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EXTENSIONS OF THE BASIC CONSTRUCTIVE LOGIC FOR NEGATION-CONSISTENCY B_{Kc4} DEFINED WITH A FALSITY CONSTANT*

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Abstract
The logic B_{Kc4} is the basic constructive logic for negation consistency (i.e., absence of any contradiction) in the ternary relational semantics without a set of designated points. In this paper, a number of extensions of B_{Kc4} defined with a propositional falsity constant are defined. It is also proved that negation-consistency is not equivalent to absolute consistency (i.e., non-triviality) in any logic included in positive intermediate logic LC plus the constructive negation of B_{Kc4} and the (constructive) contraposition axioms.

1. *Introduction*

A *theory* is a set of formulas closed under adjunction and provable entailment (cf. §2). Then, negation-consistency is defined as follows:

Definition 1: *A theory a is n-inconsistent (negation-inconsistent) iff for some wff A, $A \wedge \neg A \in a$ (A theory a is n-consistent iff it is not n-inconsistent).*

The basic constructive logic adequate to this sense of consistency in the ternary relational semantics without a set of designated points, i.e., the logic B_{Kc4} is defined in [8]. Next, in this same paper, it is shown how to extend B_{Kc4} with the strong constructive contraposition axioms

(i). $(A \to \neg B) \to (B \to \neg A)$

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and

(ii). $B \to [(A \to \neg B) \to \neg A)$

and with some strong implicative axioms up to positive intuitionistic logic J_+. It is clear that J_+ plus (i) and (ii) is minimal intuitionistic logic, J_m. Now, although even in B_{Kc4} the ECQ ('e contradictione quodlibet') axiom

(iii). $(A \wedge \neg A) \to \neg B$

is provable, in none of the logics included in J_m, the ECQ axiom

(iv). $(A \wedge \neg A) \to B$

or the EFQ ('e falso quodlibet') axioms

(v). $\neg A \to (A \to B)$

and

(vi). $A \to (\neg A \to B)$

are, of course, derivable.

So, in none of the logics included in the spectrum delimited by B_{Kc4} and J_m is n-consistency equivalent to absolute consistency (i.e., non-triviality). Consequently, all logics in the aforementioned spectrum are paraconsistent logics in the sense of [7].

In respect of these results, the aim of this paper is fourfold:

(1) The logic B_{Kc5} is axiomatized by adding (i) and (ii) to B_{Kc4}. Now, it will be proved that the weak constructive contraposition axioms

(vii). $(A \to B) \to (\neg B \to \neg A)$

and

(viii). $\neg B \to [(A \to B) \to \neg A)$

can be added to B_{Kc4}, the resulting logic being different from B_{Kc5}. This logic is dubbed $B_{Kc4'}$. Next, it is proved that $B_{Kc4'}$ can be extended with the axioms prefixing,

(ix). $(B \to C) \to [(A \to B) \to (A \to C)]$

suffixing

(x). $(A \to B) \to [(B \to C) \to (A \to C)]$

contraction

(xi). $[A \to (A \to B)] \to (A \to B)$

and the assertion rule

(xii). $\vdash A \Rightarrow \ \vdash (A \to B) \to B$

the resulting logic being different from the result of adding (ix), (x), (xi) and (xii) to B_{Kc5}.
In this way, a series of modal logics that include B_{Kc4} but neither include nor are included in Lewis' S5 are defined (cf. remark 9).

(2) The characteristic axiom of Dummett's intermediate logic LC (cf. [3])

(xiii). $(A \to B) \vee (B \to A)$

is added to J_m. The resulting logic is, intuitively, minimal intermediate logic LC_m. Thus, a series of constructive logics between B_{Kc4} and LC_m are defined. In none of these logics the ECQ axiom (iii) and the EFQ axioms (iv) and (v) are provable. In this way, it is shown that in LC_m and in all logics included in it negation-consistency is still independent of absolute consistency.

(3) Although B_{Kc4} (especially its implicative fragment) is not a strong logic, in [5] it is shown how to build a definitionally equivalent logic (the concept is explained in §5) in which negation is treated with a propositional falsity constant F instead of the unary connective. So, the third aim of this paper is to build logics with the falsity constant definitionally equivalent to those referred to in (1) and (2).

(4) Consider the following definition:

Definition 2: *Let L be a logic and a be a theory whose underlying logic is L. Then a is w-inconsistent (weakly inconsistent) iff a contains the negation of a theorem of L (a is w-consistent iff it is not w-inconsistent).*

The basic constructive logic adequate to this sense of consistency in the ternary relational semantics without a set of designated points, i.e., the logic B_{Kc1}, is defined in [10]. In this paper, the extensions of B_{Kc1} up to contractionless intuitionistic logic JW without w-consistency collapsing in n-consistency or absolute consistency are also defined. Therefore, the fourth aim of this paper is to compare B_{Kc4} and its extensions with B_{Kc1} and its extensions.

The structure of the paper is as follows. In §2, the logic B_{K+} is defined. It is the result of adding the K rule

(xiv). $\vdash A \Rightarrow \ \vdash B \to A$

to Routley and Meyer's well known logic B_+. Then, some extensions of B_{K+} with some strong implicative axioms are defined. In §3, the logics B_{Kc4} and B_{Kc5} are recalled and the logic $B_{Kc4'}$ is introduced. In §4, logics formulated with F definitionally equivalent to those defined in §3 are introduced, and in §5, the definitional equivalence is proved. In §6, all the logics treated so far are extended with some strong implicative axioms. Finally, in §7 the logics adequate to n-consistency and those adequate to w-consistency are compared. All logics introduced in this paper are proved sound and complete in respect of a modification of Routley and Meyer's ternary relational semantics for relevance logics (recall that all logics defined in this paper have the K rule (xiv)).

We end this introduction by remarking that all logics here introduced are paraconsistent logics in the sense of [7], and that they are paraconsistent in respect of a precisely defined sense of consistency, i.e., n-consistency.

2. *The positive logic* B_{K+} *and its extensions*

Firstly, the positive logic B_{K+} is defined. It can be axiomatized with

Axioms

A1. $A \to A$
A2. $(A \wedge B) \to A$ / $(A \wedge B) \to B$
A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
A4. $A \to (A \vee B)$ / $B \to (A \vee B)$
A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

The rules of inference are

$$\begin{aligned}
\text{Modus ponens (MP):}&\ \vdash (A \,\&\, \vdash A \to B) \Rightarrow\ \vdash B\\
\text{Adjunction (Adj.):}&\ (\vdash A \,\&\, \vdash B) \Rightarrow\ \vdash A \wedge B\\
\text{Suffixing (Suf.):}&\ \vdash A \to B \Rightarrow\ \vdash (B \to C) \to (A \to C)\\
\text{Prefixing (Pref.):}&\ \vdash A \to B \Rightarrow\ \vdash (C \to A) \to (C \to B)\\
\text{K:}&\ \vdash A \Rightarrow\ \vdash B \to A
\end{aligned}$$

Therefore, B_{K+} is B_+ with the addition of the K rule.

