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TENNANT ON MULTIPLE CONCLUSIONS

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1. *Introduction*

In *The Taming of the True*, Neil Tennant presents an argument against multiple-conclusion sequent calculi. In the following, I raise three objections showing that Tennant's argument is untenable. But first some context.

What is at stake? Central to the anti-realist's theory of the logical constants is the thought that the meanings of the logical operators are entirely captured by the rules governing our inferential practice — the rules (in some sense) determine the meaning of the logical operators featuring dominantly within them.[1] Anti-realists like Michael Dummett and Neil Tennant have, on the basis of such a proof-theoretic account of the meanings of the logical constants, put forward an argument against classical logic. The underlying idea is that our logical rules of inference, insofar as they confer meanings on the logical operators, must obey certain general meaning-theoretic principles, at least if the meanings so specified are to be coherent. These meaning-theoretic principles are brought to bear on the meanings of the logical constants by constraining the possible form rules of inference may take (e.g. languages must be learnable, hence the rules characterizing the meanings of the logical constants must be finitely stateable; language is molecular, therefore it must be possible to isolate the meaning of a particular logical constant; the meaning of a newly introduced operator must be coherent, therefore it has to satisfy constraints of harmony; etc.). The anti-realist then goes on to show that classical logic violates the plausible constraints so formulated, and concludes that the classical logician fails to attach coherent meanings to the logical constants. I shall call arguments of this form *proof-theoretic arguments*.

Crucial to the argument is the choice of a proof-theoretic framework: the argument relies on the demonstration that each of the possible ways in which

[1] An operator is said to feature *dominantly* within a rule of inference if it is the main connective in the conclusion of the inference schema, or, in the case of an elimination rule, if it is the operator to be eliminated occurring as the main connective in the major premise of the inference rule.

the intuitionistic natural deduction system (NJ) can be extended to yield its classical counterpart (NK) violates at least one of the constraints. Despite the argument's apparent reliance on specific structural features of natural deduction systems — in natural deduction systems, NK is obtained from NJ by adjoining to the latter one of the characteristically classical rules of inference[2] — it is assumed that the argument goes through without loss of generality. However, another of Gentzen's discoveries seems to cast doubt on the anti-realist's claim to generality. Having introduced the sequent calculus, Gentzen observes that it is possible in this system to move between the classical variant (LK) and the intuitionistic one (LJ) simply by requiring that in the intuitionistic case, the succedents be restricted to at most one formula.[3] As Gentzen puts it

> The distinction between intuitionistic and classical logic is, externally, of a quite different type in the calculi LJ and LK from that in the calculi NJ and NK. In the case of the latter, the distinction is based on the inclusion or the exclusion of the law of the excluded middle whereas for the calculi LJ and LK the difference is characterized by the restriction on the succedent (Gentzen 1934, p. 86).

Thus the question arises what grounds the anti-realist might have to privilege natural deduction over the (multiple-succedent) sequent calculus. It is to this question that Tennant seeks to provide an answer.

2. *Tennant's argument*

Tennant's approach is to reject sequent calculi that allow for multiple members in their succedent. If he could provide a general argument for the illegitimacy of sequents of this form, Tennant would be home free because he would have blocked the path to the standard sequent system of classical logic. Tennant's argument is this:

[2] The law of excluded middle, reductio ad absurdum, classical dilemma, double negation elimination, etc.

[3] I will speak of the *succedent* of a sequent to designate the set on the right-hand side of the sequent sign in order to distinguish it from the overall conclusion of the derivation, which is itself a sequent rather than a set. I shall use 'multiple succedent' to refer to systems that allow for the succedent to be of cardinality greater than one.

