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DEONTIC ALGEBRAS OF ACTIONS

THIERRY LUCAS[†]

In the present paper, we continue to explore the structure of actions in the line of our papers [VWA] and [AA]. In the first of those papers, we started from some of von Wright's ideas to isolate a concept which we present again here under the name of 'explicit algebra of actions' : one of the fundamental intuitions is to consider an action as a coherent mapping from a set of conditions to results ; conditions and results are members of boolean algebras B and C (representing states of the world) respectively and one looks at the very rich structure of coherent mappings from finite parts of B to C. In the second paper, we took a more abstract stance and presented an axiomatic version of actions under the name of 'support algebras with truth-value supports' or variants of it ; the idea was there to have a unisort version of explicit algebras of actions, embedding, so to speak, the algebra B of conditions into the algebra of actions.

The aim of the present paper is to go a few steps further into the study of actions. In the study of explicit algebras of actions, we add now the consideration of two embeddings and of their adjoints : one embedding represents a condition as the support of a 0-valued action ; its adjoint is the functor which associates to an action the union of the domain of that action ; the other embedding represents a result as the everywhere defined action having that result ; for complex actions in general, associating various results to various conditions, there are two natural adjoints to that functor : the first one forms the conjunction of the results; the second one forms the disjunction of the results.

In addition to those two embeddings and their adjoints, we add the study of the description of an action, something easily accounted for when the algebra of conditions B and the algebra of results C coincide. The idea is to describe the action α by associating with it an element $\Phi\alpha$ of the common algebra of conditions and results saying that 'such and such a condition gives such and such a result', $\Phi\alpha = \bigwedge_{\sigma\in\Sigma}(\sigma \to \alpha(\sigma))$, where Σ is the set of conditions of α. Actions and their descriptions should not be confused ; roughly speaking, the difference is that α is a mapping, while $\Phi\alpha$ is a formula.

[†]Emeritus Professor at Université catholique de Louvain.

The last step in our study consists in admitting that the common algebra of conditions and results is endowed with a necessity operator $\Box$ obeying standard conditions ; we then define the obligation $O\alpha$ to do α as the necessity of $\Phi\alpha$, $O\alpha = \Box\Phi\alpha$, formalizing the idea that the obligation to do an action α is the necessity that such and such a condition gives such and such a result associated to it by α.

The above presentation is given in some details in the first part of the present paper, avoiding however most proofs, which in general consist of easy computations. In the second part of the paper, we move towards an abstract version of explicit algebras of actions : in the line of [AA], they are support algebras with internal mappings derived from the three adjunctions given above ; truth-value supports may be considered and descriptions may be added ; if a reasonable necessity operator is present, there is also a quite acceptable concept of obligation internalizing the $O\alpha$ described above.

We have already sketched in [VWA] and in [AA] what we think are the benefits of our approach. The present paper is best considered as a deepening of the question, but it seems to us that it confirms some of the interesting features of the approach, revealing the ambiguity of some concepts and giving the instruments to make the necessary distinctions, for example : the conjunction of actions has at least two different senses ; the description of an action has to be distinguished from the action ; necessity applies to descriptions of actions, while obligation applies to actions ; obligation distributes over the conjunction of actions in one sense of the conjunction, but not in another sense ; etc.

A final word to apologize for repeating the main definitions of [VWA] and [AA] : we thought that the reader would appreciate a relatively self-contained paper, avoiding him constant reference to our preceding papers.

Part 1. Explicit algebras of actions with additional structure

Section 1.1. Explicit algebras of actions

Our starting point is that an action α should be considered as a mapping defined on a set Σ of conditions and associating to each condition $\sigma \in \Sigma$ a certain result $\alpha(\sigma)$. Conditions and results are states of the world and in the algebraic approach adopted in [AA] and pursued here, conditions live in a boolean algebra B, the *boolean algebra of conditions*, and results live in a boolean algebra C, the *boolean algebra of results* ; B and C need not be identical at this moment of our study and it may be conceptually useful to distinguish them. We repeat now the basic definitions of fundamental operations on actions and of explicit algebras. We remind the reader that we distinguish a 'long' and a 'short' conjunction ; roughly speaking, the long conjunction is defined on the union of the domains of the component actions, while the short conjunction is defined on the intersection of those domains. We make a similar distinction for disjunction. Let us also insist on the fact that negation has many different meanings, the most important ones being expressed here below by $\neg$ and $\sim$. For more comments, we refer the reader to our paper [AA].

Definition 1.1.1:
(1) An action from B to C *is a mapping* $\alpha\ : dom\alpha \longrightarrow C$, *where* $dom\alpha \subseteq B$, $dom\alpha$ *is finite and* α *satisfies a* coherence condition *: for all* $\sigma, \sigma' \in dom\alpha$, *if* $\sigma \wedge \sigma' \neq 0$, *then* $\alpha(\sigma) = \alpha(\sigma')$.
(2) Actions are pre-ordered *by the relation* $\leq$ *defined by :* $\alpha \leq \beta$ *iff* $\bigvee dom\beta \leq \bigvee dom\alpha$ *and for all* $\sigma \in dom\alpha$, $\pi \in dom\beta$, *if* $\sigma \wedge \pi \neq 0$, *then* $\alpha(\sigma) \leq \beta(\pi)$.
(3) The pre-ordering $\leq$ *induces an* equivalence $\approx$ *of actions characterized by :* $\alpha \approx \beta$ *iff* $\bigvee dom\alpha = \bigvee dom\beta$ *and for all* $\sigma \in dom\alpha$, $\pi \in dom\beta$, *if* $\sigma \wedge \pi \neq 0$, *then* $\alpha(\sigma) = \beta(\pi)$.
(4) The empty *action* 1 *is defined by :* $dom(1) = \emptyset$ *and* 1 *is the empty mapping from* $dom(1)$ *to* C.
(5) The everywhere nul *or* zero *action* 0 *is defined by :* $dom(0) = \{1_B\}$ *(*1_B *representing the greatest element of* B*) and* $0(1_B) = 0_C$ *(*0_C *representing the smallest element of* C*).*
(6) Operations $\cdot$, $-$ *and* $+$ *on finite subsets* Σ *and* Π *of* B *are defined by :*
$\Sigma \cdot \Pi = \{\sigma \wedge \pi \mid \sigma \in \Sigma, \pi \in \Pi, \sigma \wedge \pi \neq 0\}$
$-\Sigma = \{\neg \bigvee \Sigma\}$
$\Sigma + \Pi = (\Sigma \cdot \Pi) \cup (-\Sigma \cdot \Pi) \cup (\Sigma \cdot -\Pi)$.
(7) The conjunction $\alpha \wedge \beta$ *of actions* α *and* β *is defined by :* $dom(\alpha \wedge \beta) = dom\alpha + dom\beta$ *and for all* $\omega \in dom(\alpha \wedge \beta)$, $(\alpha \wedge \beta)(\omega)$ *is defined according to the form of the domain by three cases :*
(a) if $\omega \in dom\alpha \cdot dom\beta$, *then* $\omega = \sigma \wedge \pi$ *for some* $\sigma \in dom\alpha$ *and* $\pi \in dom\beta$ *and one lets* $(\alpha \wedge \beta)(\omega) = \alpha(\sigma) \wedge \beta(\pi)$ *;*
(b) if $\omega \in -dom\alpha \cdot dom\beta$, *then* $\omega = \neg \bigvee dom\alpha \wedge \pi$ *for some* $\pi \in dom\beta$ *and one lets* $(\alpha \wedge \beta)(\omega) = \beta(\pi)$ *;*
(c) if $\omega \in dom\alpha \cdot -dom\beta$, *then* $\omega = \sigma \wedge \neg \bigvee dom\beta$ *for some* $\sigma \in dom\alpha$ *and one lets* $(\alpha \wedge \beta)(\omega) = \alpha(\sigma)$.
(8) The disjunction $\alpha \vee \beta$ *of actions* α *and* β *is defined by :* $dom(\alpha \vee \beta) = dom\alpha \cdot dom\beta$ *and for all* $\omega \in dom(\alpha \vee \beta)$, $\omega = \sigma \wedge \pi$ *for some* $\sigma \in dom\alpha$ *and* $\pi \in dom\beta$ *and one lets* $(\alpha \vee \beta)(\omega) = \alpha(\sigma) \vee \beta(\pi)$.
(9) The 0-cosupport $C_0\alpha$ *of action* α *is defined by :* $dom(C_0\alpha) = -dom\alpha = \{\neg \bigvee dom\alpha\}$ *and* $(C_0\alpha)(\neg \bigvee dom\alpha) = 0_C$.
(10) The negation $\neg\alpha$ *of action* α *is defined by :* $dom(\neg\alpha) = dom\alpha$ *and for all* $\sigma \in dom\alpha$, $(\neg\alpha)(\sigma) = \neg(\alpha(\sigma))$, *the latter negation being taken in* C.