We now define the semantics for B_{K+}. A B_{K+} *model* is a triple $<K, R, \vDash>$ where K is a non-empty set, and R is a ternary relation on K subject to the following definitions and postulates for all a, b, c, $d \in K$ with quantifiers ranging over K:

d1. $a \leq b =_{\text{df}} \exists x Rxab$
d2. $R^2abcd =_{\text{df}} \exists x(Rabx \,\&\, Rxcd)$
P1. $a \leq a$
P2. $(a \leq b \,\&\, Rbcd) \Rightarrow Racd$

Finally, $\vDash$ is a valuation relation from K to the sentences of the positive language satisfying the following conditions for all propositional variables p, wff A, B and $a \in K$:

(i). $(a \leq b \,\&\, a \vDash p) \Rightarrow b \vDash p$
(ii). $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$
(iii). $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$
(iv). $a \vDash A \to B$ iff for all $b, c \in K$, $(Rabc \,\&\, b \vDash A) \Rightarrow c \vDash B$

A formula A is B_{K+} *valid* ($\vDash_{B_{k+}} A$) iff $a \vDash A$ for all $a \in K$ in all models.

Remark 1: *The postulates P3 $Rabc \Rightarrow b \leq c$, P4 $(a \leq b \,\&\, b \leq c) \Rightarrow a \leq c$ and P5 $R^2abcd \Rightarrow Rbcd$ hold in all models.*

In [10] or in [11], it is proved that B_{K+} is sound and complete in respect of this semantics.

Remark 2: *As is known, in the standard semantics for relevance logics (see, e.g., [12]), there is a set of 'designated points' in terms of which the relation $\leq$ is defined and formulas are determined to be valid. The absence of this* *set in B_{K+} semantics (and the corresponding changes in d1 and the definition of validity) are the only but crucial differences between B_+ models and B_{K+} models.*

Next, we define some positive extensions of B_{K+}. Consider the following axioms and rule of inference

A7. $(B \to C) \to [(A \to B) \to (A \to C)]$
A8. $(A \to B) \to [(B \to C) \to (A \to C)]$
A9. $[A \to (A \to B)] \to (A \to B)$
A10. $\vdash A \Rightarrow \ \vdash (A \to B) \to B$
A11. $A \to [(A \to B) \to B]$
A12. $A \to (B \to A)$
A13. $(A \to B) \vee (B \to A)$

The following logics are defined:

(1) TW_+: B_+ + A7 + A8
(2) EW_+: TW_+ + A10
(3) RW_+: TW_+ + A11
(4) JW_+: RW_+ + A12
(5) LCW_+: JW_+ + A13
(6) T_+: TW_+ + A9
(7) E_+: EW_+ + A9
(8) R_+: RW_+ + A9
(9) J_+: JW_+ + A9
(10) LC_+: LCW_+ + A9

The well known logics T_+, E_+ and R_+ are the positive fragments (without fusion $\circ$ and t) of 'Ticket Entailment', T, 'Entailment Logic', E, and 'Relevance Logic', R, respectively; and TW_+, EW_+ and RW_+ are their respective contractionless fragments. On the other hand, J_+ and LC_+ are the positive fragments of 'Intuitionistic logic', J, and 'Intermediate logic LC', LC (see [3]), respectively, and JW_+ and LCW_+ are their respective contractionless fragments. Finally, TW_{K+}, EW_{K+}, RW_{K+}, JW_{K+} and LCW_{K+}, T_{K+}, E_{K+}, R_{K+}, J_{K+} and LC_{K+} are the logics just defined plus the K rule. Now, the K rule is not, of course, independent in JW_{K+}, LCW_{K+}, J_{K+} and LC_{K+}. So, these logics will be referred to by JW_+, LCW_+, J_+ and LC_+.

We note:

Proposition 1:

(1) *RW_{K+} and JW_+ (so, R_{K+} and J_+) are deductively equivalent logics.*

(2) *TW_{K+}, EW_{K+}, RW_{K+} (= JW_+) and LCW_+, T_{K+}, E_{K+}, R_{K+} (= J_+) and LC_+ are different logics.*

Proof. (1) It is trivial and (2) it follows by well known results on relevance and intuitionistic logics (alternatively, one can use MaGIC, the matrix generator developed by J. Slaney (see [13]). □

We now turn to semantics. Consider the following set of postulates

P6. $R^2abcd \Rightarrow (\exists x \in K)(Rbcx \;\&\; Raxd)$

P7. $R^2abcd \Rightarrow (\exists x \in K)(Racx \;\&\; Rbxd)$

P8. $Rabc \Rightarrow R^2abbc$

P9. $(\exists x \in K)Raxa$

P10. $Rabc \Rightarrow Rbac$

P11. $Rabc \Rightarrow a \leq c$

P12. $(Rabc \;\;\&\;\; Rade) \Rightarrow (b \leq e$ or $d \leq c)$

Now, models for the logics introduced above are defined, similarly, as B_{K+} models except for the addition of the following postulates:

(1) TW_{K+} models: P6, P7.

(2) EW_{K+} models: P6, P7, P9.

(3) RW_{K+} models: P6, P7, P10.

(4) JW_+ models: P6, P7, P10, P11.

(5) LCW_+ models: P6, P7, P10, P12.

(6) T_{K+} models: P6, P7, P8.

(7) E_{K+} models: P6, P7, P8, P9.

(8) R_{K+} models: P6, P7, P8, P10.

(9) J_+ models: P6, P7, P8, P10, P11.

(10) LC_+ models: P6, P7, P8, P10, P12.

As in B_{K+} models, validity is defined in all cases in respect of all points of K.

We next define the canonical models (cf. [11]). We begin by recalling some definitions. A *theory* is a set of formulas closed under adjunction and provable entailment (that is, a is a theory if whenever $A, B \in a$, then $A \wedge B \in a$; and if whenever $A \to B$ is a theorem and $A \in a$, then $B \in a$); a theory a is *prime* if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; a theory a is *regular* iff all theorems belong to a. Finally, a is *null* iff no wff belong to a. Now, we define the B_{K+} canonical model. Let K^T be the set of all theories and R^T be defined on K^T as follows: for all formulas A, B and $a, b, c \in K^T$, $R^T abc$ iff if $A \to B \in a$ and $A \in b$, then $B \in c$. Further, let K^C be the set of all prime non-null theories and R^C be the restriction of R^T to K^C. Finally, let $\vDash^C$ be defined as follows: for any wff A and $a \in K^C$, $a \vDash^C A$ iff $A \in a$. Then, the B_{K+} *canonical model* is the triple $<K^C, R^C, \vDash^C>$.