> the classical logician has to treat of sequents of the form $X: Y$ where the succedent Y may in general contain more than one sentence. In general, this smuggles in non-constructivity through the back door. For provable sequents are supposed to represent acceptable arguments. In normal practice, arguments take one from premisses to a single conclusion. There is no acceptable interpretation of the 'validity' of a sequent $X: Q_1, \ldots, Q_n$ in terms of preservation of warrant to assert when X contains only sentences involving no disjunctions. If one is told that $X: Q_1, \ldots, Q_n$ is 'valid' in the extended sense for multiple-conclusion arguments just in case $X: Q_1 \vee \cdots \vee Q_n$ is valid in the usual sense for single-conclusion arguments, the intuitionist can demand to know precisely which disjunct Q_i, then, proves to be derivable from X. No answer to such a question can be provided in general with the multiple-conclusion sequent calculus of the classical logician. It behoves us, then, to stay with a natural deduction system, and to present it in sequent form only if we observe the requirement that sequents should not have multiple conclusions (Tennant 1997, p. 320).

Tennant's objection involves two steps: First, a sequent represents an acceptable argument adequately only if the sequent as a whole is interpreted as a single-succedent sequent. But — second step — when interpreted in this way (i.e. disjunctively), it is not in general the case that it can be determined which of the disjuncts in the succedent of the end-sequent holds. In other words, multiple succedent calculi fail to satisfy the disjunction property, which requires that for every proof of a disjunction, we can (at least in principle) produce a proof of at least one of the disjuncts. Therefore, multi-succedent sequents fail to conform to constructivist strictures.

3. *Objections*

3.1. *First objection*

So much for exposition; we turn now to criticism. Leaving the first step to one side, we note that allowing for sets of cardinality with more than one member in the succedent is no sufficient condition for non-constructivity. It is not true in general that we part company with the anti-realist in admitting multiple conclusions. Indeed constructivity can be restored by slightly modifying the right-hand side introduction rules for conditional, negation and for the universal quantifier.[4] To illustrate this for the propositional case, consider a sequent system obtained by taking the standard classical sequent rules for the propositional connectives (along with the usual structural rules) with the exception of the rules governing the material conditional:

$$(R\supset)\,\frac{\Gamma, \phi\colon \psi, \Delta}{\Gamma\colon \phi\supset\psi, \Delta}$$

$$(L\supset)\,\frac{\Gamma_0\colon \phi, \Delta_0 \qquad \Gamma_1, \psi\colon \Delta_1}{\Gamma_0, \Gamma_1, \phi\supset\psi\colon \Delta_0, \Delta_1}$$

We replace the right introduction rule by the following rule:

$$(R\supset)^*\,\frac{\Gamma, \phi\colon \psi}{\Gamma\colon \phi\supset\psi, \Delta}$$

The difference consists solely in the delayed introduction of the context Δ on the far right-hand side of the succedent. In the standard classical rule the context is already present in the premise. The system obtained is a multi-succedent system for propositional intuitionistic logic.[5]

Anti-realists like Tennant who advocate even more extensive logical reform in favour of intuitionistic *relevant* logic, will find fault with the explicit dilution introducing the context Δ on the far right-hand side of the conclusion. It may be objected that the proposed system, by its very constitution, violates relevantist principles, and so precludes the possibility of further revision from the outset. However, the instance of the rule of dilution in the statement of $(R\supset)^*$ turns out to be inessential. Indeed, we obtain an equivalent system if we replace the standard classical right-hand side introduction rule for $\supset$ with the usual single-succedent rule.[6] The resulting system recognizably remains a multiple-succedent system for intuitionistic logic. Moreover, it contains no 'built-in' violations of relevantist principles. Allowing sequents with multiple succedents therefore does not *per se* violate constructivist principles, not even in the face of the additional constraint imposed by intuitionistic relevant logicians like Tennant.

[4] For brevity's sake we stipulate that $\sim\phi =_{\text{def.}} \phi\supset\bot$. This allows us to dispense with extra rules for negation.

[5] The system can be extended to full intuitionistic first-order logic by amending the right introduction rule for the universal quantifier in an analogous way. See Troelstra and Schwichtenberg for details (1996, p. 69).