We recall that the equivalence relation $\approx$ is compatible with the operations $\neg$, $\wedge$, $\vee$ and C_0, a fact which allows us to work with the quotient structure, or less formally to identify equivalent actions, writing $\alpha = \beta$ instead of the more formal $\alpha \approx \beta$ and speak accordingly of the ordering $\leq$.

The set of all actions from B to C equipped with the distinguished elements 0, 1, with the operations $\neg$, $\wedge$, $\vee$, C_0 and with the relation $\leq$ will be referred to as the *explicit algebra of all actions from B to C*. For further reference and comparison, we give a formal definition of the more general notion of 'explicit algebra of actions' :

Definition 1.1.2:
An explicit algebra of actions $\mathcal{A}$ *is a triple* $< B, A, C >$ *where* B *and* C *are Boolean algebras and* A *is a* $< 0, 1, \neg, \wedge, \vee, C_0 >$*-subalgebra of the explicit algebra of all actions from* B *to* C.

Note that in explicit algebras of actions, interesting derived operations may be obtained by duality, via $\neg$: $\alpha \leq^* \beta$ iff $\neg\beta \leq \neg\alpha$, $\alpha \wedge^* \beta = \neg(\neg\alpha \vee \neg\beta)$ (the *short conjunction*, defined on $dom\alpha \cdot dom\beta$), $\alpha \vee^* \beta = \neg(\neg\alpha \wedge \neg\beta)$ (the *long disjunction*, defined on $dom\alpha + dom\beta$), $1^* = \neg 0$ (the *everywhere unit* action), $C_1\alpha = \neg C_0\neg\alpha = \neg C_0\alpha$ (the 1-*cosupport* of α). Note in particular that $\neg 1 = 1$, so that the empty action is auto-dual. Other derived notions are given by : $S_0\alpha = C_0C_0\alpha$ (the 0-*support* of α) and its dual $S_1\alpha = \neg S_0\neg\alpha = \neg S_0\alpha$ (the 1-*support* of α), $\gamma \setminus \beta = C_0\beta \wedge (\gamma \wedge^* \neg\beta)$ (the *difference* of γ and β) and its dual $\beta \rightarrow^* \gamma = C_1\beta \vee^* (\neg\beta \vee \gamma)$ (the *co-implication* from β to γ). An action α is *total* when $S_0\alpha = 0$ or equivalently $C_0\alpha = 1$.

On the other hand, there are very fundamental operations on actions which make sense in explicit algebras of actions, but do not seem to be derivable from the preceding operations. They will not play an important role in this paper, except when we deal with descriptions of actions where they find a nice a posteriori justification.

Definition 1.1.3:
(1) The 0-value 0-support *of* α, $S_0^{=0}\alpha$, *is defined by* $dom(S_0^{=0}\alpha) = \{\sigma \mid \sigma \in dom\alpha, \alpha(\sigma) = 0\}$ *and for* σ *in that domain,* $(S_0^{=0}\alpha)(\sigma) = 0$.
(2) The complex negation *of* α, $\sim\alpha$, *is defined by* $dom(\sim\alpha) = -\Sigma \cup \Sigma_{\neq 0}$, *where* $\Sigma = dom(\alpha)$ *and* $\Sigma_{\neq 0} = \{\sigma \mid \sigma \in dom\alpha, \alpha(\sigma) \neq 0\}$ *; for* $\sigma \in -\Sigma$, *i.e.* $\sigma = \neg\bigvee\Sigma$, *one lets* $(\sim\alpha)(\sigma) = 0$ *; for* $\sigma \in \Sigma_{\neq 0}$, *one lets* $(\sim\alpha)(\sigma) = \neg(\alpha(\sigma))$.
(3) The implication *from* β *to* γ, $\beta \rightarrow \gamma$, *is defined by* $dom(\beta \rightarrow \gamma) = (-\Pi \cdot \Xi) \cup (\Pi/\Xi)$, *where* $\Pi = dom\beta$, $\Xi = dom\gamma$ *and* $\Pi/\Xi = \{\pi \wedge \xi \mid \pi \in \Pi, \xi \in \Xi, \pi \wedge \xi \neq 0$ *and* $\beta(\pi) \not\leq \gamma(\xi)\}$ *; for* $\omega \in -\Pi \cdot \Xi$, *one*

has $\omega = \neg \bigvee \Pi \wedge \xi$ *for some* $\xi \in \Xi$ *and one lets* $(\beta \to \gamma)(\omega) = \gamma(\xi)$ *; for* $\omega \in \Pi/\Xi$*, one has* $\omega = \pi \wedge \xi$ *for some* $\pi \in \Pi$*,* $\xi \in \Xi$ *and one lets* $(\beta \to \gamma)(\omega) = \beta(\pi) \to \gamma(\xi)$*.*

Section 1.2. Embedding conditions as actions

Explicit algebras of actions are described in detail in [AA] but their study, be it explicit or axiomatic under the name of "support algebras", is essentially limited to the $< 0, 1, \neg, \wedge, \vee, C_0 >$-structure and to its enrichment by operations such as $S_0^{=0}$, $\sim$, or $\to$. We want to consider here a yet richer structure, induced by natural embeddings from B to $\mathcal{A}$ and from C to $\mathcal{A}$ and we make here the general proviso that our explicit algebras of actions are stable under the new operations.

Since an action α is a mapping from $dom\alpha \subseteq B$ into C, there is a natural way of representing elements ϕ of B as mappings $\overleftarrow{\phi}$ in $\mathcal{A}$: take ϕ as the unique element of $dom(\overleftarrow{\phi})$ and associate 0_C with ϕ. Conversely, given an action α, there is a natural way of associating an element $Cond(\alpha)$ of B with it by taking the disjunction of the finite domain of α. Although a bit formal, those representations find some intuitive content in expressions like 'avoid situation ϕ' (for $\overleftarrow{\phi}$) and 'here are the conditions of action α' (for $Cond\alpha$).

Definition 1.2.1:
(1) For ϕ *in B,* $\overleftarrow{\phi}$ *is the action defined by* $dom\overleftarrow{\phi} = \{\phi\}$ *and* $\overleftarrow{\phi}(\phi) = 0_C$*.*
(2) For $\alpha \in A$*,* $Cond\alpha = \bigvee dom\alpha$*.*

Here are the basic properties of $\overleftarrow{(\ )}$ and $Cond$ expressed with (a moderate use of) the terminology of functors :

Proposition 1.2.2:
(1) $\overleftarrow{(\ )}$ *is contravariant : if* $\phi \leq \psi$*, then* $\overleftarrow{\psi} \leq \overleftarrow{\phi}$*.*
(2) $Cond$ *is contravariant : if* $\alpha \leq \beta$*, then* $Cond\beta \leq Cond\alpha$*.*
(3) $\overleftarrow{(\ )}$ *and* $Cond$ *are adjoint for the relevant orderings :* $\overleftarrow{\phi} \leq \alpha$ *iff* $\phi \geq Cond\alpha$*.*
(4) $\overleftarrow{(\ )}$ *is faithful : if* $\overleftarrow{\phi} = \overleftarrow{\psi}$*, then* $\phi = \psi$*.*
(5) $Cond$ *is faithful on* S_0*-supports : if* $Cond(S_0\alpha) = Cond(S_0\beta)$*, then* $S_0\alpha = S_0\beta$*.*
(6) $Cond$ *is determined by* S_0*-supports :* $Cond(\alpha) = Cond(S_0\alpha)$*.*

It is routine work to deduce from those properties all the expected properties of $\overleftarrow{(\ )}$ and $Cond$. We give them here in the natural way of proving things :

Proposition 1.2.3:
(1) $\phi \geq Cond\overleftarrow{\phi}$; $\overleftarrow{\phi \wedge \psi} = \overleftarrow{\phi} \vee \overleftarrow{\psi}$; $\overleftarrow{1_B} = 0$.
(2) $\overleftarrow{Cond\alpha} \leq \alpha$; $Cond(\alpha \wedge \beta) = Cond\alpha \vee Cond\beta$; $Cond1 = 0_B$.
(3) If $\overleftarrow{\phi} \leq \overleftarrow{\psi}$, *then* $\psi \leq \phi$; $\phi \leq Cond\overleftarrow{\phi}$.
(4) If $CondS_0\alpha \leq CondS_0\beta$, *then* $S_0\alpha \geq S_0\beta$.
(5) $Cond\overleftarrow{\phi} = \phi$; $S_0\overleftarrow{\phi} = \overleftarrow{\phi}$; $\overleftarrow{0_B} = 1$; $\overleftarrow{\phi \vee \psi} = \overleftarrow{\phi} \wedge \overleftarrow{\psi}$; $\overleftarrow{\neg\phi} = C_0\overleftarrow{\phi}$.
(6) $\overleftarrow{CondS_0\alpha} = S_0\alpha$; $CondS_0\alpha = Cond\alpha$; $Cond0 = 1_B$; $Cond(\alpha \vee \beta) = Cond\alpha \wedge Cond\beta$; $CondC_0\alpha = \neg Cond\alpha$.
(7) $\overleftarrow{Cond\alpha} = S_0\alpha$.