Now, let L_+ be any of the extensions of B_{K+} defined above. The L_+ canonical model is defined, similarly, as the B_{K+} canonical models except that its items are referred to L_+ theories instead of B_{K+} theories. Then, we have

Proposition 2: *Given the logic B_{K+} and B_{K+} semantics, P6, P7, P8, P9, P10, P11 and P12 are the corresponding postulates to A7, A8, A9, A10, A11, A12 and A13, respectively.*

Proof. Given B_{K+} and B_{K+} semantics, we have to prove that each axiom is proved valid with the corresponding postulate and that the corresponding postulate is proved valid with the axiom. Now, that this is the case for A7 (P6), A8 (P7), A9 (P8), A10 (P9), A11 (P10) and A12 (P11) is proved in (or can easily be derived from) e.g., [12]. So, we prove that P12 is the corresponding postulate to A13.

(1) *A13 is LCW_+ valid*: Suppose $a \vDash A \to B$, $a \nvDash B \to A$ for wff A, B and $a \in K$ in some model. Then, $b \vDash A$, $d \vDash B$, $c \nvDash B$, $e \nvDash A$ for b, c, d, $e \in K$ such that $Rabc$ and $Rade$. By P12, $b \leq e$ or $d \leq c$. So, either $e \vDash A$ or $c \vDash B$, a contradiction.

(2) *P12 holds canonically*: Suppose $R^C abc$, $R^C ade$ for a, b, c, d, $e \in K^C$, and, for reductio, $b \nleq^C e$ and $d \nleq^C c$. Then, $A \in b$, $B \in d$, $A \notin e$, $B \notin c$ for some wff A, B. As a is non-null, it is regular by the K rule. So, $(A \to B) \vee (B \to A) \in a$ by A13. As a is prime, $A \to B \in a$ or $B \to A \in a$. So, either $B \in c$ or $A \in e$, a contradiction.

□

Remark 3: *The correspondence between postulates and axioms A7 (P6), A8 (P7), A9 (P8), A10 (P9) and A11 (P10) stated in proposition 2 can be proved in respect of* B_+ *instead of* B_{K+}.

Now, it is clear that, given the soundness and completeness of $\mathrm{B_{K+}}$, those of $\mathrm{TW_{K+}}$, $\mathrm{EW_{K+}}$, $\mathrm{RW_{K+}}$ (= $\mathrm{JW_+}$), $\mathrm{LCW_+}$, $\mathrm{T_{K+}}$, $\mathrm{E_{K+}}$, $\mathrm{R_{K+}}$ (= $\mathrm{J_+}$) and $\mathrm{LC_+}$ in respect of the corresponding semantics follow immediately by proposition 2.

3. *The logics* $\mathrm{B_{Kc4}}$, $\mathrm{B_{Kc4'}}$ *and* $\mathrm{B_{Kc5}}$

We add the unary connective ¬ (negation) to the positive language. Consider the following axioms:

A14. $\neg A \rightarrow [A \rightarrow (A \wedge \neg A)]$
A15. $[B \rightarrow (A \wedge \neg A)] \rightarrow \neg B$
A16. $(A \wedge \neg A) \rightarrow \neg(A \rightarrow A)$
A17. $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$
A18. $\neg B \rightarrow [(A \rightarrow B) \rightarrow \neg A]$
A19. $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$
A20. $B \rightarrow [(A \rightarrow \neg B) \rightarrow \neg A]$

Then, the logics are axiomatized as follows:

(1) $\mathrm{B_{Kc4}}$: $\mathrm{B_{K+}}$ + A14 + A15 + A16
(2) $\mathrm{B_{Kc4'}}$: $\mathrm{B_{K+}}$ + A15 + A16 +A18
(3) $\mathrm{B_{Kc5}}$: $\mathrm{B_{K+}}$ + A16 + A120

We note the following theorems of $\mathrm{B_{K+}}$ and $\mathrm{B_{Kc4}}$ (cf. [8]):

$\mathrm{T1_{B_{K+}}}$. $(A \rightarrow B) \rightarrow [A \rightarrow (A \wedge B)]$
$\mathrm{T1_{B_{Kc4}}}$. $(A \rightarrow B) \rightarrow \neg(A \wedge \neg B)$
$\mathrm{T2_{B_{Kc4}}}$. $\neg(A \wedge \neg A)$
$\mathrm{T3_{B_{Kc4}}}$. $\vdash A \Rightarrow\ \vdash (B \rightarrow \neg A) \rightarrow \neg B$
$\mathrm{T4_{B_{Kc4}}}$. $\vdash A \Rightarrow\ \vdash \neg A \rightarrow \neg B$
$\mathrm{T5_{B_{Kc4}}}$. $\neg A \rightarrow [A \rightarrow \neg(A \rightarrow A)]$
$\mathrm{T6_{B_{Kc4}}}$. $[A \rightarrow \neg(B \rightarrow B)] \rightarrow \neg A$

Next, we prove the following theorems of $B_{Kc4'}$:

$\text{T1}_{B_{Kc4'}}$. $\neg A \rightarrow [A \rightarrow (A \wedge \neg A)]$

Proof. By $\text{T1}_{B_{K+}}$,

1. $(A \rightarrow \neg A) \rightarrow [A \rightarrow (A \wedge \neg A)]$

By A18,

2. $\neg A \rightarrow [(A \rightarrow A) \rightarrow \neg A]$

By 2 and the K rule

3. $\neg A \rightarrow (A \rightarrow \neg A)$

Then, $\text{T1}_{B_{Kc4'}}$ follows by (1) and (3). □

Therefore, we have:

Proposition 3: *B_{Kc4} is deductively included in $B_{Kc4'}$ (but does not include it).*

Proof. (1) A14, A15 and A16 are theorems of $B_{Kc4'}$. (2) MaGIC. □

Next, we have

$\text{T2}_{B_{Kc4'}}$. $\neg B \rightarrow [A \rightarrow (A \wedge \neg B)]$

Proof. By A18 and the K rule,

1. $\neg B \rightarrow (A \rightarrow \neg B)$

By $\text{T1}_{B_{K+}}$,

2. $(A \rightarrow \neg B) \rightarrow [A \rightarrow (A \wedge \neg B)]$

So, $\text{T2}_{B_{Kc4'}}$ follows by (1) and (2). □

$\text{T3}_{B_{Kc4'}}$. $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$

Proof. By A18,

1. $\neg(A \wedge \neg B) \rightarrow [[A \rightarrow (A \wedge \neg B)] \rightarrow \neg A]$

By (1) and $T1_{B_{Kc4}}$,

2. $(A \rightarrow B) \rightarrow [[A \rightarrow (A \wedge \neg B)] \rightarrow \neg A]$

So, $T3_{B_{Kc4'}}$ follows by (2) and $T2_{B_{Kc4'}}$. □

$T4_{B_{Kc4'}}$. $[B \rightarrow \neg(A \rightarrow A)] \rightarrow [(A \rightarrow B) \rightarrow [A \rightarrow \neg(A \rightarrow A)]]$
$T5_{B_{Kc4}}$, $T6_{B_{Kc4}}$, A18