[6] In the case of the corresponding first-order systems, we replace the right-hand side rule for the universal quantifier with its standard single-succedent counterpart.

3.2. *Second objection*

Furthermore, Tennant's argument is question-begging. Recall that the declared aim of the anti-realist is to translate meaning-theoretic principles into constraints on inference rules, thereby furnishing *independent*, meaning-theoretic grounds against classical forms of inference and in favour of intuitionistic ones — grounds other than the standard arguments for the rejection of the classicist's assumption of bivalence. But if this is our aim, we surely cannot appeal to the very principles we set out to justify (e.g. the disjunction property) when explaining to the classical logician why appealing to the sequent calculus will not enable him to dodge the anti-realist's proof-theoretic argument. This, however, is precisely what Tennant does: according to him, the classicist's appeal to the sequent calculus is illegitimate because, in order to obtain classical systems, we have to allow for sequents involving multiple members in their succedent; this the classicist cannot do, he claims, on pain of violating constructivist principles. But these are the very principles that the proof-theoretic argument aims to establish in the first place. Tennant's appeal to the disjunction property is thus patently circular.

3.3. *Third objection*

Finally, Tennant's proposed restriction to single-succedent calculi fails to be a sufficient condition for constructivity — it does not guarantee that the system obtained will be acceptable by constructivist standards. In other words, the admission of multiple succedents is not a necessary condition for non-constructivity either. For example, it is possible to give a formulation of (propositional) classical logic in a single-succedent system. This can be achieved by adding a version of the law of the excluded middle (LEM) restricted to atomic formulas to the intuitionistic propositional single-succedent calculus. The rule in question is this:[7]

$$LEM\,\frac{\phi, \Gamma_0 : \psi \qquad \sim\phi, \Gamma_1 : \psi}{\Gamma_0, \Gamma_1 : \psi}$$

Therefore, even if it were possible to give a conclusive argument against multiple-succedent systems, the task would remain for the anti-realist to disclose additional constraints — in a non ad hoc way — on admissible single-succedent sequent systems. It might be thought that there is something fishy about this version of the law of the excluded middle. Unlike standard operational rules in the sequent calculus, this rule does not introduce a constant, but rather eliminates two formulas, and therefore behaves more like the cut

[7] See Negri and von Plato (op. cit., p. 114) for details.

rule. Perhaps there is a way out for the anti-realist. However, it is incumbent upon him to show how the intuitive aberrance of the inference rule in question can be accounted for by formulating principled constraints on inference rules of single-succedent systems; and this is no trivial task.

4. *Conclusion*

We have found Tennant's argument against mutiple-succedent sequent calculi wanting. As it stands, Tennant's proof-theoretic argument for intuitionistic logic does not possess the desired generality, and thus, in the absence of other arguments, seems to rely on arbitrary features characteristic of natural deduction systems. Yet it has not been shown that no such arguments could be forthcoming. Indeed, there are two avenues the anti-realist may wish to pursue. The first concerns Tennant's initial point that multiple-conclusion sequents may only be interpreted disjunctively. If this is correct, multiple-conclusion sequents presuppose an understanding of disjunction. A system containing ineliminable occurrences of sequents of this form could therefore not be taken to be constitutive of the meanings of all the logical constants; it would fail to qualify as an adequate proof-theoretic framework. This point has been made by Dummett (1991, p. 187). However, it is not obvious that sequents with two or more members in their succedents *must* be interpreted disjunctively. For instance, an intuitively compelling reading of multiple-succedent sequents is available to those willing to countenance a notion of denial alongside that of assertion.[8] The second, more promising, more philosophically fruitful, but also more laborious route for the anti-realist to take would be that of producing a detailed independent argument for the primacy of natural deduction systems. This, however, is again not a trivial matter — that natural deduction is natural is not a pleonasm.[9]

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[8] The notion of denial as a distinct form of linguistic force has been discussed for independent reasons by Timothy Smiley (1996) and by Ian Rumfitt (2000) among others.

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