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A PARADOX REMAINS

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The attempt to formalise modalities such as necessity or the a priori as predicates struggles with several paradoxes. Montague [4] e.g. famously proves that a theory Γ is inconsistent if it contains the following axioms and rule:

(M1) Robinson's arithmetic Q
(M2) α'D' $\rightarrow$ D
(M3) derive α'D' from D

It has been attempted to overcome the difficulties by formal as well as informal arguments. It turns out however that none of them leads to a theory that is consistent as well as satisfactory, for too many paradoxes can be found. The major strategy is to combine the theories of two modalities; in many cases the compound theory is inconsistent, although both of the two original theories are consistent. A paradox put forward by Halbach[1] [1] is of this kind and will be considered more closely below. An important result concerning tense logic along the same lines has been proven by Horsten and Leitgeb [2].

I'd like to introduce a new paradox of this kind. Actually, what I shall present is only a slightly changed version of Montague's paradox. But the slight change will have a great effect: It makes the paradox immune to a solution available to both Montague's original paradox and Halbach's paradox. Two further remarkable features with respect to which the new paradox differs from the other two will be emphasised at the close of the paper. The philosophical significance of the paradox depends on the claim that the axioms from which it is derived do indeed express properties of the modalities of necessity and apriority[2]. The most part of this paper will be concerned with establishing the plausibility of this claim.

[1] I'd like to thank Volker Halbach for discussing his results with me.

[2] Apriority should be understood as a special kind of actual knowledge, viz. knowledge that is justificationally independent of sense experience. It will not suffice to read apriority as justification, since being justified in believing p does not entail the truth of p, whereas knowledge of p does. Apart from this, the results do not depend on accepting a particular account of the a priori.

Consider now the first order theory Π with Robinson's arithmetic Q as a sub-theory and

(T) T'D' ↔ D, if D does not contain T
(P1) derive α'D' from D, if D does not contain T
(P2) α'D' → T'D', if D does not contain T

as further axioms. Then Π is inconsistent.

Proof :

01. ¬α'D' ↔ D	Diagonalisation
02. ¬D	for reductio
03. α'D'	from 1.
04. T'D'	from 3. with (P2)
05. D	from 4. with (T)
06. ⊥	from 2. and 5.
07. D	from 2. to 6.
08. α'D'	from 7. with (P1)
09. ¬D	from 1. and 8.
10. ⊥	from 7. and 9. Q.E.D.

It needs to be made clear why Π should consist of exactly these axioms. The theory combines the truth predicate with another predicate that is appropriately characterized by (P1) and (P2), and furthermore unspecified. The discussion will show that N (necessity) and Ap (a priori) are such predicates. With respect to its combining two predicates, Π resembles the theory Σ Halbach [1] has put forward and proven inconsistent. The starting point for his proof is the T-Scheme: (T) is the hierarchical formalisation of the truth predicate in Σ (and Π); since the predicate is typed, (T) is consistent if it is taken by itself. That is to say that (T) is not subject to Montague's paradox. However, the combination of truth and necessity will yield a further contradiction. This combination is desirable because one wants to prove sentences like "all mathematical *truths* are *necessary*" in a theory of truth. Consider now Σ. If it contains Robinson's Q and the following axioms and rule, it will be inconsistent.

(T) T'D' ↔ D, if D does not contain T
(N1) N'D' → D, if D does not contain N
(N2) Derive N'D' from D, if D does not contain N.

Proof :

1. D ↔ ¬T'N'D"	Diagonalisation
2. T'N'D" ↔ ¬D	from 1.
3. N'D' → ¬D	(T)

4. $N`D' \rightarrow D$	(N1)
5. $\neg N`D'$	from 3. and 4.
6. $\neg T`N`D''$	(T)
7. D	from 1. and 6.
8. N'D'	(N2)
9. $\perp$	from 5. and 8.

(T) as well as (N1) and (N2) are typed and so consistent if they remain separate, as Halbach [1, 278] remarks. What his paradox shows is thus that the hierarchy is not capable of securing the consistency of a theory in all circumstances but only of defending the theory against the liar. However, a way around Halbach's paradox exists and will be examined later. At any rate, considering Σ makes clear that (T) is in its typed form not — by itself — responsible for an inconsistency and can therefore be part of Π.