We note:

Proposition 4: *Let* $B_{Kc4'(b)} = B_{Kc4} + A17$. *Then,* $B_{Kc4'(b)}$ *and* $B_{Kc4'}$ *are deductively equivalent.*

Proof. We have

$T1_{B_{Kc4'(b)}}$. $\neg B \rightarrow (A \rightarrow \neg B)$

Proof. By A14 and A17,

1. $\neg A \rightarrow [\neg(A \wedge \neg A) \rightarrow \neg A]$

Then, $T1_{B_{Kc4'(b)}}$ follows by $T2_{B_{Kc4}}$ and the K rule. □

$T2_{B_{Kc4'(b)}}$. $\neg B \rightarrow [A \rightarrow (A \wedge \neg B)]$

Proof. Similar to that of $T2_{B_{Kc4'}}$ with $T1_{B_{Kc4'(b)}}$. □

Finally, we have

$T3_{B_{Kc4'(b)}}$. $\neg B \rightarrow [(A \rightarrow B) \rightarrow \neg A]$

Proof. By $T2_{B_{Kc4'(b)}}$ and A17,

1. $\neg B \rightarrow [\neg(A \wedge \neg B) \rightarrow \neg A]$

Then, the theorem follows by $\mathrm{T1}_{\mathrm{B_{Kc4}}}$. □

Thus, $\mathrm{B_{Kc4'}}$ can intuitively be described as the result of adding the weak constructive contraposition axioms A17 and A18 to $\mathrm{B_{Kc4}}$. □

We remark the following:

Proposition 5: (1) *B_{Kc4} and $B_{Kc4'}$ are included in B_{Kc5} (but do not include it).*
(2) *B_{Kc4} and $B_{Kc4'}$ are different logics.*
(3) *B_{Kc4}, $B_{Kc4'}$ and B_{Kc5} are well axiomatized in respect of B_{K+} (that is, the negation axioms are, in each case, mutually independent).*

Proof. (1) See [8]. (2), (3) by MaGIC. □

We now turn to the semantics. Consider the following postulates

P13. $(Rabc \;\&\; c \in S) \Rightarrow a \in S$

P14. $a \in S \Rightarrow (\exists x \in S)Raax$

P15. $(R^2abcd \;\&\; d \in S) \Rightarrow (\exists x \in K)(\exists y \in S)(Racx \;\&\; Rbxy)$

P16. $(R^2abcd \;\&\; d \in S) \Rightarrow (\exists x \in K)(\exists y \in S)(Rbcx \;\&\; Raxy)$

P17. $(R^2abcd \;\&\; d \in S) \Rightarrow (\exists x \in S)R^2acbx$

P18. $(R^2abcd \;\&\; d \in S) \Rightarrow (\exists x \in S)R^2bcax$

A B_{Kc4} *model* is a quadruple $\langle K, S, R, \vDash\rangle$ where S is a non-empty subset of K, and K, R and $\vDash$ are defined, in a similar way, as in $\mathrm{B_{K+}}$ models, except for the addition of the following clause

(v). $a \vDash \neg A$ iff for all $b, c \in K$, $(Rabc \;\&\; c \in S) \Rightarrow b \nvDash A$

and postulates P13 and P14. Then, $B_{\mathrm{Kc4'}}$ *models* (B_{Kc5} *models*) are, similarly, defined as $\mathrm{B_{Kc4}}$ models, save for the addition of P15, P16 (P17, P18). In the three cases validity is defined in respect of all points of K.

The B_{Kc4} *canonical model* is the quadruple $\langle K^C, S^C, R^C, \vDash^C\rangle$ where K^C, R^C and $\vDash^C$ are defined in a similar way to which they are defined in the $\mathrm{B_{K+}}$ canonical model, and S^C is interpreted as the set of all non-null prime negation-consistent theories. A theory a is *n-inconsistent* (negation-inconsistent) iff for some wff A, $A \wedge \neg A \in a$. A theory a is *n-consistent* (negation-consistent) iff it is not n-inconsistent (cf. definition 1). The $B_{\mathrm{Kc4'}}$ *canonical model* and the B_{Kc5} *canonical model* are defined, similarly, as the $\mathrm{B_{Kc4}}$ canonical model, its items being referred now, of course, to $\mathrm{B_{Kc4'}}$ and $\mathrm{B_{Kc5}}$ theories, respectively.

Remark 4: *Clause (v) is an adaptation of the negation clause characteristic of minimal intuitionistic logic in binary relational semantics. The intuitionistic clause reads*

$$a \vDash \neg A \textit{ iff } (Rab \;\&\; b \in S) \Rightarrow b \nvDash A$$

That is, a formula of the form $\neg A$ is true at point a iff A is false in all consistent points accessible from a –'inconsistent' is here understood in the (minimal) intuitionistic way–. So, in ternary relational semantics, the (minimal) intuitionistic clause would be translated as clause (v). That is, a formula of the form $\neg A$ is true in point a iff A is false in all points b such that $Rabc$ for all consistent points c –'consistent' is here understood as n-consistent–.

Now, we recall that a theory is w-inconsistent iff it contains the negation of a theorem (cf. definition 2 in §1). We prove:

Proposition 6: *Let a be a B_{Kc4} theory. Then, a is n-consistent iff it is w-inconsistent.*

Proof. (1) Suppose that for some wff A, $A \wedge \neg A \in a$. By A16, a is w-inconsistent. (2) Suppose that a is w-inconsistent, i.e., $\neg A \in a$, A being a theorem. By the K rule, $\neg A \rightarrow A$ is a theorem. So, $A \in a$ and consequently, $A \wedge \neg A \in a$. □

Therefore, in B_{Kc4} (and in all logics including it), n-consistency is equivalent to w-consistency.

In [8] it is proved that B_{Kc4} and B_{Kc5} are sound and complete in respect of the corresponding semantics defined above. So, we shall prove the soundness and the completeness of $B_{Kc4'}$.

We first prove a useful proposition stating that n-consistency of theories is preserved when they are extended to prime theories (this proposition is implicitly used in what follows). Let $B_{K+,\neg}$ be any extension of B_{K+} in which the rule contraposition

con. $\vdash A \rightarrow B \Rightarrow \neg B \rightarrow \neg A$

and

A16. $(A \wedge \neg A) \rightarrow \neg(A \rightarrow A)$

hold. We note that the following De Morgan law

dm. $\vdash (\neg A \vee \neg B) \rightarrow \neg(A \wedge B)$

is provable in $B_{+,\neg}$ (A2, A5, con.). We also remark that con. is provable in B_{Kc4} (cf. [8]).