To summarize many of those properties, we can state that $\overleftarrow{(\ )}$ and $Cond$ establish a pair of isomorphisms between the structure $< B, 0_B, 1_B, \wedge, \vee, \neg >$ and the structure $< S_0A, 1, 0, \vee, \wedge, C_0 >$, where $S_0A = \{\alpha \mid S_0\alpha = \alpha\}$. We can also interpret property (7) as saying that the triple induced by the adjunction reduces to S_0.

Section 1.3. Embedding results as actions

We now turn to the natural way of embedding C into $\mathcal{A}$: associate with $\phi \in C$ the action $\overrightarrow{\phi}$ which is everywhere defined and gives the result ϕ, an action which finds its intuitive content in an expression like ‘in any case, obtain ϕ’. In the other direction, given an action α, which in general assigns different results to different conditions, there are two natural ways of associating one result ; the first one, to be denoted by $Res^{\wedge}\alpha$, is the conjunction of the different results of α ; the other one, to be denoted by $Res^{\vee}\alpha$, is the disjunction of the different results of α. $Res^{\wedge}\alpha$ represents so to speak the strongest result, the result one has to obtain to be sure to perform the action in every circumstance ; of course, this could easily lead to impossible actions, should for example α be described by $dom\alpha = \{\sigma_1, \sigma_2\}, \sigma_1 \wedge \sigma_2 = 0, \sigma_1 \neq 0, \sigma_2 \neq 0, \alpha(\sigma_1) = \neg\alpha(\sigma_2)$, in which case $Res^{\wedge}\alpha = \alpha(\sigma_1) \wedge \alpha(\sigma_2) = 0$. Dually, $Res^{\vee}\alpha$ represents the weakest result ; obtaining the disjunction of all transformations recommended in all circumstances by α is certainly implied by performing α and represent the ‘laziest’ way of approximating α ; it could also amount in many circumstances to obtaining nothing new ; take for example the same α as before, in which case $Res^{\vee}\alpha = \alpha(\sigma_1) \vee \alpha(\sigma_2) = 1$. A more significant case is when

α is constant (same result, whatever the condition) ; disregarding the case of the empty action, $Res^{\wedge}\alpha$ and $Res^{\vee}\alpha$ then coincide with the unique result of α (see section 1.4 for a more thorough discussion).

Here are the precise definitions :

Definition 1.3.1:
(1) For $\phi \in C$, $\overrightarrow{\phi}$ is the action defined by dom $\overrightarrow{\phi} = \{1_B\}$ and $\overrightarrow{\phi}(1_B) = \phi$.
(2) For $\alpha \in A$, $Res^{\wedge}\alpha = \bigwedge_{\sigma \in dom\alpha} \alpha(\sigma)$.
(3) For $\alpha \in A$, $Res^{\vee}\alpha = \bigvee_{\sigma \in dom\alpha} \alpha(\sigma)$.

The basic properties of $\overrightarrow{(\ )}$, $Res^{\wedge}$ and $Res^{\vee}$ are given in the following proposition.

Proposition 1.3.2:
(1) $\overrightarrow{(\ )}$ is covariant : if $\phi \leq \psi$, then $\overrightarrow{\phi} \leq \overrightarrow{\psi}$.
(2) $Res^{\wedge}$ is covariant : if $\alpha \leq \beta$, then $Res^{\wedge}\alpha \leq Res^{\wedge}\beta$.
(3) $\overrightarrow{(\ )}$ and $Res^{\wedge}$ are left and right adjoints for the relevant orderings : $\overrightarrow{\phi} \leq \alpha$ iff $\phi \leq Res^{\wedge}\alpha$.
(4) $\overrightarrow{(\ )}$ is faithful : if $\overrightarrow{\phi} = \overrightarrow{\psi}$, then $\phi = \psi$.
(5) $\neg\overrightarrow{\phi} = \overrightarrow{\neg\phi}$.
(6) If $\alpha \neq 1$, then $Res^{\wedge}(\overrightarrow{\phi} \vee S_0\alpha) \leq \phi$.
(7) $Res^{\vee}\alpha = \neg Res^{\wedge}\neg\alpha$.

To interpret property (6), remember that $\beta \vee S_0\alpha$ works as a restriction of β to the domain of α and observe that for $\alpha \neq 1$, the equality $Res^{\wedge}(\overrightarrow{\phi} \vee S_0\alpha) = \phi$ easily follows ; property (6) then means not only that $Res^{\wedge}$ applied to $\overrightarrow{\phi}$ gives back ϕ, but also that the same is true of every non-empty restriction of $\overrightarrow{\phi}$. Property (7) expresses that $Res^{\vee}$ is the dual $(Res^{\wedge})^*$ of $Res^{\wedge}$, in a sense given by $\neg$ and recalled in section 1.1 : $\alpha \leq^* \beta$ iff $\neg\beta \leq \neg\alpha$, $\alpha \wedge^* \beta = \neg(\neg\alpha \vee \neg\beta)$, etc. It follows that all properties of $Res^{\wedge}$ may be dualized to give properties of $Res^{\vee}$. For example, $Res^{\vee}$ and $\overrightarrow{\phi}$ are left and right adjoints for the relevant orderings : $Res^{\vee}\alpha \leq \phi$ iff $\alpha \leq^* \overrightarrow{\phi}$.

Here are properties routinely deduced from the basic ones :

Proposition 1.3.3:
(1) $\phi \leq Res^{\wedge}\overrightarrow{\phi}$; $\overrightarrow{\phi \vee \psi} = \overrightarrow{\phi} \vee \overrightarrow{\psi}$; $\overrightarrow{0_B} = 0$.
(2) $\overrightarrow{Res^{\wedge}\alpha} \leq \alpha$; $Res^{\wedge}(\alpha \wedge \beta) = Res^{\wedge}\alpha \wedge Res^{\wedge}\beta$; $Res^{\wedge}1 = 1_C$.
(3) If $\overrightarrow{\phi} \leq \overrightarrow{\psi}$, then $\phi \leq \psi$; $Res^{\wedge}\overrightarrow{\phi} \leq \phi$; $Res^{\wedge}0 = 0_C$.

(4) $\overrightarrow{\phi \wedge \psi} = \overrightarrow{\phi} \wedge \overrightarrow{\psi}$ *;* $\overrightarrow{1_C} = 1^*$ *;* $S_0\overrightarrow{\phi} = 0$ *;* $C_0\overrightarrow{\phi} = 1$ *;* $Res^\wedge 1^* = 1_C$ *;* $Res^\wedge S_1\alpha = 1_C$.
(5) If $\overrightarrow{\phi} \leq S_0\alpha \neq 1$*, then* $\overrightarrow{\phi} = 0$ *or, equivalently,* $\phi = 0_C$ *; if* $\alpha \neq 1$*, then* $Res^\wedge\alpha \leq Res^\vee\alpha$ *; if* $\alpha \neq 1$*, then* $Res^\wedge S_0\alpha = 0_C$.

Section 1.4. Approximations of actions

The structure of triple induced by the embedding $\overrightarrow{(\ )}$ and its two adjoints is now less trivial than in the case of the embedding $\overleftarrow{(\ )}$ and it does not reduce to already known operations. Here are the definitions.

Definition 1.4.1:
(1) $\alpha^\wedge = \overrightarrow{Res^\wedge\alpha}$.
(2) $\alpha^\vee = \neg(\neg\alpha)^\wedge$.

