

## AN EPISTEMIC CONSTRUCTIVE DEFINITION OF INFORMATION\*

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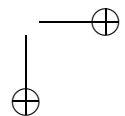
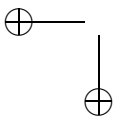
### *Abstract*

The present paper formulates an Epistemic Constructive Definition of Information (ECDI), based on the rejection of the alethic nature ascribed to declarative objective semantic information (DOS). ECDI reformulates the principles holding for the non-alethic Standard Definition of Information (SDI), it is based on the Verificationist Principle of Truth and it refers to the typical constructive distinction between judgemental act and propositional content. The resulting framework defines two distinct and complementary epistemic acts: justifications and conditions of knowledge. Finally, the concept of information is formally presented within the syntactic-semantic method of Constructive Type Theory (CTT).

### 1. *A constructive Approach for the Philosophy of Information*

The currently accepted concept of “semantic information”, notoriously introduced by Bar-Hillel and Carnap in [7], is the result of different formulations and frameworks, collected in the emergent field of the so-called Philosophy of Information. The general intuitive notion, referred to as *declarative objective semantic* information (DOS), is defined by alethic values (see e.g. [14]). This characterization becomes stronger if the definition is explicitly based on truth-values, as it is done by Barwise and Seligman in [8]. Later, such a definition has been restricted in terms of the *standard definition of information* (SDI), which gives up with alethic values (see [19], [11], [14]). The definition of information as well-formed meaningful data maintained by SDI is notoriously based on the following principles:

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- *Typological Neutrality*: information cannot be dataless, and everything can be a datum;
- *Taxonomical Neutrality*: a datum is a relational entity, so is information;
- *Ontological Neutrality*: data implementing information are physical;
- *Genetical Neutrality*: data (and therefore information) can have a semantics independently of any informer;
- *Alethic Neutrality*: meaningful and well-formed data qualify as information, no matter whether they represent or convey a truth or a falsehood or have no alethic value at all.

These principles produce various regrettable consequences and a number of alternative approaches have been considered. In particular, critiques apply to the principle of Alethic Neutrality. Among the alternatives, Floridi in [14] and [16] rejects the thesis of the supervenience of truth-values on SDI, suggesting that information should encapsulate "truthfulness": as a result, "true information" is a redundant expression and "false information" a misleading one.

The *Epistemic Constructive Definition of Information* (ECDI) here introduced interprets the notion of information from an epistemic perspective. The resulting interpretation in respect to truth is different from those already mentioned. In the first section a number of theories are surveyed: they represent the background of the here proposed interpretation both from an epistemic and a logical viewpoint. In the next section, information is defined as the content of a peculiar agent's epistemic state, connected with — but distinct from — the constructive notion of knowledge. On the basis of this definition, the standard principles of SDI are revised according to a constructive interpretation. In the end, the general framework of Constructive Type Theory is introduced, in order to provide a formal structure in which the interpretation of epistemic information can be formulated.

## 2. The Agent-based Approaches: the background of ECDI

The Epistemic Constructive Definition of Information (from now on ECDI for short) interprets the concept of information in the frame of an agent-based approach. A number of theories can be accounted as its background. A first definition of epistemic information based on agents' knowledge states is contained in [23]: it defines information as a datum perceived by an agent, leading to a modification of her knowledge state. The notion of informativity as an intrinsic property of the datum is rejected, replaced by a subjective notion related to the agent's knowledge state. In the same spirit, the constructive

frame at the basis of ECDI underlines the subjective and context-dependent nature of informational contents.

A different agent-based approach is presented in [5]: it provides a formal interpretation of information as the property of "being informative". The objectual notion is replaced by the operation of handling information-like objects; the classical framework of the semantic theory is abandoned in place of arguments in favor of logical pluralism and consequently the formal approach is inspired by non-classical logics.

Following the rejection of a classical framework for a logical notion of information, a constructive approach has been already developed in [35], where information is the foundation for decision-making procedures. In that context, it is argued that neither preference nor subjective probability can be assumed as given: those properties need to be generated instead by the relevant information available to the agent in a given situation. According to such a constructive approach, the meaning of information relies essentially on the value ascribed to rational procedures, i.e. in terms of the way agents understand judgeable contents.

Recently, a formulation of a constructive logic for the flow of information has been provided in [4]: it presents a formal language, based on Nelson's constructive logic with strong negation, defining an *infon* as a discrete item of information, and a *situation* as some part of a world built up by infons. Such a logic is characterized by the basic constraints on infon logic, namely partiality of information contents, persistence and situated reasoning. In the constructive version of infon logic, infons are interpreted as proofs or disproofs in a constructive setting, and a situation can be interpreted as a set of constructively formed infons (a piece of information).