We have

Proposition 7: *Let a be a $B_{K+,\neg}$ n-consistent theory. Then, there is some prime n-consistent theory x such that $a \subseteq x$.*

Proof. Define from a a maximal n-consistent theory x such that $a \subseteq x$. If x is not prime, then $A \vee B \in x$, $A \notin x$, $B \notin x$ for some wff A, B. Define the theories $[x, A] = \{C \mid \exists D[D \in x \ \& \vdash_{B_{+,\neg}} (A \wedge D) \rightarrow C]\}$, $[x, B] = \{C \mid \exists D[D \in x \ \& \vdash_{B_{+,\neg}} (B \wedge D) \rightarrow C]\}$ that strictly include x. By the maximality of x, $[x, A]$ and $[x, B]$ are n-inconsistent. So, $C \wedge \neg C \in x$, $D \wedge \neg D \in x$ for some wff C, D. By definitions, $\vdash_{B_{+,\neg}} (A \wedge G) \rightarrow (C \wedge \neg C)$, $\vdash_{B_{+,\neg}} (B \wedge G') \rightarrow (D \wedge \neg D)$ for G, $G' \in x$. By A16, $\vdash_{B_{+,\neg}} (A \wedge G) \rightarrow \neg(C \rightarrow C)$, $\vdash_{B_{+,\neg}} (B \wedge G') \rightarrow \neg(D \rightarrow D)$. Then, by B_+, $\vdash_{B_{+,\neg}} [(A \vee B) \wedge (G \wedge G')] \rightarrow [\neg(C \rightarrow C) \vee \neg(D \rightarrow D)]$. As $(A \vee B) \wedge (G \wedge G') \in x$, $\neg(C \rightarrow C) \vee \neg(D \rightarrow D) \in x$. By dm., $\neg[(C \rightarrow C) \wedge (D \rightarrow D)] \in x$, but $\vdash_{B_{+,\neg}} (C \rightarrow C) \wedge (D \rightarrow D)$ by A1 and Adj. Therefore, x is n-inconsistent by proposition 6, which is impossible. Consequently, x is prime. ☐

Thus, in any logic including B_{K+} plus con. and A16, n-consistent theories can be extended to prime n-consistent theories.

We prove

Proposition 8: *Given the logic $B_{Kc4'}$ and $B_{Kc4'}$ semantics,*

(1) *P15 is the corresponding postulate to A17, and*
(2) *P16 is the corresponding postulate to A18.*

Proof. We prove, e.g., case 2. The proof of case 1 is similar.

A17 is $B_{Kc4'}$ valid: Suppose $a \vDash \neg B$, $a \nvDash (A \rightarrow B) \rightarrow \neg A$ for wff A, B and $a \in K$ in some model. So, $b \vDash A \rightarrow B$, $d \vDash A$ for b, c, $d \in K$ and $e \in S$ such that $Rabc$ and $Rcde$. By d2, R^2abde, and by P16, $Rbdz$ and $Razu$ for $z \in K$ and $u \in S$. On the other hand, by $a \vDash \neg B$, $(Raxy \ \& \ y \in S) \Rightarrow x \nvDash B$ for all $x \in K$ and $y \in S$. So, $z \nvDash B$. But $z \vDash B$ ($b \vDash A \rightarrow B$, $Rbdz$, $d \vDash A$).

P16 holds canonically: it follows immediately from the following lemma:

Lemma 1: *Let a, b, c be non-null elements in K^T and d a non-null n-consistent member in K^T such that $R^{T2}abcd$. Then, there are non-null x in K^T and some non-null n-consistent y in K^T such that R^Tbcx and R^Taxy.*

Proof. Let a, b, c be non-null elements in K^T and d a n-consistent element in K^T such that $R^{T2}abcd$, i.e., by d2, R^Tabz and R^Tzcd for some $z \in K^T$. Define the non-null theories $x = \{B \mid \exists A[A \to B \in b \,\&\, A \in c]\}$, $y = \{B \mid \exists A[A \to B \in a \,\&\, A \in x]\}$ such that R^Tbcx and R^Taxy. We prove that y is n-consistent. Suppose it is not. Then, $\neg A \in y$, A being a theorem (cf. proposition 6). So, $B \to \neg A \in a$, $C \to B \in b$ for some wff B and $C \in c$. As A is a theorem, by T3$_{\mathrm{B_{Kc4}}}$, $\neg B \in a$. By A18, $(C \to B) \to \neg C \in a$. So, $\neg C \in z$ (R^Tabz), whence, by A14, $C \to (C \wedge \neg C) \in z$ and consequently, $C \wedge \neg C \in d$ (R^Tzcd, $C \in c$), contradicting the n-consistency of d. □

□

Now, given the soundness and completeness of B$_{\mathrm{Kc4}}$, by proposition 8, it follows:

Theorem 5: (soundness and completeness of B$_{\mathrm{Kc4'}}$) $\vdash_{B_{Kc4'}} A$ *iff* $\vDash_{B_{Kc4'}} A$.

4. *The logic* B$_{\mathrm{Kc4}F}$ *and its extensions*

We add the propositional falsity constant F to the positive language together with the definition

D¬: $\neg A \leftrightarrow A \to F$

Now, consider the following axioms:

A21. $F \to (A \to F)$
A22. $[A \wedge (A \to F)] \to F$
A23. $(A \to B) \to [(B \to F) \to (A \to F)]$
A24. $(B \to F) \to [(A \to B) \to (A \to F)]$
A25. $[A \to (B \to F)] \to [B \to (A \to F)]$
A26. $B \to [[A \to (B \to F)] \to (A \to F)]$

Then, the following logics are defined:

(1) B$_{\mathrm{Kc4}F}$: B$_{\mathrm{K+}}$ + A21 + A22
(2) B$_{\mathrm{Kc4}F'}$: B$_{\mathrm{K+}}$ + A21 + A22 + A24
(3) B$_{\mathrm{Kc5}F}$: B$_{\mathrm{K+}}$ + A22 + A26

Remark 6: *The logic B_{Kcr} is the main logic defined in [11]. It is axiomatized by adding to B_{K+} A23, A24 and the special law of reductio in the form $[A \to (A \to F)] \to (A \to F)$. We note that B_{Kcr} is deductively included in (but does not include) $B_{Kc4F'}$.*

We shall prove that B_{Kc4F} and B_{Kc4}, $B_{Kc4F'}$ and $B_{Kc4'}$, and B_{Kc5F} and B_{Kc5} are definitionally equivalent. So, the relations between the logics stated in proposition 5 hold correspondingly for the definitionally equivalent logics defined with the falsity constant. Moreover, we remark that B_{Kc4F}, $B_{Kc4F'}$ and B_{Kc5F} are well axiomatized in respect of B_{K+} (MaGIC, cf. proposition 5).