Definition (2) gives $\alpha^\vee$ as the dual of $\alpha^\wedge$, and it is clear that $\alpha^\vee = \overrightarrow{Res^\vee\alpha}$. Actions $\alpha^\wedge$ and $\alpha^\vee$ are easily described : $dom(\alpha^\wedge) = dom(\alpha^\vee) = \{1_B\}$, $\alpha^\wedge(1_B) = \bigwedge_{\sigma\in dom\alpha}\alpha(\sigma)$ and $\alpha^\vee(1_B) = \bigvee_{\sigma\in dom\alpha}\alpha(\sigma)$. Their interpretation is clear : $\alpha^\wedge$ is the best approximation of α by total actions from below for the ordering $\leq$ and dually, $\alpha^\vee$ is the best approximation of α from above for the ordering $\leq^*$. Intuitively speaking, $\alpha^\wedge$ and $\alpha^\vee$ underly such recommendations as 'in any case, do everything recommended by α' and 'in any case, do at least something recommended by α'.

Basic properties are given in the following proposition :

Proposition 1.4.2:
(1) $\alpha^\wedge \leq \alpha$ *; if* $\alpha \leq \beta$*, then* $\alpha^\wedge \leq \beta^\wedge$ *;* $\alpha^\wedge \wedge \beta^\wedge \leq (\alpha \wedge \beta)^\wedge$ *;* $\alpha^\wedge \leq \alpha^{\wedge\wedge}$.
(2) $S_0\alpha^\wedge = 0$ *; if* $\alpha \neq 1$*, then* $(S_0\alpha)^\wedge = 0$ *; for* $\alpha = 1$, $(S_0\alpha)^\wedge = 1^\wedge = 1^*$.

The following properties are routinely derived :

Proposition 1.4.3:
(1) $(\alpha \wedge \beta)^\wedge = \alpha^\wedge \wedge \beta^\wedge$ *;* $\alpha^{\wedge\wedge} = \alpha^\wedge$.
(2) $0^\wedge = 0$ *;* $1^\wedge = 1^*$ *;* $(1^*)^\wedge = 1^*$.
(3) $(C_1\alpha)^\wedge = 1^*$ *;* $(S_1\alpha)^\wedge = 1^*$.
(4) If $C_0\alpha \neq 1$ *(i.e.* $S_0\alpha \neq 0$*), then* $(C_0\alpha)^\wedge = 0$ *; if* $C_0\alpha = 1$ *(i.e.* $S_0\alpha = 0$*), then* $(C_1\alpha)^\wedge = 1^*$ *; if* $\alpha \neq 1$*, then* $(S_0\alpha)^\wedge = 0$ *; if* $\alpha = 1$*, then* $(S_0\alpha)^\wedge = (S_01)^\wedge = 1^*$.
(5) $C_0\alpha^\wedge = 1$ *;* $C_1\alpha^\wedge = 1$ *;* $S_1\alpha^\wedge = 1^*$.

As already hinted at in section 1.3, an interesting point in considering the functors $Res^{\wedge}$, $Res^{\vee}$, $(\ )^{\wedge}$ and $(\ )^{\vee}$ is that they allow us to recover constant actions and to characterize them in different ways. Looking back at the basic situation with actions as functions, constant actions may be defined as those which give the same result whatever the condition :

Definition 1.4.4:
The action α is constant *iff for all σ, $\sigma' \in dom\alpha$, $\alpha(\sigma) = \alpha(\sigma')$.*

The empty action 1 is constant and the other constant actions may be described as defined on a unique condition and associating with it a unique result ; remember indeed that our actions are defined up to the equivalence $\approx$: $\alpha \approx \overline{\alpha}$, where $\overline{\alpha}$ is in this case defined on $\bigvee dom\alpha$ and $\overline{\alpha}(\bigvee dom\alpha)$ is the common $\alpha(\sigma)$. Using $Res^{\wedge}$ and $Res^{\vee}$, we have thus another more abstract way of characterizing constant actions : α is constant iff $\alpha = 1$ or $Res^{\wedge}\alpha = Res^{\vee}\alpha$. Indeed, if α associates the result $\alpha(\sigma)$ to the unique condition σ, then clearly $Res^{\wedge}\alpha = Res^{\vee}\alpha = \alpha(\sigma)$; conversely, if $Res^{\wedge}\alpha = Res^{\vee}\alpha$, then for all conditions $\sigma, \sigma' \in dom\alpha$, $\alpha(\sigma) = \alpha(\sigma')$. This discussion proves the first of the following properties :

Proposition 1.4.5:
(1) The action α is constant iff $\alpha = 1$ or $Res^{\wedge}\alpha = Res^{\vee}\alpha$.
(2) The action α is constant iff $\alpha = 1$ or $\alpha^{\wedge} = \alpha^{\vee}$.
(3) If α is constant, then $\neg\alpha$ is constant, $Res^{\wedge}(\neg\alpha) = \neg Res^{\vee}\alpha$, $Res^{\vee}(\neg\alpha) = \neg Res^{\wedge}\alpha$, $(\neg\alpha)^{\wedge} = \neg\alpha^{\vee}$ and $(\neg\alpha)^{\vee} = \neg\alpha^{\wedge}$.
(4) The action 0 is constant, $Res^{\wedge}0 = Res^{\vee}0 = 0_C$ and $0^{\wedge} = 0^{\vee} = 0$.
(5) For all α, $S_0\alpha$ is constant ; for $\alpha \neq 1$, $Res^{\wedge}(S_0\alpha) = Res^{\vee}(S_0\alpha) = 0_C$ and $(S_0\alpha)^{\wedge} = (S_0\alpha)^{\vee} = 0$; for $\alpha = 1$, $S_0\alpha$ is the constant 1.
(6) If α is constant, then $\alpha = \alpha^{\wedge} \vee S_0\alpha$.
(7) If α is constant, then so is $\alpha \vee S_0\beta$ for every β.
(8) If α is constant, then $\alpha^{\wedge}$ is total and constant.
(9) The action α is constant iff $\alpha = \alpha^{\wedge} \vee S_0\beta$ for some β.
(10) If α and β are constant, then so is $\alpha \vee \beta$.
(11) If α and β are constant and $\alpha \vee \beta \neq 1$, then $(\alpha \vee \beta)^{\wedge} = \alpha^{\wedge} \vee \beta^{\wedge}$.

We omit the proofs, since the less trivial of them will be given later in an axiomatic context. A nice way of expressing a good deal of those properties is to state that $\alpha^{\wedge}$ is the best total constant approximation of α from below : for all total constant β, $\beta \leq \alpha$ iff $\beta \leq \alpha^{\wedge}$. For the proof of that fact, use (6). Dually, $\alpha^{\vee}$ is the best total constant approximation of α from above in a sense given by the dual ordering : for all total constant β, $\alpha \leq^{*} \beta$ iff $\alpha^{\vee} \leq^{*} \beta$.

Section 1.5. Representing conditions as results and results as conditions

When we deal with a pair of adjoint functors, the induced "triple" structure carries fundamental information with it. In section 1.2, we saw that the triple induced by $\overleftarrow{(\ )}$ and $Cond$ coincides with S_0 : $\overleftarrow{Cond\alpha} = S_0\alpha$. In sections 1.3 and 1.4, we saw that the triple induced by $\overrightarrow{(\ )}$ and $Res^\wedge$ gives something new, denoted by $\alpha^\wedge$: $\alpha^\wedge = \overrightarrow{Res^\wedge\alpha}$; dually, $\alpha^\vee = \overrightarrow{Res^\vee\alpha}$. If we now consider the quite natural case where the algebra of conditions coincides with the algebra of results, it makes sense to consider "mixed" compositions :

Definition 1.5.1:
For any action α, $\alpha^c = \overrightarrow{Cond\alpha}$, $\alpha^r = \overleftarrow{Res^\wedge\alpha}$ and $\alpha^{r^*} = \overleftarrow{Res^\vee\alpha}$.

We use the letter 'c' to remind conditions, the letter 'r' to remind results and the asterisk to remind duality. The mappings $(\ )^c$, $(\ )^r$ and $(\ )^{r^*}$ are endo-maps of the algebra of actions and they play an important role in structuring it. The first one associates with an action α the total action α^c whose result is $Cond\alpha$; besides $\overleftarrow{Cond\alpha}$, which coincides with $S_0\alpha$, it is thus another way of representing the conditions of an action, this time as a total constant action, expressed by something like 'in any case put yourself in the conditions of α'. The second map associates with an action α the action α^r whose domain is $\{Res^\wedge\alpha\}$ and value is 0_C ; besides $\overrightarrow{Res^\wedge\alpha}$, which coincides with $\alpha^\wedge$, it is also another way of representing the (conjunction of the) results of the action, this time as an element of $S_0\mathcal{A}$, expressed by something like 'avoid the conjunction of results of α'. Similar considerations apply with the third mixed composition α^{r^*}.