The mentioned approaches provide an interpretation of informational processes in terms of agents and contexts (situations). In [15] and [6] the same kind of perspective is assumed. In the first approach, the problem of defining an information logic is interpreted in terms of a comparison with epistemic and doxastic logics: it translates their axiom schemata in a logic of "being informed". In the second approach, a similar problem is addressed: the Adaptive Logics' framework is used to interpret the notion of "being locally or partially informed". In both cases, the core problem is represented by the reformulation of the standard notions of the Philosophy of Information, in a situated and context-dependent frame.

Along the lines of these alternative frameworks, ECDI rejects classical logic as the basis of the semantic frame. It insists on interpreting information as an epistemic state, reflecting the notion of "being-informed". The epistemic definition is given in terms of a set of conditions on whose basis knowledge is formulated by a situated agent and the notion introduced is therefore clarified as a component of an epistemic process. Unlike other constructive or agent-based interpretations, ECDI provides a reformulation

of the principles holding for SDI, which makes this definition general and apt to describe various of the previously mentioned agent-based approaches. The related results can be summarized as follows:

- by restricting the principle of Typological Neutrality (TyN.) one avoids the identity of values for different, conflicting informational contents;
- by reformulating the principle of Genetical Neutrality (GN.), the ontology of data is made dependent on the role of the epistemic source;
- by reconsidering the principle of Taxonomical Neutrality (TaxN.) the evaluation of informational contents requires the essential role of the receiver;
- finally, the constructive formulation of the principle of Alethic Neutrality (AN.) allows to reconsider the problem of truthfulness for informational data.

Like in the constructive interpretation of infon logic from [4], the present framework relies explicitly on the relation between proofs and contents; unlike other approaches, it defines informational contents as distinct from (linguistic) descriptions of given, actual situations in the world. The final aim is to provide a context-dependent definition of information, based on the agent's conditions for formulating and accepting contents of possible knowledge. This implies that the condition on persistency is explicitly abandoned and a procedure of revision on knowledge contents from a constructive perspective is made possible<sup>1</sup>.

ECDI is formalized in the framework of Constructive Type Theory (CTT). In this formal system the notion of information has been used already at least with two different meanings:

1. for the distinction between the polymorphic and the monomorphic versions of type theories: whenever one considers the different types (e.g. the type of sets and the type of propositions) and their different properties and rules, one is referring to the polymorphic version of the theory. By "forgetting" the *information* related to the kind of type one is working with, one formulates a more general interpretation which, under the Curry-Howard isomorphism (see [10], [18]) is

<sup>1</sup> For the revision procedure of a constructive-based notion of belief, see [27].

equally valid for different types (monomorphic version). This process is at hand in the case of the transition from different programming languages to the general formulation of an extensional set theory, and it provides the relevant understanding of the notion of abstraction for type theories (see [28]). It is usually referred to as the "forgetting-restore principle"([33], [34]);

2. for typed computer systems, the operation of specification of data can be identified with the formulation of a type, and a program satisfying the specification is an element of that type; the *information* saying that a program satisfies a certain specification is not relevant to the program, i.e. it is not computationally significant. This information can be discharged if it can be retrieved when needed; this is a specific formulation of the previous case, which is done via some formal syntactic operator (see e.g. [32]).

The interpretation introduced in this paper is a general formulation of these constructive principles. Nonetheless, it is more appropriate and explicitly formulated for describing knowledge processes, because it formalizes a set of operations on informational contents for epistemic agents, rather than purely formal instructions. In the following, the structure of CTT will be used to show how a general interpretation of the epistemic notion of information holds for typed systems and constructive logics. In this way, two goals are achieved: first, a formally interpreted notion of information is provided for the constructive epistemology; second, a unique meaning is given to such a notion within the formal structure of constructive logics, providing a satisfactory explanation of both its syntactical and semantical aspects.

### 3. *What is Information and where it comes from?*

Everything is "a piece of information", as long as it conveys some kind of meaning: my father's opinion on the rise of violence in modern societies, (as much as) a textbook of sociology or politics treating the same topic; the forecasts on television, (as much as) my grandmother's pain in the bones saying the weather is changing; a prime minister's discourse in Parliament illustrating the reasons of the last war, (as much as) the pacifists demonstrating against the same war, obviously with different reasons. Each of these situations (and the statements which can be formulated to describe or refer to them) consists in a certain amount of data, producing information. The principle of Typological Neutrality holding for the semantic theory of information says essentially that *any possible kind of data is information*. On the other hand, a unique interpretation is given in terms of *conceptual contents*

which are linguistically expressible, referred to as "declarative objective semantic content"<sup>2</sup>.

