A → B) → B

TA & (A → B)		T*A & (A → B)	
FB		F*B	
TA		T*A	
TA → B		T*A → B	
FA	TB	FA	T*B
x	x	?	x

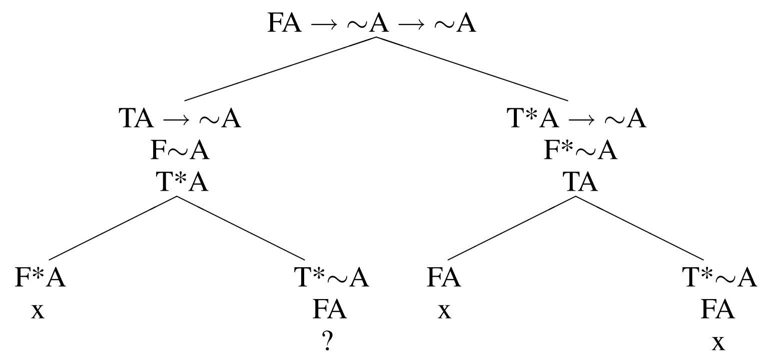
Here, T*A ⇒ TA would close the third branch.
For an M1-logic, T*A → B ⇒ TA → B would enable the two right-hand branches to close.

A ∨ ∼A. (for M1- and M2-logics)

FA ∨ ∼A
FA
F∼A
T*A
?

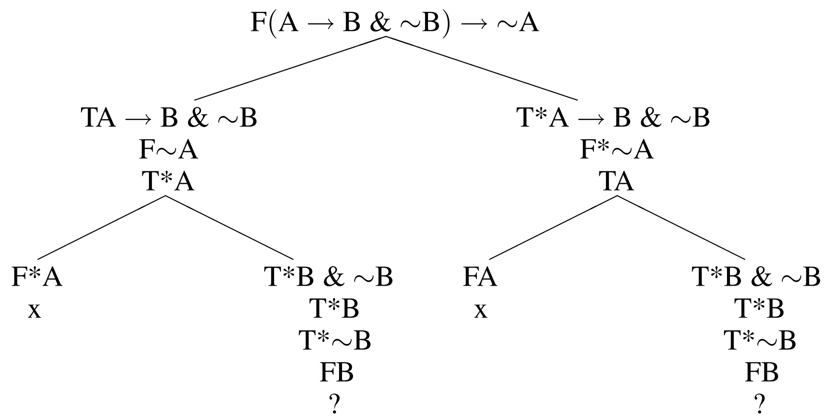
Here, T*A ⇒ TA would close the branch.

A → ∼A → ∼A. (for an M2-logic)



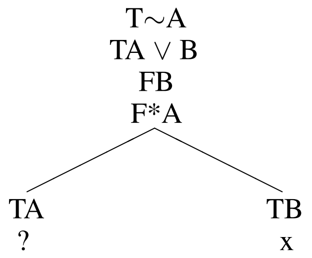
Here, T*A ⇒ TA would close the branch.
For an M1-logic, T*A → ∼A ⇒ TA → ∼A and TA ⇒ T*A would also be needed to close the right-hand branches.

(A → B & ∼B) → ∼A. (for an M2-logic)



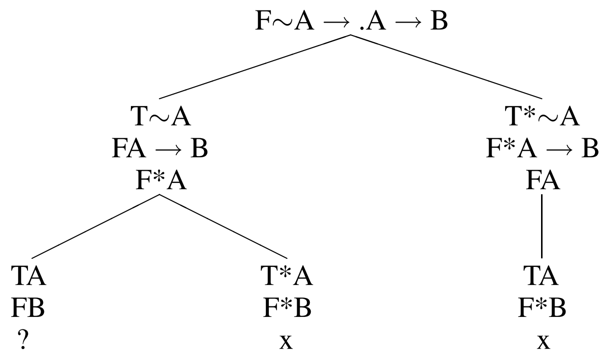
Here, T*B $\Rightarrow$ TB would close the branches.
For an M1-logic, T*A $\rightarrow$ B & $\sim$B $\Rightarrow$ TA $\rightarrow$ B & $\sim$B and then TB $\Rightarrow$ T*B would close the far right-hand branch.

$\sim$A, A $\vee$ B $\Rightarrow$ B. (For M1- and M2-logics)



Here, TA $\Rightarrow$ T*A would close the branch.

$\sim$A $\rightarrow$ .A $\rightarrow$ B. (for an M2-logic)



Here, TA $\Rightarrow$ T*A would close the branch.
For an M1-logic, F*A $\rightarrow$ B would close the right-hand branch sooner.

A $\Rightarrow$ $\sim$(A $\rightarrow$ $\sim$A), in an M1-logic.

TA
F$\sim$(A $\rightarrow$ $\sim$A)
T*A $\rightarrow$ $\sim$A
?

Here, T*A $\rightarrow$ $\sim$A $\Rightarrow$ TA $\rightarrow$ $\sim$A and then TA $\Rightarrow$ T*A would close the branches.