We now define the semantics (cf. [5]). A *B_{Kc4F} model* is a quadruple $\langle K, S, R, \vDash\rangle$ where K, S, R and $\vDash$ are defined, in a similar way, as in a B_{Kc4} model, postulates P13 and P14 hold, but clause (v) is substituted by the clauses

(vi). $(a \leq b \quad \& \quad a \vDash F) \Rightarrow b \vDash F$

and

(vii). $a \vDash F$ iff $a \notin S$

Then, *$B_{Kc4F'}$ models* (*B_{Kc5F} models*) are defined similarly as B_{Kc4F} models save for the addition of P15 and P16 (P17, P18). In the three cases validity is defined in respect of all points of K.

Next, the *B_{Kc4F} canonical model* cf. [5]) is the quadruple $\langle K^C, S^C, R^C, \vDash^C\rangle$ where K^C, R^C and $\vDash^C$ are defined in a similar way to which they are defined in the B_{Kc4} (or B_{K+}) canonical model, and S^C, as before, is the set of all non-null prime consistent theories, but now a theory is *consistent* iff $F \notin a$ (a theory a is *inconsistent* iff $F \in a$). The *$B_{Kc4F'}$ canonical model* and the *B_{Kc5F} canonical model* are defined similarly, but with its items referred to $B_{Kc4F'}$ theories and B_{Kc5F} theories, respectively.

We prove:

Proposition 9: *Let a be a B_{Kc4F} theory. Then, a is inconsistent iff a is n-inconsistent.*

Proof. (1) Let $F \in a$. As $\neg F$ is a theorem (A1, D¬), $F \wedge \neg F \in a$. Suppose $A \wedge \neg A \in a$ for some wff A. Then, $F \in a$ by A22. □

Therefore, in B_{Kc4F} (and in all logics included in it), consistency (understood as the absence of F) and n-consistency are equivalent.

Now, in [5] it is proved that $\mathrm{B}_{\mathrm{Kc}4F}$ is sound and complete in respect of the semantics defined above. So, we shall prove the soundness and completeness of $\mathrm{B}_{\mathrm{Kc}4F'}$ and $\mathrm{B}_{\mathrm{Kc}5F}$.

As in the case of $\mathrm{B}_{\mathrm{Kc}4}$, a proposition on the preservation of consistency in building prime theories is provable. Let $\mathrm{B}_{+,F}$ be the result of extending the positive language of B_{+} with the propositional falsity constant F, no new axioms, however, being added. We have:

Proposition 10: *Let a be a consistent $B_{+,F}$ theory. Then, there is some prime consistent theory x such that $a \subseteq x$.*

Proof. Define from a a maximal consistent theory x such that $a \subseteq x$. If x is not prime, then $A \vee B \in x$, $A \notin x$, $B \notin x$ for some wff A, B. Define, similarly, as in proposition 7 the theories $[x, A]$ and $[x, B]$ strictly including x. Then, $[x, A]$ and $[x, B]$ are inconsistent, i.e., $F \in [x, A]$, $F \in [x, B]$ whence, by definitions, $\vdash_{\mathrm{B}_{+,F}} (A \wedge C) \to F$, $\vdash_{\mathrm{B}_{+,F}} (B \wedge C') \to F$ for $C \in x$, $C' \in x$. Then, $F \in x$ (cf. proposition 7), which is impossible. Therefore, x is prime. □

Thus, in any logic including $\mathrm{B}_{+,F}$, consistent theories can be extended to prime consistent theories.

We now prove

Proposition 11: *Given the logic B_{Kc4F} and B_{Kc4F} semantics, P15, P16, P17 and P18 are the corresponding postulates to A23, A24, A25 and A26, respectively.*

Proof. We prove, e.g., that P18 is the corresponding postulate to A26. The rest of the cases are proved similarly and are left to the reader.

A26 is $B_{\mathrm{Kc}4F}$ valid: suppose $a \vDash B$, $a \nvDash [A \to (B \to F)] \to (A \to F)$ for wff A, B and $a \in K$ in some model. Then, $b \vDash A \to (B \to F)$, $d \vDash A$, $e \nvDash F$ for a, b, c, d, $e \in K$ such that $Rabc$ and $Rcde$. By d2, R^2abde, and as $e \in S$, by P18, $Rbdx$ and $Rxay$ for $x \in K$ and $y \in S$. So, $x \vDash B \to F$ and then, $y \vDash F$, i.e., $y \notin S$ (clause (vii)), a contradiction.

P18 holds canonically: It follows immediately from the following lemma:

Lemma 2: *Let a, b, c be non-null members in K^T and d a non-null consistent member in K^T such that $R^{T2}abcd$. Then, there are non-null y in K^T and non-null consistent x in K^T such that R^Tbcy and R^Tyax, i.e., $R^{T2}bcax$.*

Proof. Suppose non-null a, b, c in K^T and non-null consistent d in K^T such that $R^{T2}abcd$, i.e., R^Tabz and R^Tzcd for some (non-null) $z \in K^T$. Define the non-null theories $y = \{B \mid \exists A[A \to B \in b \,\&\, A \in c]\}$, $x = \{B \mid$

$\exists A[A \to B \in y \,\&\, A \in a]\}$ such that $R^T bcy$ and $R^T yax$. We prove that x is consistent. Suppose it is not. Then, $F \in x$. So, $B \to (A \to F) \in b$ for some $A \in a$, $B \in c$. By A26, $[B \to (A \to F)] \to (B \to F) \in a$. So, $B \to F \in z$ ($R^T abz$) and so, $F \in d$ ($R^T zcd$), contradicting the consistency of d. □

□

Now, given the soundness and completeness of $\mathrm{B}_{\mathrm{Kc}4F}$, by proposition 11, it follows:

Theorem 7: (soundness and completeness of $\mathrm{B}_{\mathrm{Kc}4F'}$ and $\mathrm{B}_{\mathrm{Kc}5F}$)
(1) $\vdash_{B_{Kc4F'}} A$ *iff* $\vDash_{B_{Kc4F'}} A$
(2) $\vdash_{B_{Kc5F}} A$ *iff* $\vDash_{B_{Kc5F}} A$

5. *The definitional equivalence between* $\mathrm{B}_{\mathrm{Kc}4}$ *and* $\mathrm{B}_{\mathrm{Kc}4F}$ *and their respective extensions*

Firstly, we introduce F by definition in $\mathrm{B}_{\mathrm{Kc}4}$ (cf. [5]). Note that for any formulas A, B, $\neg(A \to A)$ and $\neg(B \to B)$ are equivalent by $\mathrm{T4}_{\mathrm{B}_{\mathrm{Kc}4}}$. Then, we state:

Let A be a wff. Then,

$$\mathrm{D}F\text{: } F \leftrightarrow \neg(A \to A)$$

That is, F replaces any wff of the form $\neg(A \to A)$ (note that the defining formula does not depend on the choice of A). We note:

Proposition 12: *Let a be a B_{Kc4} theory. Then, a is n-inconsistent iff for some wff A, $\neg(A \to A) \in a$.*

Proof. Proposition 6. □

Therefore, in $\mathrm{B}_{\mathrm{Kc}4}$ (and in all logics including it) a theory is n-inconsistent iff it contains F as defined above. More precisely, in $\mathrm{B}_{\mathrm{Kc}4}$ (and in all logics included in it) a theory is n-inconsistent iff it is w-inconsistent iff it contains F (cf. propositions 6 and 12).