Hence, information is conveyed by propositional contents. According to the constructive epistemology, propositions are understood as contents of judging acts. I will start from this basic distinction between propositional contents and judgmental acts to show how it applies to the mentioned principle of Typological Neutrality. The consequences of such an interpretation suggest a major change in the common understanding of the semantic nature of information.

### 3.1. Which knowledge, which information

Knowledge processes can be described in a constructive model according to the basic distinction between

- knowledge states;
- informational states.

I will start by considering the standard concept of knowledge holding within the constructive epistemology and thus defining the former of these states. The notion of informational state represents a major innovation for the constructive epistemology, and ECDI is designed to describe the nature of such states<sup>3</sup>.

The notion of constructive knowledge is essentially based on the definition of truth. This definition is given in terms of proofs by the Verificationist Principle of Truth (VPT):

*Principle 1: (Verificationist Principle of Truth) Knowledge of the Truth amounts to the Existence of a Proof (of the propositional content whose truth is asserted in a judgment).*

The notion of knowledge is defined in the constructive frame as an epistemic state towards a certain propositional content, expressed by a judgemental act and produced by the possession of a justification (proof object, instance). Therefore, to know something amounts to being in a certain epistemic state, in terms of the verification of some propositional content.

<sup>2</sup>Notoriously, the translation into linguistic contents of a situation is represented in the following terms: "a's being of the type F, conveys the information that b is G".

<sup>3</sup>For more on the formulation and use of informational states within constructive epistemology and the connected philosophical issues, see [29].

This explanation of the first element in our epistemic structure allows for a further comparison with epistemic logics, namely with models of (multi-agent) knowledge processes. These logics have notoriously their origin in the works of von Wright ([36]) and, after the development of the semantics of modalities by Kripke, have been studied by Hintikka ([17]), aiming at formalizing expressions like "agent  $i$  knows that  $x$ " and "agents  $j$  believes that  $y$ " by means of the doxastic operators  $K$  and  $B$ , and a set of epistemic axioms. Usually, the basic systems with a primitive K-axiom interpreted by the schema

$$K_i(\phi \rightarrow \psi) \rightarrow (K_i\phi \rightarrow K_i\psi)$$

are extended to  $S4$  by the schema (positive introspection)

$$K_i\phi \rightarrow K_iK_i\phi$$

and to  $S5$  by the schema (negative introspection)

$$\neg K_i\phi \rightarrow K_i\neg K_i\phi.$$

An important step in the evolution of epistemic logics has been represented by the interpretation of the system  $S5$  in terms of states of a process in a distributed system<sup>4</sup>. It is obvious to try a comparison between these extremely powerful formalizations for knowledge processes and the constructive model. In the following I will make some remarks on the nature of the semantic structure involved, the relations of accessibility and the role of modalities.

In the system described by Fagin et al. in [13], a formal semantic model for knowledge is presented, based on Kripke-semantics. The basic intuition is that uncertainty or incomplete knowledge can be translated as possible worlds candidates for the agent, and knowledge of a fact consists in validity of that fact at all possible worlds. This establishes a distinction between the actual state of affairs of an agent (given her current information) and the alternatives to it: the latter is represented as a binary relation among actual and possible states, capturing a possibility relation for the agent. According to this model, an agent considers possible something in her world if that requires the same information content of her actual world. This standard semantics is completed by a definition of truth for connectives, which uses the mentioned possibility (accessibility) relations among worlds for the valuation of epistemic formulas. The important variant on this theme introduced

<sup>4</sup> See for example [13].

in [13] is represented by the Aumann structures, in which the focus is on the notion of event, corresponding to a set of states. These structures do not have truth assignment functions, and instead of possibility relations they use a partition in the information set of an agent. By means of the Aumann structures, knowledge is therefore defined in terms of events; agents are accounted in a local state, an agent knowing an event at a certain state meaning that the event holds at every state the agent considers possible on the partition of that state. The distributed system of knowledge for multiple agents is a global state produced by the local ones: the behavior of the system for possible executions produces its global state, eventually ordered by a time-indexing function.