A, $\sim$B $\Rightarrow$ $\sim$(A $\rightarrow$ B), in an M1-logic.

$$\begin{array}{c} \text{TA} \\ \text{T} \sim \text{B} \\ \text{F}{\sim}(\text{A} \rightarrow \text{B}) \\ \text{F*B} \\ \text{T*A} \rightarrow \text{B} \\ ? \end{array}$$

Here, T*A $\rightarrow$ B $\Rightarrow$ TA $\rightarrow$ B and then TB $\Rightarrow$ T*B (or TA $\Rightarrow$ T*A) would close the branches.

We would have to assume $\sim$-consistency or $\sim$-completeness or both to close the branches of these non-theorems. As can readily be seen, both these properties are single negation properties, thus indicating that each of these non-theorems are so as a result of one or two single negation properties. In the case of the last two rules, this presents a case for favouring M1-logics over M2-logics, as these key rules for M2-logics, which serve to differentiate the two types of logics, are seen to depend on two single negation properties. The problem with establishing negated entailments is that very often relevance reasons would apply, but these are not capturable as such within the sort of formal framework that we are using.

9. *The Four- and Three-Valued Matrix Models*

Once the '$\vdash$ A $\rightarrow$ B' clause has been removed from the metavaluation conditions, they take the shape of inductive modelling conditions and, because four combinations can be generated from M(A) and M*(A) being true or false, this should result in a 4-valued matrix logic. And, there should be slightly different matrices for the M1- and M2-logics, with the difference occurring in the $\rightarrow$-matrix. In this section, we will introduce and explore these two matrix logics. We will also see that each of these reduces to a three-valued matrix logic when the metavaluations for sentential variables are taken into account. We include these matrices as a matter of interest, and indeed those obtained for M2-logics are already familiar.

First, we put the four combinations of M and M* metavaluations together, with their matrix valuations, as follows:
t: M(A) = T and M*(A) = T.
b: M(A) = T and M*(A) = F.
n: M(A) = F and M*(A) = T.
f: M(A) = F and M*(A) = F.
Due to Lemmas 1 and 2, the matrix valuation v will satisfy the following:
v(A) = t iff $\vdash$A and not-$\vdash$ $\sim$A;

v(A) = b iff ⊢A and ⊢ ∼A;
v(A) = n iff not-⊢A and not-⊢ ∼A;
v(A) = f iff not-⊢A and ⊢ ∼A.
As we can see, the t and f values represent the classical provability of A and ∼A, respectively, whilst the b and n values represent non-classical provability, the b standing for both provabilities and the n for neither.

We start by establishing the matrices for M2-logics, using the metavaluational conditions for M and M* for the various connectives. We get the following matrices:

∼	
t	f
b	b
n	n
f	t

&	t	b	n	f
t	t	b	n	f
b	b	b	f	f
n	n	f	n	f
f	f	f	f	f

v	t	b	n	f
t	t	t	t	t
b	t	b	t	b
n	t	t	n	n
f	t	b	n	f

→	t	b	n	f
t	t	f	n	f
b	t	b	n	f
n	t	n	t	n
f	t	t	t	t

The designated values are t and b, i.e. where M(A) = T, and validity is defined as usual in terms of these. The matrices can be recognized as those for the logic BN4, introduced and axiomatized in Brady [1982], pp. 10,21–3. As can be seen from the proof of Theorem 2, which follows that of Theorem 1, the M2-logic RW, and thus all weaker logics, are valid in the matrix logic BN4. One can also check the axioms and rules against the matrices. However, BN4 itself is not metacomplete since it contains disjunctive axioms such as A ∨ ∼B ∨ (A → B). Recall that we checked the axioms and rules of RW to see that '⊢ A → B' could be removed from the metavaluation condition for M(A → B), but it cannot be removed when evaluating A ∨ ∼B ∨ (A → B). There may be stronger logics than RW which are M2-metacomplete, but we would have to check whether '⊢ A → B' can be removed, and in such a case the logic would be contained in BN4.

We can take this further. In all of this modelling, we have not used the metavaluations for the sentential variables p: M(p) = F and M*(p) = T, which is the matrix value n above. This amounts to the common assignment of all sentential variables to n. When we do this, we can see that the value t is obtainable from p → q and f is obtainable from ∼(p → q), but the value b cannot be reached as one needs the value b to start with. So, we can replace the 4-valued matrices by 3-valued ones without the 'b', to obtain the following:

∼	
t	f
n	n
f	t

&	t	n	f
t	t	n	f
n	n	n	f
f	f	f	f

v	t	n	f
t	t	t	t
n	t	n	n
f	t	n	f

→	t	n	f
t	t	n	f
n	t	t	n
f	t	t	t

t is the only remaining designated value. Clearly then, the matrix logic is the Lukasiewicz 3-valued logic, L3, which is stronger than BN4, as any formula invalid in L3 can also be made invalid in BN4 using the same assignment of values.