The basic properties of $(\ )^c$, $(\ )^r$ and $(\ )^{r^*}$ are given in the following proposition.

Proposition 1.5.2:
(1) The mapping $(\ )^c$ is contravariant: if $\alpha \leq \beta$, then $\beta^c \leq \alpha^c$.
(2) The mapping $(\ )^r$ is contravariant: if $\alpha \leq \beta$, then $\beta^r \leq \alpha^r$.
(3) The mappings $(\ )^c$ and $(\ )^r$ are adjoint for the relevant orderings : $\alpha^c \leq \beta$ iff $\alpha \geq \beta^r$.
(4) $(C_0\alpha)^c = \neg\alpha^c$; $(\alpha \vee \beta)^c = \alpha^c \wedge \beta^c$; $(\neg\alpha)^c = \alpha^c$; $S_0\alpha^c = 0$.
(5) If $S_0\alpha = 0$, then $(C_0\alpha)^r = 1$ and if $S_0\alpha \neq 0$, then $(C_0\alpha)^r = 0$; $S_0\alpha^r = \alpha^r$.
(6) If $\alpha \neq 1$, then $(\gamma^c \vee S_0\alpha)^r = \gamma^{cr}$.
(7) $\alpha^{cr} = S_0\alpha$; $\alpha^{rc} = \alpha^\wedge$; $\alpha^{r^*c} = \alpha^\vee$; $\alpha^{r^*} = C_0(\neg\alpha)^r$.
(8) The operation α is constant iff $\alpha = 1$ or $\alpha^r = \alpha^{r^*}$.

Starting from those basic properties, we may apply routine computations to derive all kinds of relations between $(\ )^c$, $(\ )^r$ and $(\ )^{r^*}$. We refer the reader to the second part of this paper, where those properties will be stated and some of them explicitly proved in an axiomatic context.

We anticipate also the second part of this paper to mention that many of the properties of $(\ )^c$ and $(\ )^r$ are embodied in the fact that those functors induce a pair of inverse homomorphisms between the structure $S_0\mathcal{A} =< S_0A, \leq, \wedge, 1, 0, C_0 >$ of S_0-supports of the algebra A of actions and the structure $Tc\mathcal{A} =< TcA, \geq, \vee, 0, 1^*, \neg >$ of total constant elements of A, TcA denoting the set of total constant elements of A.

Section 1.6. Description of actions

We go on assuming that $B = C$ and introduce the description of an action. As already explained in the introduction, the idea is to introduce a functor Φ associating with each action α an element $\Phi(\alpha)$ of the common algebra $B(= C)$ expressing that such and such a condition implies such and such a result :

Definition 1.6.1:
For any action α, the description *of α is defined by $\Phi(\alpha) = \bigwedge_{\sigma \in dom\alpha}(\sigma \to \alpha(\sigma))$.*

Here are the basic properties of Φ.

Proposition 1.6.2:
(1) $\Phi(\alpha \wedge \beta) = \Phi\alpha \wedge \Phi\beta$; $\Phi(\alpha \vee \beta) = \Phi\alpha \vee \Phi\beta$.
(2) $\Phi(0) = 0_B = 0_C$; $\Phi(1) = 1_B = 1_C$.
(3) $\Phi(\neg\alpha) = \neg\Phi(\alpha) \vee \neg\Phi(C_0\alpha)$.
(4) $\Phi(\overleftarrow{\phi}) = \neg\phi$; $\Phi(\overrightarrow{\phi}) = \phi$.
(5) $\Phi(C_0\alpha) = Cond\alpha$.

Note that putting together (3) and (5) gives $\Phi(\neg\alpha) = \neg Cond\alpha \vee \neg\Phi(\alpha) = (Cond\alpha \to \neg\Phi(\alpha))$, the rightarrow designating here the implication of the common algebra of conditions and results ; the description of $\neg\alpha$ appears thus quite naturally as the negation of the description of α restricted to the conditions of α.

It may be of interest to introduce the dual Φ^* of Φ :

Definition 1.6.3:
For any action α, the dual description *of α is defined by $\Phi^*(\alpha) = \neg\Phi\neg\alpha$.*

Here are derived properties of descriptions of actions.

Proposition 1.6.4:
(1) If $\alpha \leq \beta$*, then* $\Phi\alpha \leq \Phi\beta$.
(2) $\neg\Phi\alpha = \Phi(\neg\alpha \wedge C_0\alpha)$.
(3) $\Phi(C_0C_0\alpha) = \neg\Phi(C_0\alpha)$.
(4) If $\Phi(S_0\alpha) = \Phi(S_0\beta)$*, then* $S_0\alpha = S_0\beta$.
(5) $(Cond\alpha \rightarrow Res^{\wedge}\alpha) \leq \Phi\alpha \leq (Cond\alpha \rightarrow Res^{\vee}\alpha)$.
(6) If α *is constant, then* $\Phi\alpha = (Cond\alpha \rightarrow Res^{\wedge}\alpha) = (Cond\alpha \rightarrow Res^{\vee}\alpha)$.
(7) $\Phi(1^*) = 1_B = 1_C$ *;* $\Phi(S_1\alpha) = 1_B = 1_C$.
(8) $\Phi(\beta \rightarrow \gamma) = (\Phi\beta \rightarrow \Phi\gamma)$.
(9) $\Phi(\sim\alpha) = \neg\Phi\alpha$.
(10) $\Phi(\sim\sim\alpha) = \Phi\alpha$ *;* $\Phi(\sim\neg\alpha) = \Phi^*\alpha$.

Most of those properties are easily derived, except perhaps property (8), which will be proved later in a slightly transposed axiomatized context. We draw the readers's attention to the descriptions $\Phi(\sim\alpha)$ and $\Phi(\beta \rightarrow \gamma)$, which are equal to $\neg\Phi\alpha$ and $(\Phi\beta \rightarrow \Phi\gamma)$ respectively : they are a particularly striking a posteriori justification of the fundamental character of the complex negation $\sim$ and of the implication $\rightarrow$ between actions. Property (6) should also be noted, because it expresses in a particularly compact way the idea that the description of an action is the assertion that "condition implies result".

Looking forward to the axiomatization of the structure of actions without algebra of conditions B nor algebra of results C, it seems natural to "embed" Φ in the structure $\mathcal{A}$ via the functor $\overleftarrow{(\ )}$ so as to mimick the effect of Φ. Another possibility would be to use the functor $\overrightarrow{(\ )}$, but that would not bring different informations and we prefer to think of the image of $\overleftarrow{(\ )}$, which is S_0A, as a kind of substitute of $B(= C)$. Taking contravariance into account, the natural definition will be :

Definition 1.6.5:
The internalized description *of* α *is defined by* $\Phi'(\alpha) = \overleftarrow{\neg\Phi\alpha}$.

Using properties of $\overleftarrow{(\ )}$ and of Φ, we prove :

Proposition 1.6.6:
(1) $\Phi'(C_0\alpha) = C_0\alpha$ *;* $\Phi'(S_0\alpha) = S_0\alpha$.
(2) $\Phi'(\alpha \wedge \beta) = \alpha \wedge \beta$ *;* $\Phi'(\alpha \vee \beta) = \alpha \vee \beta$.
(3) $\Phi'0 = 0$ *;* $\Phi'1 = 1$.

(4) $\Phi'(\neg\alpha) = C_0\Phi'\alpha \vee S_0\alpha$ *;* $\Phi' 1^* = 1$.
(5) $\Phi' \overleftarrow{\phi} = \overleftarrow{\phi}$ *;* $\Phi' \overrightarrow{\phi} = C_0 \overleftarrow{\phi}$.
(6) $\Phi'\alpha^c = C_0\alpha$ *;* $\Phi'\alpha^\wedge = C_0\alpha^r$ *;* $\Phi'\alpha^r = \alpha^r$.