Some comparisons might be drawn now with respect to a constructive knowledge system. The modal semantics for the intuitionistic propositional logic has been formulated since the seminal work of Prawitz [26]. The related interpretation of constructive Kripke models has been explored to a great extent. In the standard interpretation this is done by relating modalities to quantifiers on the accessibility relations (see e.g. [12], [25]); the type-theoretical operations on Kripke structures are considered — more extensively than here — in [31] and [29]. In the following I will briefly reconsider some basic intuitions of Kripke structures in the constructive interpretation, essentially in terms of the basic relation between justification and truth. An accessibility relation  $Rww'$  between the actual and some possible world can be defined in terms of the agent's state as the formulation of a construction for a knowledge content in the actual world  $c : C \in w$  based on the condition of another construction in a possible world [ $c' : C' \in w'$ ]. The kind of condition here considered reverses the standard relation between actual and possible worlds: the evaluation of these formulas might be expressed as saying that a certain proposition  $c : C$  is true (which is a judgmental act) if and only if some other propositional content  $c' : C'$  is true, for all those cases in which a conditional relation holds between these two contents, i.e.  $c : C[c' : C']$ . In other words, under condition of the possessed information that the content expressed by  $c' : C'$  is true, the knowledge of  $c : C$  is stated. I shall show that this conditional relation is the peculiar constructive understanding of actuality and possibility<sup>5</sup>. In turn, this is translated by extending the standard notion of *justification of knowledge* with the (conceptually prior) notion of *condition for knowledge*. In the case of Constructive Type Theory (CTT), this is obtained in the formalism by means of the so-called syntactic/semantic method of expression, which provides the interpretation of information via the introduction of the conditions of knowledge.

<sup>5</sup> See [21].



The relation between potentiality and actuality is also central in interpreting the mentioned modification of modal structures by means of temporal operators. A temporal relation among states is usually interpreted as a partial order on the accessibility relations from one state to another. In the language of CTT no temporal operator is introduced, and no other modality is defined as an operator. The ability to express ordered knowledge states is preserved by the conceptual distinction between potential and actual knowledge: the latter is proper tensed and dependent knowledge defined in terms of proofs; the former might be considered as independent and tenseless content, but it relies only on the satisfaction of basic conditions. Moreover, this distinction provides a solution to the well-known logical omniscience problem (LOP), which occurs in many epistemic systems. Notoriously, the LOP expresses the counterintuitive implication according to which an agent knows all the consequences of her knowledge: given any normal modal logic  $M$  containing the operator  $K_i$  to formalize the expression "the agent  $i$  knows that", for any formula  $\phi$  in the language of  $M$ , if  $K_i\phi$  and  $\phi \vdash_M \psi$ , then  $K_i\phi \vdash_M K_i\psi$ . This strong formulation is modified in terms of different weaker ones<sup>6</sup>:

- Knowledge of valid formulae: agent  $i$  knows all logical truths (Necessitation Rule);
- Closure under logical implication: if agent  $i$  knows  $\phi$  and if  $\phi$  logically implies  $\psi$  (i.e.,  $\phi \rightarrow \psi$  is valid), then agent  $i$  knows  $\psi$  (Monotonicity Rule),
- Closure under logical equivalence: if agent  $i$  knows  $\phi$  and if  $\phi$  and  $\psi$  are logically equivalent (i.e.,  $\phi \leftrightarrow \psi$  is valid), then agent  $i$  knows  $\psi$  (Congruence Rule);
- Closure under material implication: if agent  $i$  knows  $\phi$  and if agent  $i$  knows  $\phi \rightarrow \psi$  then agent  $i$  knows  $\psi$  (axiom K);
- Closure under conjunction: if agent  $i$  knows  $\phi$  and if agent  $i$  knows  $\psi$  then agent  $i$  knows  $\phi \wedge \psi$  (axiom C).

In CTT, the distinction between actual and potential knowledge clarifies the LOP. An agent's (proper) knowledge consists in everything for which a proper construction is provided; potential knowledge is the content for which (proper) conditions have been stated. For the latter, the agent cannot state to have knowledge, rather only that its content suffices to provide conditions on the construction for some other content. Therefore, in the constructive epistemology the introduction of informational states can be illustrated as a method of providing accessibility on points of a structure (model): this

<sup>6</sup>Cf. [13], p.311.

is epistemically different than defining the accessibility itself, which is intended as accessibility by proving. The notion of information introduced is not accounted simply as the content of any possible knowledge act (i.e. of any accessibility relation on points), rather as a peculiar form of extending knowledge states, namely by expressing conditions of accessibility on further points in the structure.

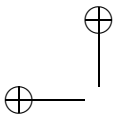
One of the most relevant epistemic and philosophical distinction for the constructive frame concerns the role of the agent and the awareness of her knowledge state, established in standard epistemic logics via the mentioned axioms for introspection in  $S4$  and  $S5$ . The constructive frame (and in particular the formalization of CTT) does not consist in a *description* of knowledge processes, i.e. it does not use a *third-person perspective* to formalize the relation between an agent and the content of knowledge. Knowledge is expressed explicitly by judging acts whose propositional contents are claimed to be known true by the agent. In this way, the content of knowledge is not independent from the agent's act of getting it known, and it is not something which possibly remains hidden to the agent performing those acts. The constructive epistemology provides thus a *first-person perspective*: every content of knowledge has a corresponding act of getting that content known, and therefore someone claiming to know it by verifying its content. In this sense, the operator  $K$  of epistemic logics is a way to provide a pure, independent from conditions, subjectless content of knowledge; on the other hand, the axioms introduced by  $S4$  and  $S5$  restore the necessity of an aware subject for the epistemic acts. The epistemic system of CTT, based on the distinction between act of judging and propositional content of knowledge, relies explicitly on the presence of the knower, and therefore any content of judgment is subject-dependent. This means that there is no need to restore the awareness of the subject by axiomatic means<sup>7</sup>.