The same must be shown for (P1) and (P2). (P1) resembles (N2), but the axioms have different restrictions on what the valid instances of the axiom scheme are. (N1) and (N2) would be inconsistent by Montague's paradox if they allowed for N in D. So the question is if (P1), and (P2) with it, should be restricted like (N1), disallowing α in D rather than T. First of all, the actual restrictions have been made only to make sure that the classical liar cannot be derived from Π. But why should further restrictions be made? (N1) and (N2) have their restrictions in order to show that the hierarchy does not achieve a lot. (The question of whether the hierarchical approach is a satisfactory formalisation of the natural notion of truth or necessity is left aside; Priest [5] argues for a negative answer.) However, that Halbach imposes restrictions on the axioms of Σ for a certain purpose — viz. showing that they don't help — does not mean that one should allow for these restrictions always and everywhere without further argument. The mere fact that a contradiction can be derived without them is certainly not such an argument, and another one is not apparent. On the contrary, it seems to be an especially random move to restrict (P1) or (P2), since unlike (N1) and (N2), they are, by themselves, consistent without any restriction. Only the combination with (T) makes the theory inconsistent. Moreover, every account that does not allow D to contain α as an instance of the axioms will have to explain why the particular predicate for which α stands is supposed to be hierarchical. If Π is a theory of, say, the a priori, one would have to answer the question as to why any knowledge, and in particular knowledge a priori should be typed. There is hence no reason why the axioms should be changed.

The next reaction to the inconsistency of Π might be the attempt to reject (P1) or (P2) completely. Taking 'a priori' or 'necessary' for α, (P2) can hardly be refused. Even if someone should disagree with all of the informal arguments that can be given in favour of (P2), the following simple formal argument will convince her. Rejecting (P2) one will have to endorse its negation which has a particularly unwelcome consequence:

1. $\neg(\alpha$‘D’ $\rightarrow$ T‘D’$)$	$\neg$(P2)
2. α‘D’ $\wedge \neg$T‘D’	by sentential logic

Every sentence to which (P2) does not apply is α but not true; this is fatal if α is e.g. ‘a priori’ or ‘necessary’. Thus, (P2) is part of a theory of necessity and of the a priori (and probably some other modalities).

(P1) seems just as secure as (P2). Notice that it is identical with (M3) and (N2), modulo the restrictions. Thus if there is an argument for dropping (P1) for a certain modality, this modality will also be safe from Montague’s and Halbach’s paradox. The axiom is certainly appropriate for truth and necessity. But one might attempt to argue that it does not express a property of the a priori: one could deny that all truths of arithmetic are a priori, and then argue that — in contradiction of this — with (P1) and Robinson’s Q in the theory, all truths of arithmetic that follow from Q *would* come out as a priori. It is questionable that there are provable truths of arithmetic that are not a priori, but for the sake of argument let us accept this claim and try to drop (P1) due to the above argument.

We then arrive at a theory of the a priori, call it Θ, that contains Robinson’s arithmetic Q but not (P1). Since we are now also concerned with Montague’s and Halbach’s paradox, Θ should contain the axioms (M2) [$\approx$ N1] with ‘a priori’ for α and (T)[3]. Neither Montague’s nor Halbach’s paradox can be derived from the theory Θ. Unfortunately, however, Θ is consistent with the claim that no sentence is a priori. One can check this by extending a model for Q, (T) and (M2) by taking the empty set for the extension of the predicate Ap. Thus, since we lack a rule that allows for the introduction of the predicate Ap, Θ is trivially consistent and can hardly be considered as a theory of the a priori. If we want to add such a rule of introduction and preserve consistency at the same time, we need a rule that is weaker than (P1):

(P1*) derive Ap‘D’ from D, if D does not contain either Ap or T.

The above contradictions, in particular Halbach’s, still cannot be derived from the extended theory, but the whole point of combining truth and the a priori is lost, for sentences like ‘It is true that it is a priori that x’ or ‘It is knowable a priori that x is true’ will never be provable in a theory containing (P1*). But this is what we want to achieve by the combination of the two

[3] For reasons that will soon become clear, Θ should in fact contain α‘α‘D’ $\rightarrow$ D’ $\wedge$ $\neg\alpha$‘D’ rather (M2). But this does not change what else is to be said about Θ.

predicates. Therefore, the theory Θ will either be trivial (without (P1*)), or too weak (with (P1*)) or inconsistent (with (P1)).

Since a trivial theory is of no use, a theory of the a priori finally must contain a rule for introducing the predicate Ap, just as the theories of truth and necessity must contain a corresponding rule.

Such a rule does three things: First of all, it is stronger than (P1*) and makes thus sure that one can prove sentences in the theory that combine two predicates. Secondly, it makes the theory of the a priori judge as a priori all mathematical truths that are provable from the theory; although some people might not want this, they will have to admit that a theory without (P1), such as the original Θ, is no option since it will be trivial. Thirdly, it brings the theory close to Montague’s and Halbachs paradox. In other words: if it contains (M2), as Θ does, it will be inconsistent. Notice that (M2) is not part of the theory Π.