Section 1.7. Deontic notions

In order to introduce deontic notions, it remains to assume that instead of simply being a boolean algebra, the common algebra of conditions and results $B(= C)$ is equipped with a necessity operator $\Box$ obeying reasonably strong axioms, say K-axioms, with the possible addition of axiom $\Box 0_B = 0_B$, algebraic version of axiom $P\Box$ $(\neg\Box\bot)$, equivalent of the more familiar axiom D $(\Box A \rightarrow \Diamond A)$ in the system K (see Chellas [MLI] for terminology and basic properties). As already explained at the beginning of this paper, it suffices then to consider the obligation to do action α as the assertion of the necessity of the description of α :

Definition 1.7.1:
The obligation to do α *is defined by* $O\alpha = \Box\Phi\alpha$.

The following typical properties have already been proven in a not very different context (see [VWA]) :

Proposition 1.7.2:
(1) If $\alpha \leq \beta$*, then* $O\alpha \leq O\beta$.
(2) $O1 = 1_B$.
(3) $O(\alpha \wedge \beta) = O\alpha \wedge O\beta$.
(4) $O\alpha \leq O(\alpha \vee \beta)$.
(5) $O(\alpha \rightarrow \beta) \leq (O\alpha \rightarrow O\beta)$.
(6) $O0 = 0_B$ *(assuming axiom* $\Box 0_B = 0_B$*).*
(7) $O \sim \alpha \leq \neg O\alpha$ *(assuming axiom* $\Box 0_B = 0_B$*).*
(8) $O(\alpha \wedge \beta) \leq O(\alpha \wedge^* \beta)$.
(9) $O\alpha \wedge O\beta \leq O(\alpha \wedge^* \beta)$.

But $O(\alpha \wedge^* \beta) \leq O\alpha$ is not valid. See [VWA] for other examples of non valid formulas and further details.

Once again, we want to reproduce inside the structure of actions similar phenomena. This will be done in part 2 where $\Box$ will be essentially acting on $S_0 A$ and obligation will be defined using the internalized description of action. Considering the question of iterated obligation, we observe that $\Box\Phi(\Box\Phi\alpha)$ does not make sense here because $\Box\Phi\alpha$ is an element of B and Φ is defined only on actions. If we want to iterate obligation, it is essential in the present approach that $\Box\Phi\alpha$ which represents a proposition, be considered as an action on which Φ may again be applied ; that is the reason why the internalized description Φ' will be used instead of Φ ; Φ' maps indeed actions to 'actions coding a proposition'.

Part 2. Deontic algebras of actions

We want to show here that a careful selection of the notions put forward in Part 1 allows us to define in reasonably economical terms an extremely rich structure of actions containing its own deontic notions. That structure of actions will be dealt with here axiomatically. Since the motivation was given in Part 1, we give here the axioms without many comments and concentrate on a few less immediate proofs of derived properties. Looking backwards to Part 1, those proofs may in turn be used to give synthetic proofs of steps which were not explicitely supplied there.

Section 2.1. Definition of deontic algebras of actions

Here is the definition of the full structure. We will call it "deontic algebra of actions" for short, but a detailed study could profitably distinguish some reducts of the structure, which are of interest in themselves. For the sake of reference, we repeat the definition of support algebra with truth-value support given in [AA].

Definition 2.1.1:
A deontic algebra of actions *is determined by*

(1) a support algebra $\mathcal{A} =< A, 0, 1, \neg, \wedge, \vee, C_0, \leq>$ *with truth-value supports* $S_0^{=0}$*, i.e. a structure where* A *is a set,* 0 *and* 1 *are elements of* A*,* $\neg$*,* C_0 *and* $S_0^{=0}$ *are unary operations on* A*,* $\wedge$ *and* $\vee$ *are binary operations on* A *and* $\leq$ *is a binary relation on* A *satisfying the following properties :*

(Group 0) $< A, 0, 1, \wedge, \vee, \leq>$ *is a distributive lattice with smallest element* 0 *and greatest element* 1 *;*

$C_0\alpha \leq \beta$ *iff* $1 \leq \alpha \vee \beta$ *;*
$C_0\alpha \wedge C_0\beta \leq C_0(\alpha \vee \beta)$ *;*
$\alpha \wedge \neg\alpha = C_0C_0\alpha$ *;*
$\neg\neg\alpha = \alpha$ *;*
$\neg 1 = 1$ *;*
$\neg(\neg\alpha \wedge \neg\beta) = (\alpha \vee \beta) \wedge (\alpha \vee C_0\beta) \wedge (C_0\alpha \vee \beta)$ *;*
$\neg(\alpha \vee C_0\beta) \leq \neg\alpha \vee C_0\beta$ *;*
$S_0^{=0}\alpha \leq \beta$ *iff* $\alpha \leq S_0\beta$ *;*

(2) unary operations $(\ )^c$, $(\ )^r$, Φ', $\Box$ *on A satisfying the following three groups of axioms :*

(Group 1) *if* $\alpha \leq \beta$*, then* $\beta^c \leq \alpha^c$ *;*
if $\alpha \leq \beta$*, then* $\beta^r \leq \alpha^r$ *;*
$\alpha^c \leq \beta$ *iff* $\beta^r \leq \alpha$ *;*
$(C_0\alpha)^c = \neg\alpha^c$ *;*
$(\alpha \vee \beta)^c = \alpha^c \wedge \beta^c$ *;*
$(\neg\alpha)^c = \alpha^c$ *;*
if $S_0\alpha = 0$*, then* $(C_0\alpha)^r = 1$ *and if* $S_0\alpha \neq 0$*, then* $(C_0\alpha)^r = 0$ *;*
$S_0\alpha^r = \alpha^r$ *;*
$\alpha^{cr} = S_0\alpha$ *;*
if $\alpha \neq 1$*, then* $(\gamma^c \vee S_0\alpha)^r = \gamma^{cr}$ *;*

(Group 2) $\Phi'(\alpha \wedge \beta) = \Phi'\alpha \wedge \Phi'\beta$ *;*
$\Phi'(\alpha \vee \beta) = \Phi'\alpha \vee \Phi'\beta$ *;*
$\Phi'(\neg\alpha) = C_0\Phi'(\alpha \wedge C_0\alpha)$ *;*
$\Phi'(C_0\alpha) = C_0\alpha$ *;*
$\Phi'(\alpha^c) = C_0\alpha$ *;*

(Group 3) $\Box\alpha = \Box S_0\alpha = S_0\Box\alpha$ *;*
$\Box(\alpha \wedge \beta) = \Box\alpha \wedge \Box\beta$ *;*
$\Box 1 = 1.$

Axioms of Group 0 are taken from [AA]. We give comments and consequences of the axioms of the other groups in the following sections.

Section 2.2. About axioms of Group 1

Group 1 concerns the representation of conditions and results and may be supplemented with the following definitions of duals and derived notions :

Definition 2.2.1:
(1) $\alpha^{r^*} = C_0(\neg\alpha)^r$.
(2) $\alpha^{\wedge} = \alpha^{rc}$.
(3) $\alpha^{\vee} = \alpha^{r^*c}$.
(4) α *is constant iff* $\alpha = 1$ *or* $\alpha^r = \alpha^{r^*}$.

Here are the main properties derived from Group 1 and given in a natural order of derivation. We give the proofs of (10), (12) and (15), which are less immediate than the other properties.

Proposition 2.2.2:
(1) $\alpha^{cr} \leq \alpha$ *;* $\alpha^{rc} \leq \alpha$ *;* $\alpha^{crc} = \alpha^c = (S_0\alpha)^c$ *;* $\alpha^{rcr} = \alpha^r = \alpha^{\wedge r}$.
(2) $S_0\alpha^{r^*} = \alpha^{r^*}$.