Both the epistemic notions used in the constructive epistemic model are given in terms of basic grounds for assertion: the notion of knowledge state requires the explicit use of constructions (proofs) for the propositional contents whose truth is asserted; informational states require instead the formulation of assertion conditions.

Another obvious last comparison is due, namely one to the so-called epistemic logic with justification<sup>8</sup>. Such a logic, along with the usual knowledge operator, introduces explicitly justifications for contents of knowledge. The

<sup>7</sup>The distinction between first- and third-person perspective in relation to the debate between logical realism and anti-realism has been introduced by Göran Sundholm; I am indebted to him for its formulation in the present context.

<sup>8</sup>This kind of epistemic logic has been introduced since the early 1990, by Artemov e.g. in [1], [2], [3].



aim of this logic, namely to express explicitly and in a formal way justifications for assertions, is based on the following assumptions:

1. each axiom has a justification;
2. justifications are checkable;
3. the assertion of a justification for a statement implies knowledge of this statement (which in turn satisfies  $S4$ );
4. any justification is compatible with any other justification.

It is not difficult to see that each of these assumptions is simply and naturally interpreted in a system of type-theoretical constructive logic:

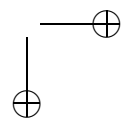
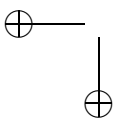
- 1c. axioms are immediate known judgments (known by themselves);
- 2c. the construction of a proof for a propositional content is analytically contained in its type, and therefore can be checked;
- 3c. according to the first-person perspective principle, the formulation of a judgment with the related construction implies the existence of the subject aware of the judged content;
- 4c. the analytic development of constructions is monotonic (if not, either an error can be found, or the extension is synthetic<sup>9</sup>).

The aim behind the constructive epistemology and the epistemic logic with justification is similar, namely to formalize a system of knowledge explicitly based on justifications. The main difference between the two approaches can be identified in the order of priority between formalization and epistemic description: whereas the epistemic logic with justification extends standard logics by making proofs explicit, the constructive formalism provides primarily the use of justifications, on whose basis the conceptual system is grounded.

### 3.2. *Introducing Informational States*

In the present section, the notion of information is defined for the constructive epistemology. On the basis of an account of justifications for propositional contents, informational states are formulated as a new epistemic notion expressing assertion conditions for the judgeable contents. The introduction of the notion of information within our constructive knowledge system corresponds to the formulation of a certain epistemic state:

<sup>9</sup>For this result, see [27].



*Definition 1: (Informational State)* An agent's informational state  $S_{inf}$  is an interconnected collection of propositional contents that

1. are accepted on the basis of their meaningfulness;
2. are accessible from previously acquired information;
3. can be syntactically and semantically extended.

The epistemic role of such an informational state is to formulate the conditions on which it is possible for the agent to acquire proper knowledge.

The idea behind this definition is that a content of knowledge expressed in a judgment  $J$  consists in the assertion of the truth of a content  $C$  under justification  $c$ <sup>10</sup>:

$$J = c : C$$

Such a justification  $c$  is only partially satisfied by a proper analytic construction, i.e. by a proof object  $p_o$ . Part of the content of the construction  $c$  shall be expressed by the *assertion conditions* for that construction, here contained in  $S_{inf}$ :

$$J = c\{p_o[S_{inf}]\} : C$$

To be in a certain knowledge state means therefore for an agent to formulate a set of informational statements, on whose basis the former is obtained. The role of these statements is to establish that contents of knowledge are meaningful and their basic conditions are satisfied. The *Epistemic Constructive Definition of Information* (ECDI) is based on the conceptual difference between the notions of knowledge and information, the latter being formulated as the meaningful content presupposed and required by an assertion of the former.

In order to specify which kind of data satisfies the role and the properties stated in the Definition 1, I will consider in the following the expressions formulating conditions for knowledge<sup>11</sup>:

1. *presuppositions* for meaningful predications. These expressions provide the agent with new *meaningful concepts*, which can be asserted

<sup>10</sup> By the *set-as-props-as-types* holding in CTT under the Curry-Howard isomorphism, the content  $C$  can be understood as a propositional content with  $c$  a proof, or as a set (abstracted property, concept) with  $c$  intended as an instance.