Next, we turn to the proof of the definitional equivalence. We shall understand the notion as 'definitional equivalence via translations' (see, e.g., [6]). We have to prove the following two propositions (cf. [2]):

Proposition 13: (1) $B_{Kc4F} \subseteq B_{Kc4} \cup \{DF\}$
(2) $B_{Kc4} \subseteq B_{Kc4F} \cup \{D\neg\}$

Proposition 14: (1) *D¬ is provable in* $B_{Kc4} \cup \{DF\}$
(2) DF *is provable in* $B_{Kc4F} \cup \{D\neg\}$

Propositions 13 and 14 are proved in [5]. So, in order to prove the definitional equivalence between $B_{Kc4'}$ and $B_{Kc4F'}$, B_{Kc5} and B_{Kc5F}, it suffices to prove propositions 15 and 16 that follow:

Proposition 15: (1) $B_{Kc4'} \subseteq B_{Kc4F'} \cup \{D\neg\}$
(2) $B_{Kc4F'} \subseteq B_{Kc4'} \cup \{DF\}$

Proof.

(1) A18= A24, by D¬.
(2) $T4_{B_{Kc4'}}$= A24, by DF.

□

Proposition 16: (1) $B_{Kc5} \subseteq B_{Kc5F} \cup \{D\neg\}$
(2) $B_{Kc5F} \subseteq B_{Kc5} \cup \{DF\}$

Proof.

(1) A20= A26, by D¬.
(2) Firstly, we note that $B \to \{[A \to [B \to \neg(A \to A)]] \to [A \to \neg(A \to A)]\}$ is a theorem of B_{Kc5} by A20, $T5_{B_{Kc4}}$ and $T6_{B_{Kc4}}$. Then, this theorem is equivalent to A26, by DF.

□

6. *Strengthening the positive logic*

We take up again the extensions of B_{K+} defined in §2. Now, negation can be introduced in these logics in a similar way to which it was introduced in B_{K+}. Thus, the following logics can be defined:

(1) TW_{Kc4}, EW_{Kc4}, RW_{Kc4} (= JW_{c4}), LCW_{c4}

(2) $TW_{Kc4'}$, $EW_{Kc4'}$, $RW_{Kc4'}$ (= $JW_{c4'}$), $LCW_{c4'}$

(3) TW_{Kc5}, EW_{Kc5}, RW_{Kc5} (= JW_{c5}), LCW_{c5}

(4) T_{Kc4}, E_{Kc4}, R_{Kc4} (= J_{c4}), LC_{c4}

(5) $T_{Kc4'}$, $E_{Kc4'}$, $R_{Kc4'}$ (= $J_{c4'}$), $LC_{c4'}$

(6) T_{Kc5}, E_{Kc5}, R_{Kc5} (= J_{c5}), LC_{c5}

It is clear that, given propositions 13-16, the logics definitionally equivalent to those in groups 1-6, can be defined:

1'. $\mathrm{TW}_{\mathrm{Kc}4F}$, $\mathrm{EW}_{\mathrm{Kc}4F}$, $\mathrm{RW}_{\mathrm{Kc}4F}$ (= $\mathrm{JW}_{\mathrm{c}4F}$), $\mathrm{LCW}_{\mathrm{c}4F}$

2'. $\mathrm{TW}_{\mathrm{Kc}4F'}$, $\mathrm{EW}_{\mathrm{Kc}4F'}$, $\mathrm{RW}_{\mathrm{Kc}4F'}$ (= $\mathrm{JW}_{\mathrm{c}4F'}$), $\mathrm{LCW}_{\mathrm{c}4F'}$

3'. $\mathrm{TW}_{\mathrm{Kc}5F}$, $\mathrm{EW}_{\mathrm{Kc}5F}$, $\mathrm{RW}_{\mathrm{Kc}5F}$ (= $\mathrm{JW}_{\mathrm{c}5F}$), $\mathrm{LCW}_{\mathrm{c}5F}$

4'. $\mathrm{T}_{\mathrm{Kc}4F}$, $\mathrm{E}_{\mathrm{Kc}4F}$, $\mathrm{R}_{\mathrm{Kc}4F}$ (= $\mathrm{J}_{\mathrm{c}4F}$), $\mathrm{LC}_{\mathrm{c}4F}$

5'. $\mathrm{T}_{\mathrm{Kc}4F'}$, $\mathrm{E}_{\mathrm{Kc}4F'}$, $\mathrm{R}_{\mathrm{Kc}4F'}$ (= $\mathrm{J}_{\mathrm{c}4F'}$), $\mathrm{LC}_{\mathrm{c}4F'}$

6'. $\mathrm{T}_{\mathrm{Kc}5F}$, $\mathrm{E}_{\mathrm{Kc}5F}$, $\mathrm{R}_{\mathrm{Kc}5F}$ (= $\mathrm{J}_{\mathrm{c}5F}$), $\mathrm{LC}_{\mathrm{c}5F}$

We prove some propositions on the relations between these logics:

Proposition 17: *TW_{Kc4} and $TW_{Kc4'}$ are deductively equivalent logics. So, EW_{Kc4} and $EW_{Kc4'}$, RW_{Kc4} (= JW_{c4}) and $RW_{Kc4'}$ (= $JW_{c4'}$) and LCW_{c4} and $LCW_{c4'}$, T_{Kc4} and $T_{Kc4'}$, E_{Kc4} and $E_{Kc4'}$, R_{Kc4} (= J_{c4}) and $R_{Kc4'}$ (= $J_{c4'}$) and LC_{c4} and $LC_{c4'}$are deductively equivalent logics.*

Proof. A17 is derivable by A8, A14 and A15; A18 is derivable by A7, A14 and A15. □

Proposition 18: *RW_{Kc4} (= JW_{c4}) and RW_{Kc5} (= JW_{c5}) and LCW_{c4} and LCW_{c5} are deductively equivalent logics. So, R_{Kc4} (= J_{c4}) and R_{Kc5} (= J_{c5}) and LC_{c4} and LC_{c5} are deductively equivalent logics.*