(3) $(\neg\alpha)^r = C_0\alpha^{r^*}$ *;* $(\alpha \wedge \beta)^c = \alpha^c \vee \beta^c$ *;* $(\alpha \wedge \beta)^r = \alpha^r \vee \beta^r$*.*
(4) $1^c = 0$ *;* $1^r = 0$ *;* $1^{r^*} = 1$ *;* $1^\wedge = 1^*$*.*
(5) $(S_0\alpha)^c = \alpha^c$*.*
(6) $0^c = 1^*$ *;* $0^r = 1$ *;* $0^{r^*} = 0$ *;* $1^\vee = 0$*.*
(7) If $\alpha = 1$*, then* $(S_0\alpha)^r = 0$ *and if* $\alpha \neq 1$*, then* $(S_0\alpha)^r = 1$*.*
(8) $\alpha^\vee = \neg(\neg\alpha)^\wedge$ *;* $\alpha^{r^*cr} = \alpha^{r^*} = \alpha^{\vee r}$ *;* $\alpha^{\vee\wedge} = \alpha^\vee$ *;* $\alpha^{\wedge\wedge} = \alpha^\wedge$ *;* $\alpha^{cr^*} = \alpha^{cr} = S_0\alpha$ *;* $\alpha^{r^*cr^*} = \alpha^{r^*} = \alpha^{\vee r^*}$*.*
(9) α *is constant iff* $\alpha = 1$ *or* $\alpha^\wedge = \alpha^\vee$*.*
(10) All elements of the form $\gamma^c \vee S_0\alpha$ *are constant and if* $\alpha \neq 1$*, then* $(\gamma^c \vee S_0\alpha)^r = (\gamma^c \vee S_0\alpha)^{r^*} = \gamma^{cr} = \gamma^{cr^*} = S_0\gamma$*.*
(11) $(\gamma^\wedge \vee S_0\alpha)^\vee = \gamma^\wedge = \gamma^{\wedge\vee}$ *;* $(\gamma^\vee \vee S_0\alpha)^\vee = \gamma^\vee = \gamma^{\vee\vee}$ *;* $(\gamma^\wedge \vee S_0\alpha)^\wedge = \gamma^\wedge = \gamma^{\wedge\wedge}$ *;* $(\gamma^\vee \vee S_0\alpha)^\wedge = \gamma^\vee = \gamma^{\vee\wedge}$*.*
(12) α *is constant iff* $\alpha = \alpha^\wedge \vee S_0\alpha$*.*
(13) α *is total constant iff* $\alpha = \alpha^\wedge$ *;* α *is total constant iff* $\alpha = \alpha^\vee$*.*
(14) If α *is constant, then* $\neg\alpha$ *is constant and* $(\neg\alpha)^r = C_0\alpha^r$*.*
(15) If α *and* β *are constant, then* $\alpha \vee \beta$ *is constant,* $(\alpha \vee \beta)^r = (\alpha \vee \beta)^{r^*} = \alpha^r \wedge \beta^r = \alpha^{r^*} \wedge \beta^{r^*}$ *and* $(\alpha \vee \beta)^\wedge = (\alpha \vee \beta)^\vee = \alpha^\wedge \vee \beta^\wedge = \alpha^\vee \vee \beta^\vee$*.*

Proof of (10). If $\alpha = 1$, then $\gamma^c \vee S_0\alpha = \gamma^c \vee 1 = 1$ which is constant. If $\alpha \neq 1$, then $(\gamma^c \vee S_0\alpha)^r = \gamma^{cr}$ by the last axiom of Group 1, and

$(\gamma^c \vee S_0\alpha)^{r^*}$	$= C_0(\neg(\gamma^c \vee S_0\alpha))^r$	by definition 2.2.1 (1)
	$= C_0(\neg\gamma^c \wedge^* S_1\alpha)^r$	by properties of support algebras
	$= C_0(\neg\gamma^c \vee S_0\alpha)^r$	by theorem 2.16 (1) of [AA]
	$= C_0((C_0\gamma)^c \vee S_0\alpha)^r$	since $(C_0\gamma)^c = \neg\gamma^c$
	$= C_0(C_0\gamma)^{cr}$	by the last axiom of Group 1
	$= C_0S_0(C_0\gamma)$	by the last but one axiom of Group 1
	$= S_0\gamma$	by properties of support algebras
	$= \gamma^{cr}$	by the last but one axiom of Group 1

□

Proof of (12). For $\alpha = 1$, we have $\alpha^\wedge \vee S_0\alpha = 1^\wedge \vee S_01 = 1^* \vee 1 = 1$. For $\alpha \neq 1$, $\alpha^\wedge \vee S_0\alpha \leq \alpha$ is easy ; to prove the other direction, observe that $\alpha^\wedge \vee S_0\alpha$ and α have the same support, so that it suffices to prove $\alpha \leq^* \alpha^\wedge \vee S_0\alpha$: $\alpha \leq^* \alpha^\vee$, $\alpha \leq^* \alpha^\wedge$ (using the constancy of α and property (9)), $\alpha \vee S_0\alpha \leq^* \alpha^\wedge \vee S_0\alpha$ (by monotony, see theorem 2.14 (2) of [AA]) and $\alpha \leq^* \alpha^\wedge \vee S_0\alpha$ (since $\alpha = \alpha \vee S_0\alpha$). Conversely, $\alpha^\wedge \vee S_0\alpha = \alpha^{rc} \vee S_0\alpha$, which is constant by property (10). □

Proof of (15). The cases $\alpha = 1$ or $\beta = 1$ are trivial. Assume then that α and β are constant and both different from 1. By (12), $\alpha = \alpha^\wedge \vee S_0\alpha$ and $\beta = \beta^\wedge \vee S_0\beta$, so that

$$\alpha \vee \beta = \alpha^\wedge \vee S_0\alpha \vee \beta^\wedge \vee S_0\beta$$
$$= \alpha^\wedge \vee \beta^\wedge \vee S_0(\alpha \vee \beta)$$

$= \alpha^{rc} \vee \beta^{rc} \vee S_0(\alpha \vee \beta)$
$= (\alpha^r \wedge \beta^r)^c \vee S_0(\alpha \vee \beta)$,
which shows that $\alpha \vee \beta$ has the form $\gamma^c \vee S_0\delta$, a constant action by property (10). Moreover, using the preceding equation,
$(\alpha \vee \beta)^r = ((\alpha^r \wedge \beta^r)^c \vee S_0(\alpha \vee \beta))^r$
$= (\alpha^r \wedge \beta^r)^{cr}$ by property (10)
$= S_0(\alpha^r \wedge \beta^r)$
$= S_0\alpha^r \wedge S_0\beta^r$
$= \alpha^r \wedge \beta^r$
and
$(\alpha \vee \beta)^{r^*} = ((\alpha^r \wedge \beta^r)^c \vee S_0(\alpha \vee \beta))^{r^*}$
$= (\alpha^r \wedge \beta^r)^{cr}$ by property (10)
$= \alpha^r \wedge \beta^r$ as above.
The remaining equalities are easy to prove. □

A substantial part of those properties is synthesized in the following theorem :

Theorem 2.2.3:
(1) The mapping ( )^c is a homomorphism from the structure of actions $< A, \leq, \wedge, 1, \vee, 0, C_0, \neg >$ *onto the structure of total constant actions* $< TcA, \geq, \vee, 0, \wedge, 1^*, \neg, id >$ *(TcA is the set of total constant actions of A and id designates the identity).*
(2) The mapping ( )^r is a homomorphism from the structure of actions $< A, \leq, \wedge, 1, 0 >$ *onto the structure* $< S_0A, \geq, \vee, 0, 1 >$ *(S_0A is the set of 0-supports of elements of A) and it satisfies $(C_0\alpha)^r = 1$ if α is total and $(C_0\alpha)^r = 0$ if α is not total.*
(3) The mapping ( )^c induces by restriction an isomorphism from the structure $< S_0A, \leq, \wedge, 1, 0, C_0 >$ *to the structure* $< TcA, \geq, \vee, 0, 1, \neg >$ *and its inverse is given by the adequate restriction of ( )^r.*

Section 2.3. About axioms of group 2

Group 2 concerns descriptions of actions and may be supplemented with the definition of the dual Φ'^* of Φ' :

Definition 2.3.1:
$\Phi'^*\alpha = C_0\Phi'(\neg\alpha)$.

Here are the main properties derived from Group 2 and given in a natural order of derivation :