<sup>11</sup> See also [22].

to be known (declared true) in terms of their instances (according to the VPT). Using CTT as the formal representation of knowledge processes, these expressions are formulated as *type-declarations*:

$$\begin{aligned} &< \alpha : type > \\ &a : \alpha \end{aligned}$$

i.e. such an expression declares a certain object  $\alpha$  to be a type ( $< \alpha : type >$ ). Out of the formalism, types amount to categories of predication, i.e. concepts: the assertion that a concept is meaningful is necessary for any real predication using that concept. Correspondingly, type declarations are necessary *presuppositions* for every possible (analytical) judgment using that type ( $a : \alpha$ ), i.e. they allow possible (right or wrong) predications within that concept<sup>12</sup>. Under the identity of propositions with the type of their proofs ( $prop : type$ ), one can state the truth of a certain propositional content (“*proposition A is true*”) by showing a proof (construction) for it ( $a : A$ ), this holding under the basic presupposition  $A : prop$ ;

2. *assumptions* for some knowledge content. These expressions represent the conditions formulated in hypothetical judgments; their value is alethic in order that a knowledge assertion may occur. In the chosen formal language, a dependent judgment is represented in the form “object  $a$  is of the type  $\alpha$ ” holding provided that certain other judgments are true, respectively “object  $x_1$  is of the type  $\alpha_1$ ” up to “object  $x_n$  is of the type  $\alpha_n$ ”:

$$\frac{[x_1 : \alpha_1, \dots, x_n : \alpha_n]}{a : \alpha}.$$

In this second case, one considers the alethic role of assumptions. This exemplifies the difference with the standard notion of presupposition, reformulated for CTT by the previous point. In the case of assumptions, one refers to some propositional content which is assumed to be true in order for something else to be known. Under the mentioned definition of propositions as types, knowledge of the truth of the proposition  $A$  is obtained by providing a proof  $a : A$ , under the assumed truth of certain other propositional contents: “if  $A_1, \dots, A_n$  are true, then  $A$  is true (by  $a : A$ )”. Also in this case, the

<sup>12</sup> See [9] for the usual epistemic notion of presupposition as something that must be known, in order for something else to be known; for the relation to dynamic logics, see [24].

presuppositions  $A_1 : type, \dots, A_n : type$  need to be formulated in order for the content of the assumptions to be meaningful.

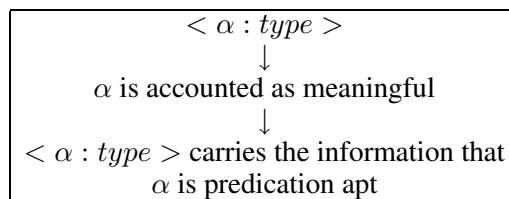
### 3.3. On Knowledge and its Conditions

According to the description provided, when referring to the type-theoretical constructive notion of knowledge, one is considering judgments providing analytic constructions within types, i.e. proper predications:

the judgment “ $A$  is true” is justified by the assertion of a construction for the proposition  $A$ , i.e. the judgment  $a : A$ .

The notion of knowledge, based on the VPT, links justification and truth: any known propositional content whose truth is correctly asserted is justified by a proper proof-object. Consequently, provability establishes the truth of knowledge contents. In this context, one has to preserve the possibility that the assertion of truth of a certain propositional content might be wrongly made. This is assured by the notion of predication aptness. To describe something as apt to be (rightly or wrongly) predicated leads directly to the description of the conditions under which such a predication is performed. Therefore, knowledge is based both on the notions of justification and condition: the former represents the acceptability criterion for the content of knowledge, defined in terms of application (presence of a construction or instantiation) and identity (equality of constructions); the latter determines the notion of constructive information here introduced. Let us reconsider here these conditions intended as declarations of meaningfulness and hypotheses:

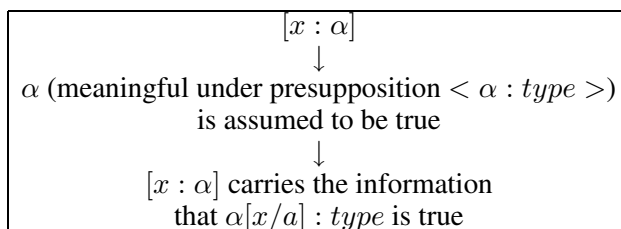
- meaningfulness by type-declarations represents a criterion of acceptability, anticipating the determination of proper meaning provided by the formulation of analytic judgments (according to the meaning-is-use slogan, holding for the intuitionistic-constructive frame):



Hence, meaningfulness can be defined as the ability of determining a certain concept as endowed with meaning. This can be done e.g. in terms of a nominal definition, providing a collection of defining properties for the concept at hand. On the other hand, the ability of

judging a proper instance for that concept relies on the analytic process of construction, which expresses the agent's proper knowledge state;

- assumptions of truth provide the background information under which some propositional content become judgeable:



The information conveyed in this case includes the meaningfulness of the concepts involved, thus restating this latter notion as the core of the constructive notion of information.