Proof. Firstly, note that A17 and A18 are derivable. Next, by A11 and A17,

1. $A \to [\neg A \to \neg(A \to A)]$

By 1 and $\mathrm{T6}_{\mathrm{B}_{\mathrm{Kc4}}}$,

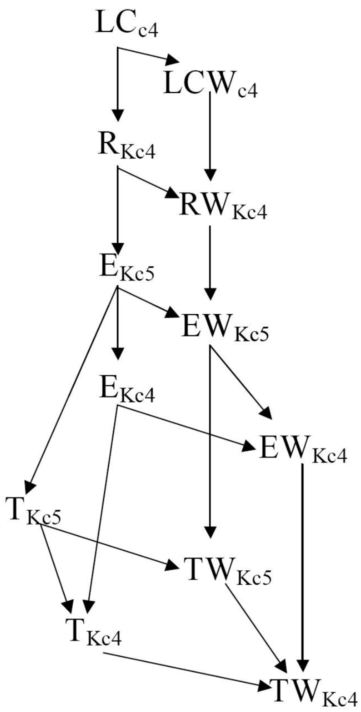
2. $A \to \neg\neg A$

Then, A19 and A20 are easily provable with, respectively, A17 and A18 together with introduction of double negation (2). □

Proposition 19: *E_{Kc4} and E_{Kc5} are different logics, the former being included in the latter. So, T_{Kc4} and T_{Kc5}, EW_{Kc4} and EW_{Kc5} and TW_{Kc4} and TW_{Kc5} are different logics, the first member of each pair being included in the second.*

Proof. By proposition 5 and MaGIC. □

By well known results on relevance and intuitionistic logics (alternatively, one can use MaGIC), the contractionless logics here defined are different from their respective counterparts plus the contraction axiom. So, the relations between these logics can be summarized in the following diagram (L $\rightarrow$ L' means that L is included in L').



Diagram

A similar diagram is, of course, obtained for the definitionally equivalent logics defined with the propositional falsity constant.

Remark 8: *Recall that LCW_{c4} (LC_{c4}), RW_{Kc4} (R_{Kc4}), EW_{Kc5} (E_{Kc5}) and TW_{Kc5} (T_{Kc5}) are the result of adding the strong constructive contraposition axioms A19 and A20 to LCW_+ (LC_+), RW_{K+} (R_{K+}), EW_{K+} (E_{K+}) and TW_{K+} (T_{K+}), respectively, and that EW_{Kc4} (E_{Kc4}) and TW_{Kc4} (T_{Kc4}) are EW_{K+} (E_{K+}) and TW_{K+} (T_{K+}), respectively, plus the weak constructive contraposition axioms A17 and A18.*

Remark 9: *E_{Kc5}, E_{Kc4}, EW_{Kc5}, EW_{Kc4}, T_{Kc5} and T_{Kc4}, TW_{Kc5} and TW_{Kc4} are constructive modal logics (the arrow in these logics is some kind of strict* *implication). But we note that these logics are not included in, e.g., Lewis' modal S5 as axiomatized by Hacking [4] (and, of course, neither do they include it): A14, for example, is not a theorem of S5. On the other hand, we remark that a necessity operator* $\Box$ *can be introduced (as in [1], §4.3) in* E_{Kc5} *and* E_{Kc4}, EW_{Kc5} *and* EW_{Kc4} *via the definition* $\Box A =_{df} (A \to A) \to A$. *Generally speaking, the operator thus introduced has the characteristic properties of the necessity operator of Lewis' S4 but with interesting relations with a possibility operator* $\Diamond$ *definable from it, due to the absence of elimination of double negation and its accompanying theses. The analysis of this question cannot, however, be pursued here.*

Regarding soundness and completeness of the logics introduced in this section, it is obvious that they follow immediately from those of the positive logics and B_{Kc4} (B_{Kc4F}), $B_{Kc4'}$ ($B_{Kc4F'}$) and B_{Kc5} (B_{Kc5F}).

We end this section with the following proposition:

Proposition 20: *Though the ECQ axiom (iii)* $(A \wedge \neg A) \to \neg B$ *is a theorem of* B_{Kc4} *(cf. [8]), the ECQ axiom (iv)* $(A \wedge \neg A) \to B$ *and the EFQ axioms (v)* $\neg A \to (A \to B)$, *(vi)* $A \to (\neg A \to B)$ *(cf. Introduction) are not provable in* LCW_{c4}.

Proof. By MaGIC. $\Box$

Therefore, in LCW_{c4} (and in all logics included in it), n-consistency is not equivalent to absolute consistency. Consequently, LC_{c4} (and all logics included in it) are paraconsistent logics in the sense of [7].

7. *A comparison between* B_{Kc1} *and* B_{Kc4} *and their respective extensions*

The basic constructive logic adequate to w-consistency in the ternary relational semantics without a set of designated points, i.e., B_{Kc1} (cf. Introduction) can be axiomatized by adding to B_{K+} the axioms

A27. $\neg A \to [A \to \neg(A \to A)]$

and

A28. $[B \to \neg(A \to A)] \to \neg A$

The logic $B_{Kc1'}$ is B_{Kc1} plus the weak constructive contraposition axioms A17 and A18, and the logic B_{Kc2} is B_{Kc1} plus the strong constructive contraposition axioms A19 and A20. Then, in [9], [10], a number of extensions of B_{Kc1}, $B_{Kc1'}$ and B_{Kc2} with the positive axioms A7, A8 and A10-A13 are introduced. However, no extensions with the contraction axiom A9 are considered, since its addition even to B_{Kc1} would cause w-consistency to be equivalent to n-consistency. Let us now compare these logics with the ones defined in this paper. Firstly, we have:

Proposition 21: *B_{Kc1}, $B_{Kc1'}$ and B_{Kc2} are included in (but do not include) B_{Kc4}, $B_{Kc4'}$ and B_{Kc5}, respectively.*

Proof. (1) A27 and A28 ($T5_{B_{Kc4}}$ and $T6_{B_{Kc4}}$, respectively, cf. §3) are theorems of B_{Kc4}. (2) By MaGIC. □

Now, let S_{Kc1} ($S_{Kc1'}$ and S_{Kc2}) be any extension of B_{Kc1} ($B_{Kc1'}$ and B_{Kc2}) defined by adding any selection of the axioms A7, A8, A10-A13; and let S_{Kc4} ($S_{Kc4'}$ and S_{Kc5}) be the extension of B_{Kc4}, $B_{Kc4'}$ and B_{Kc5} defined by the same selection. We have:

Proposition 22: *S_{Kc1} ($S_{Kc1'}$ and S_{Kc2}) is included in (but does not include) S_{Kc4} ($S_{Kc4'}$ and S_{Kc5}).*

Proof. (1) By proposition 21. (2) Let LCW_{Kc2} be the result of adding A7, A8 and A10-A13 to B_{Kc2}. Although A14 is provable in LCW_{Kc2}, A15 and A16 are not (MaGIC). □

Thus, for example, JW_{c2} and JW_{c5} are the result of adding to contractionless positive intuitionistic logic JW_+ (i.e., B_+ plus A7, A8, A11 and A12) A19 and A20, and A16 and A20, respectively. Now, JW_{c2} is included in (but does not include) JW_{c5}.

Finally, note that the relations stated in propositions 21, 22 hold, of course, correspondingly between the definitionally equivalent logics defined with the falsity constant

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