For M1-logics, we obtain the following matrices:

$\sim$	
t	f
b	b
n	n
f	t

&	t	b	n	f
t	t	b	n	f
b	b	b	f	f
n	n	f	n	f
f	f	f	f	f

v	t	b	n	f
t	t	t	t	t
b	t	b	t	b
n	t	t	n	n
f	t	b	n	f

$\rightarrow$	t	b	n	f
t	t	n	n	n
b	t	t	n	n
n	t	n	t	n
f	t	t	t	t

As above, the designated values are t and b, with validity defined as usual. We will call the matrix logic BN4-1, as it models the M1-logics in a similar manner to the modelling of M2-logics by BN4. Indeed, as for M2-logics and BN4, the M1-logics TJ, and thus all weaker logics, are valid in the matrix logic BN4-1, by using the proof of Theorem 1.

Note that the $\rightarrow$-matrix of BN4-1 has designated values in the same places as for BN4, but of course the values differ. Further, the Smiley matrices in [ENT1], pp. 161–2, differ from those of BN4-1 only in that the occurrences of 'n' in the $\rightarrow$-matrix of BN4-1 are replaced by 'f' in Smiley's matrices and designation is removed from b. The upshot of this is that Smiley's matrices, BN4 and BN4-1 are all characteristic for the first-degree entailment fragment E_{fde} of E and indeed of all logics listed in section 6.

Since $\rightarrow (t, f)$ is assigned the non-classical value n, an interesting question arises as to whether there are any formulae that are valid in BN4-1, but are not valid in classical logic. It can easily be seen that $\sim(A \rightarrow B) \rightarrow .C \rightarrow D$ is such a formula. To see this, note that $\sim(A \rightarrow B)$ takes f or n only and $C \rightarrow D$ takes t or n only.

As for the M2-logics, these matrices reduce to a 3-valued logic without the value 'b' in the same manner. These differ from L3 only in the assignment $\rightarrow (t, f)$.

10. *Concluding Remarks*

We first note that the propagation of single negations can occur through an '$\rightarrow$'-theorem. If $A \rightarrow B$ is a theorem and A has a single negation, so must B. To see this, the tree rule for $FA \rightarrow B$ yields TA and FB, as well as T*A and F*B. Since A has a single negation, this must be cancelled out by a single negation in B, through the use of a common variable, which relevant logics have. The same must apply to a rule $A \Rightarrow B$, whose tree starts with TA and FB. It follows that single negation needs to be established outside of the logic, as the logic only propagates negation, once it is already obtained.

Indeed, by all the above argument, we conclude that negation is at base a non-logical concept, which through its interaction with the connectives and quantifiers becomes a logical concept in part, through a doubling up of negation. That is, the concept of negation is bifurcated between the intensional concept, as introduced for the content semantics in [UL], and the classical extensional concept, formally captured by the LEM and the DS. It is this split personality that makes negation an especially difficult concept to pin down and enables its over- and under-determination, despite having an intensional concept in part. Classical negation, within the simplicity of classical logic, is an attempt at being a fully logical concept, but it turns out to be quite limited in its scope, as can be seen in [4BLI] and [WWCDA].

However, I believe we can go further. Since De Morgan negation is the intensional negation, which is at base four-valued, concepts need to be both positively and negatively characterised by the axiomatization. Unlike for classical logic, where the negative is automatically determined from the positive with help from the LEM and the DS, just like a default setting, the negative here needs to be independently built up in parallel with the positive, to give negativity a chance to develop. The negative characterisation is dependent on the concept(s) involved as the definition of the scope of the negation can differ from concept to concept. As part of this process, classical negation is appropriate for the large part of the physical world, which in turn relies on the material exclusion of space-time with the negativity of not being in two places at the same time and being somewhere or other. Other concepts, which could be fictional or infinite, could have different negations, depending on how the concepts are introduced. This especially applies to concepts introduced prescriptively like laws and theoretical concepts, to concepts with great generality like sets, and to incompletely specified concepts like vague concepts. Consider infinite sets, with both recursive and non-recursive properties, primarily relying on intension, expressed using entailment, and definition, expressed using co-entailment. The negative membership of such a set could be determined using non-membership of a recursive superset. Further, as pointed out in [4BLI], classical negation could be justified for fictional settings through the use of one-one mappings from the physical world. Also, as used in [WWCDA], the Peano axioms of arithmetic contain not only the usual negative axiom, $\sim 0 = n'$, but also the rule, $\sim m = n \Rightarrow \sim m' = n'$, which complements the positive rule, $m' = n' \Rightarrow m = n$.

Thus, once we consider negativity for the various concepts, it can bring the Boolean and De Morgan negations into greater perspective. The De Morgan negation always holds as a frame into which can be slotted single negations that apply for particular concepts. Boolean negation is such a negation which is appropriate for a range of concepts and situations, whilst other concepts will require a single negation more specific to them. So, it seems that

Boolean negation is just one of a number of single negation concepts with its own range of applicability.

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