In the next section, the principles holding for these expressions building informational states shall be considered.

#### 4. Principles for ECDI

The main conceptual basis for the constructive definition of information is represented by the reformulation of the principle of Alethic Neutrality in terms of meaningfulness, defined as a condition for truth. The principle of Typological Neutrality is to be reconsidered according to the expressions defining informational states, introduced in the previous section. Finally, by stressing the idea of the first-person perspective, the principle of Genetical Neutrality turns to be pivotal for the epistemic perspective.

##### 4.1. Restricting TyN.

In the examples mentioned at the beginning of the previous section, informational contents are produced by different sources. According to those examples, it seems reasonable and intuitive to model informational contents as producing different epistemic states in different receivers, which means that information is user-dependent. Hence, not every content is (equally) "informative" for an agent: rather, everything is a judgeable content (for any agent) provided its basic conditions of assertability (meaningfulness) and provability (constructions) are met. This underlines the basic epistemic difference between *known contents* and *informational contents*, the latter being

formulated to provide useful and necessary conditions in order some knowledge act to be performed. Following these lines, a restricted version of the principle of Typological Neutrality can be reformulated as follows:

*Principle 2: (TyN. Constructive Version) The conceptual and linguistic data  $D$  contained in an agent's informational state  $S_{inf}$  introduce meaningful concepts and provide conditions for the knowability of some judgments  $J_1, \dots, J_n$ .*

This restricted version of TyN. maintains the essential feature of the standard definition of information: information is built by data conveying meanings. The relation of conceptual priority between type-declarations, assumptions and analytic judgments recalls in a quite intuitive manner the process of introducing meaningful data allowing one to formulate proper judgments.

#### 4.2. Reformulating GN.

According to the principle of Genetical Neutrality for SDI, the semantics of informational data is independent from their source, which in turn amounts to saying that any source is entitled to produce information. This implies that contrasting and even contradicting sources provide equally valuable information from the semantic point of view. Obviously, this is not true from an agent-based perspective. Not only in the standard account one has to accept the counterintuitive principle according to which every kind of source can be informative, moreover one is bound to accept infamous consequences, such as the Bar-Hillel-Carnap Paradox, saying that "*a self-contradictory sentence, hence one which no ideal receiver would accept, is regarded as carrying with it the most inclusive information*"<sup>13</sup>. For the same reason, tautologies are not informative at all. Therefore, SDI considers the alethic determination of informational data irrelevant, and on the basis of other counterintuitive consequences (e.g. that by accumulating any contingent proposition one always makes the informational content larger, even if these propositions are mutually inconsistent) the notion of information has been remodelled. In [14] truthfulness is the defining property of informational contents and the related argument for the so-called *Veridicality Thesis* in [16] says: *p* counts as information if and only if *p* is (constituted by) *meaningful, well-formed and veridical data*.

On the basis of the constructive version of TyN., the nature of propositional contents admitted as information is of a different kind, and their connection to truth is unambiguous. By means of informational states, the agent

<sup>13</sup> Cf. [7], p.229.



expresses the meaningfulness attributed to conceptual terms and the assumptions of truth needed to formulate the related knowledge contents. It seems reasonable that everyone is entitled to consider some information more reliable than other information: I might consider my father's opinions more realistic than a textbook, and I would probably hold my grandmother's assertions on tomorrow's weather as trustworthy as the forecasts (because I experienced she never fails!), whereas someone else will consider these very same sources unreliable. On the other hand, an agent pondering the (eventually contradicting) contents of two different sources of information, would not account both of them as equally informative (assuming that both sources provide equal novelty in respect to the previous state of knowledge of that agent, e.g. because the agent has no previously formed opinion on the matter at hand).

The epistemic approach maintains the property of informational sources to produce possibly different or even contradicting contents. It provides a model in which sources are different according to their informational value and can be even non-informative to agents. An agent recognizes declarative contents as informative *dependently* from the source they come from, and those contents are evaluated in connection to the receiver they are provided for (i.e. the agents themselves). This also means that contents truthfully provided by any possible source and plainly accepted by any possible agent qualify no longer as information in the epistemic sense, rather they are described as proper knowable contents. These contents have to be justified in such a way that everyone is entitled to formulate and accept them. This interpretation has a nice consequence: tautologies, being simply true for any agent, have no informational content, rather they provide knowledge in terms of analytic contents; on the other hand, contradictions are not informative in such that they are recognized as false in any possible interpretation, simply because they fail to be meaningful (basic condition for informativeness).