(M2) is however contained in Γ and, with a restriction, also in Σ and is needed for the derivation of the respective contradictions. But it can be shown that (M2) should not be an axiom of Σ and — for some α — also not of Γ. Consider the following theory Λ which has all axioms of Robinson’s Q and the following axioms for all sentences D:

(L1)	α‘D’ $\rightarrow$ α‘α‘D”	(Iteration)
(L2)	α‘D $\rightarrow$ E’ $\rightarrow$ (α‘D’ $\rightarrow$ α‘E’)	(Distribution)
(L3)	derive α‘D’ from D	(Necessitation)

Löb [3] has shown that under these assumptions the following rule is always valid: If α‘D’ $\rightarrow$ D is a theorem, so is D. For a *single* sentence D for which

(L4) α‘α‘D’ $\rightarrow$ D’ $\wedge$ $\neg\alpha$‘D’

holds suffices to render Λ inconsistent. Thus its negation,

(L4$'$) α‘α‘D’ $\rightarrow$ D’ $\rightarrow$ α‘D’,

follows from (L1) to (L3) and should be accepted as an axiom, rather than (M2). As I said, if one makes this change, Halbach’s paradox will no longer hold, since 4. is not derivable any more. Montague’s paradox is avoided, too, but only for those predicates that are adequately formalised with the axioms (L1) to (L3). Necessity is a predicate of this kind but truth is not since allowing iteration for the truth predicate will lead to the paradox of the liar. I’ll leave open the question as to whether (L1) to (L3) really apply to the a priori. In any event, (M2) is inappropriate for the formalisation of some predicates. They can thus be rescued from Montague’s and Halbach’s paradox, but not from the one I have presented. This is the strength of this paradox. It is also the only real difference between this ‘new’ paradox and

Montague's: While the latter can be avoided by replacing (M2), this solution is not applicable to the former.

Two further features of the contradiction derivable from Π are noteworthy as strong points of the paradox. The first is this: after presenting his paradox, Halbach discusses a way of restoring the consistency of Σ by a further restriction of the legitimate instances of (T). The suggestion concerns the so-called indirect occurrence of the truth predicate. The proposal argues that N'D' contains the truth predicate indirectly since the sentence D contains it (by diagonalization). Thus, T'N'D'' $\leftrightarrow$ N'D' (the step from 2. to 3.) and $\neg$N'D' $\leftrightarrow$ $\neg$T'N'D'' (the step from 5. to 6.) shouldn't be valid instances of (T). If these steps are prohibited, the proof can no longer be executed. Therefore disallowing all sentences with indirect occurrences of T as instances of (T) is a way around Halbach's paradox. It might not be a very promising one, since the required technical details can be expected to be complex, as Halbach [1, 278] points out. But it is not a solution at all for the contradiction derivable from Π, since a sentence with an indirect occurrence of T is not required for this proof.

The second remarkable feature of Π is that (P1) and (P2) taken by themselves are consistent, even if they are given in their strongest form, i.e. without any restriction. This can be seen as follows: Λ (with (L4$'$) rather than (L4)) is consistent, i.e. it has a model and α has a certain extension. Since (L3) = (P1), modulo the restriction, the latter is in Λ already. Adding (P2) to the theory will only mean that the predicate T has the same extension as α. Thus (P1) and (P2) by themselves are consistent. Moreover, they seem to be two randomly chosen axioms in the sense that they are not, other than (N1) and (N2), a theory of their own. However since they express fundamental properties of e.g. the a priori and necessity, they are certainly part of the theories of these modalities. And these theories might well be consistent, even if they impose no restrictions on their axioms; in this respect they are more 'solid' than a theory containing (N1) and (N2) which tend to be inconsistent anyway. Hence, (P1) and (P2) are part of theories that could serve as a theory of, say, necessity — until they are extended to a theory of necessity and truth. What the presented paradox thus shows is that combining the theories of two different predicates leads to contradiction, even if one of the theories — other than (N1) + (N2) — is solid in the said respect.

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REFERENCES

[1] Halbach, Volker, How not to state T-sentences, *Analysis* 66 (2006), 276–280.
[2] Horsten, Leon and Leitgeb, Hannes, No Future, *Journal of Philosophical Logic* 30 (2001), 259–265.
[3] Löb, M., Solution of a problem of Leon Henkin, *Journal of Symbolic Logic* 20 (1955), 115–118.
[4] Montague, Richard, Syntactical treatment of modality, with corollaries on reflection principles and finite axiomatizability, reprinted in: *Formal philosophy: Selected papers of Richard Montague*, ed. Richmond Thomason, New Haven and London 1974, 286–302.
[5] Priest, Graham, *In Contradiction*, Oxford ^{2}2006, cap. 1.5 and 1.6.