Proposition 2.3.2:
(1) $\Phi'(\alpha^{\wedge}) = C_0\alpha^r$ *;* $\Phi'(\alpha^{\vee}) = C_0\alpha^{r^*}$*.*
(2) $\Phi'1 = 1$ *;* $\Phi'0 = 0$*.*
(3) $S_0(\Phi'\alpha) = \Phi'\alpha$ *;* $\Phi'(S_0\alpha) = S_0\alpha$*.*
(4) $\Phi'(\alpha^r) = \alpha^r$ *;* $\Phi'(\alpha^{r^*}) = \alpha^{r^*}$*.*
(5) If $\alpha \leq \beta$*, then* $\Phi'\alpha \leq \Phi'\beta$*.*
(6) $C_0(\Phi'\alpha) = \Phi'(\neg\alpha \wedge C_0\alpha) = \Phi'(\neg\alpha) \wedge C_0\alpha$*.*
(7) If $\Phi'(S_0\alpha) = \Phi'(S_0\beta)$*, then* $S_0\alpha = S_0\beta$*.*
(8) $\Phi'(1^*) = 1$*.*
(9) $C_0\alpha^r \leq \Phi'\alpha$ *;* $S_0\alpha \leq \Phi'\alpha$*.*
(10) $S_0\alpha \vee C_0\alpha^r = (C_0\alpha \to C_0\alpha^r) \leq \Phi'\alpha \leq (C_0\alpha \to C_0\alpha^{r^*})$*.*
(11) If α *is constant, then* $\Phi'\alpha = (C_0\alpha \to C_0\alpha^r) = (C_0\alpha \to C_0\alpha^{r^*})$*.*
(12) $\Phi'(S_0^{=0}\alpha) = S_0^{=0}\alpha$ *;* $\Phi'(S_0^{\neq 0}\alpha) = S_0^{\neq 0}\alpha$ *;* $\Phi'(S_0^{\neq 1}\alpha) = S_0^{\neq 1}\alpha$*.*
(13) Letting $\neg'\delta = \neg\delta \vee S_0^{\neq 0}\delta$*, one has :* $\Phi'(\neg'\delta) = \Phi'(\neg\delta)$*.*
(14) $\Phi'(\sim\alpha) = C_0\Phi'\alpha$*.*
(15) $\Phi'(\beta \to \gamma) = \Phi'\beta \to \Phi'\gamma$*.*
(16) $\Phi'(\sim\sim\alpha) = \Phi'\alpha$ *;* $\Phi'(\sim\neg\alpha) = \Phi'^*\alpha = \Phi'\alpha \wedge C_0\alpha$*.*
(17) $\Phi'^*\alpha \leq \Phi'\alpha$*.*

Comment on properties (10) and (11). Allowing for contravariance, those properties approximate the idea that $\Phi\alpha$ is a description of action α, saying that 'conditions imply results'.

Proof of (13). Here is the computation leading to the result :

$$\begin{aligned}
\Phi'(\neg'\delta) &= \Phi'(\neg\delta \vee S_0^{\neq 0}\delta) \\
&= \Phi'(\neg\delta) \vee \Phi'(S_0^{\neq 0}\delta) \\
&= C_0\Phi'\delta \vee S_0\delta \vee S_0^{\neq 0}\delta \\
&= C_0\Phi'\delta \vee S_0\delta \vee C_0S_0^{=0}\delta \vee S_0\delta \quad && \text{since } S_0^{\neq 0}\delta = C_0S_0^{=0}\delta \vee S_0\delta \\
&= C_0(\Phi'\delta \wedge S_0^{=0}\delta) \vee S_0\delta \\
&= C_0(\Phi'\delta \wedge \Phi'(S_0^{=0}\delta)) \vee S_0\delta \\
&= C_0\Phi'(\delta \wedge S_0^{=0}\delta) \vee S_0\delta \\
&= C_0\Phi'\delta \vee S_0\delta && \text{since } \delta \wedge S_0^{=0}\delta = \delta \\
&= C_0\Phi'\delta \vee C_0C_0\delta \\
&= C_0(\Phi'\delta \wedge C_0\delta) \\
&= C_0(\Phi'\delta \wedge \Phi'(C_0\delta)) \\
&= C_0\Phi'(\delta \wedge C_0\delta) \\
&= \Phi'(\neg\delta).
\end{aligned}$$

□

Proof of (15). Using the notation of (13), recall that $\beta \to \gamma = (C_0\beta \vee \gamma) \wedge \neg'(\beta \wedge^* \neg\gamma)$, so that

$$\begin{aligned}\Phi'(\beta \to \gamma) &= \Phi'(C_0\beta \vee \gamma) \wedge \Phi'(\neg'(\beta \wedge^* \neg\gamma))\\ &= \Phi'(C_0\beta \vee \gamma) \wedge \Phi'(\neg(\beta \wedge^* \neg\gamma)) \quad \text{by property(13)}\\ &= \Phi'(C_0\beta \vee \gamma) \wedge \Phi'(\neg\beta \vee \gamma))\\ &= \Phi'((C_0\beta \vee \gamma)) \wedge (\neg\beta \vee \gamma))\\ &= \Phi'((\neg\beta \wedge C_0\beta) \vee \gamma)\\ &= \Phi'(\neg\beta \wedge C_0\beta) \vee \Phi'\gamma\\ &= C_0\Phi'\beta \vee \Phi'\gamma\\ &= \Phi'\beta \to \Phi'\gamma.\end{aligned}$$

The last equality follows from the observation that implication between elements of S_0A behaves classically, with C_0 playing the role of negation : $(S_0\delta) \to (S_0\varepsilon) = C_0(S_0\delta) \vee (S_0\varepsilon)$. □

Many of those properties may be summarized by observing that

Theorem 2.3.3:
Φ' is a homomorphism mapping the structure $< A, \leq, \wedge, 1, \vee, 0, 1^*, \sim, \to >$ onto the structure $< S_0A, \leq, \wedge, 1, \vee, 0, 1, C_0, \to >$ and reducing to identity on S_0A.

Section 2.4. About axioms of group 3

Group 3 concerns modality and will be used to introduce the basic deontic modality of obligation :

Definition 2.4.1:
For any action α, the obligation to do α *is defined by $O\alpha = \Box\Phi'\alpha$.*

Note that the first axiom of Group 3 is but a technical device conveying the idea that $\Box$ is essentially defined on S_0A. The following properties are routine and the really interesting ones are those which cannot be proved! Such is typically $O(\alpha \wedge^* \beta) \leq O\alpha$; we refer the reader to our [VWA] for further details and analogous observations in a similar context.

Proposition 2.4.2:
(1) $S_0(O\alpha) = (O\alpha)$.
(2) If $\alpha \leq \beta$, then $O\alpha \leq O\beta$.
(3) $O1 = 1$.
(4) $O(\alpha \wedge \beta) = O\alpha \wedge O\beta$.
(5) $O1^* = 1$.
(6) $O(\alpha \wedge \beta) \leq O(\alpha \wedge^* \beta)$.

In the presence of axiom $\Box 0 = 0$, one proves $O0 = 0$ and $O\sim\alpha \leq \sim O\alpha = C_0O\alpha$. Property (1) means that $O\alpha$ is indeed a 'proposition', i.e.

an element of S_0A, and it is an immediate consequence of the first axiom of Group 3. Another easy consequence of this and which we think of great interest is that in the present approach, the quite natural modal axiom '4', $\Box\Box\alpha = \Box\alpha$, automatically forces a reduction of the deontic modality O :

$$\begin{aligned} OO\alpha &= \Box\Phi'(\Box\Phi'\alpha) \\ &= \Box\Phi'(S_0\Box\Phi'\alpha) && \text{by property (1)} \\ &= \Box S_0\Box\Phi'\alpha && \text{since } \Phi' S_0\delta = S_0\delta \text{ (proposition 2.3.2 (3))} \\ &= \Box\Box\Phi'\alpha && \text{by property (1)} \\ &= \Box\Phi'\alpha && \text{using axiom '4'} \\ &= O\alpha. \end{aligned}$$

This does not yet give arguments for or against non trivial iterations of modalities, but shows that the presuppositions underlying our approach lead naturally to a reduction of OO to O. If one does not appreciate the reduction, one should then criticize the presuppositions, or abandon axiom '4', or complexify the approach to get a good explanation of non trivial iteration.

Conclusion

Looking back at what we have presented in this paper and in the preceding ones, [VWA] and [AA], we think that we have reached an interesting account of the structure of actions and deontic modalities. Our starting assumption is very simple : action is considered as a coherent mapping from finite sets of conditions to results. We look at the explicit structure deriving from that assumption and obtain an extremely rich structure which carries with it interesting distinctions between different kinds of conjunction, disjunction, negation, implication, etc.

To that structure, already studied in [VWA] and [AA], we add here the consideration of embeddings of conditions and results in the structure of actions and implement two other basic intuitions : first, one should distinguish action and description of action ; secondly, obligation applies to action and is defined by applying a classical necessity operator to the description of the action.

When we turn to the axiomatic part of our study and more specially to the second part of this paper, we think that a notable feature is that we keep a simple unisort structure of action, without however loosing the structure of conditions and results : they appear internalized as a basic building block of the structure, i.e. as S_0A or TcA, which may be thought of as representing propositions. In agreement with this, descriptions are internalized and description as well as obligation transform actions into propositions : Φ' and O map actions to S_0A.

Connections of our approach with other ones are given in the conclusion of [AA] to which we refer the reader.

UCL
Institut Supérieur de Philosophie
Collège Désiré Mercier
place du Cardinal Mercier 14
1348 Louvain-la-Neuve
E-mail: Lucas@risp.ucl.ac.be

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