Under this view, GN. has a completely different meaning. The semantics of information is no longer independent from any informer (which it cannot be, because of the mentioned first-person perspective), and the principle is reformulated as follows:

*Principle 3: (GN. Constructive Version) The set of informational data  $D$  of an agent's informational state  $S_{inf}$  is determined dependently on a certain informer, but it is not relevant the nature of the informer; any source is entitled to produce information, but the informational content produced is evaluated in connection to that source.*

Information is such because there is a source producing it and this source can be different from other possible sources. To make any sense of this assertion, there must be an epistemic counterpart, i.e. something whose value

is actually *independent from any possible source* producing it: the epistemic notion of knowledge (under VPT) plays this role. This leads directly to the clarification of the mentioned relation between source and receiver and, in turn, to the reformulation of the principle of Alethic Neutrality.

#### 4.3. *TaxN. and the role of the receiver*

In its original formulation, the principle of Taxonomical Neutrality says simply that information is a relational entity. This property is basically given by the nature of data conveying meanings about something being something else. According to this basic principle, information cannot exist in a disembodied form but at the same time it is not reducible to matter. Data are the information's substrate, as such they are always received in relation to other data of analogous origin. Clearly, according to TaxN. information establishes a relation between incoming data and the receiver, intended as the interpreter of those material data: this is also the characteristic the epistemic approach will focus on. By means of the constructive version of GN, it has been expressed the principle according to which the role and the nature of the source is essential for determining a certain content as "information": correspondingly, only by considering the role of the receiver of that content one is properly entitled to speak of information.

The role of the receiver introduced by the constructive version of TaxN. is established as determining that data are (or are not) information in terms of the operations the agent performs on them. In other words, the relational nature of informational data consists in their being declarations of meaning and conditional expressions in relation to other data the agent states as knowledge. The constructive reformulation of TaxN. sounds therefore as follows:

*Principle 4: (TaxN.- Constructive Version) A set of data D is determined as informational data by any epistemic agent according to the way those contents are accepted and used in relation the knowledge state formulated on their basis.*

The independence of the informational value of data from the user is completely rejected, there is no "pure information" flowing independently from the agent predisposed to receive and to interpret it. Information is always produced for a certain type or category of agents which are able to treat and convey those contents to other agents. The epistemic value of informational contents is therefore determined by the epistemic status the agent ascribes to those data, in connection to the source from which the agent chooses to accept informations. In this sense, a content recognized as proper knowledge

by a certain user (i.e. a content for which she is able to provide a justification in terms of proofs) might be informational content for another user, who is not able to provide justifications for it, nonetheless is willing to consider them meaningful and true. This interpretation underlines once more the proper nature of the epistemic definition of information: if informational contents flow from a specific source to a specific agent accepting it, it is unlikely that such contents can always be produced by any possible source and accepted by any possible receiver.

#### 4.4. *Not true nor false: reformulating AN.*

In order to solve the inconsistency holding between probability, truth and the degree of informativity revealed by the BHC-Paradox, the SDI has detached the notion of information from its relation to truth. In the present context, this leads essentially to a reformulation of the value of informational contents in terms of meaningfulness, whereas truth values (truth and falsity) are intended as properties strictly holding only for contents of knowledge acts. Hence, Alethic Neutrality needs to be reformulated in order to express the relation between the ontological nature of informational contents (as expressed according to the restricted version of TyN.) and their truthfulness.

In the formulation of assumptions for dependent judgments, truth is assumed to hold for informational contents: a content for which truth is only assumed (but not explicitly justified) qualifies as informative to an epistemic agent. The condition for assuming the truth of some propositional content is the meaningfulness of the concepts conveyed: whenever there is an assumption of truth, there is also in the first instance a presupposition of meaningfulness. Hence, provided that only judgmental knowledge conveys truths or falsehoods in the proper sense, its epistemic counterpart is defined as information in terms of the meaningfulness of data. Once the notion of judgmental knowledge is defined (by VPT), presuppositions and assumptions amount to what is needed to be known in order the former to be stated: this explanation allows us to refer to *assertion conditions* as the basic description of informational data. The principle of Alethic Neutrality can be reformulated as follows:

*Principle 5: (AN.- Constructive Version) The set of data  $D$  in an agent's informational state  $S_{inf}$  is identified in terms of meaningfulness: this represents the basic property needed to formulate the assertion conditions for some judgment  $J$ , independently of the stated truth or falsity of such a judgment.*

The constructive version of AN. states that the alethic value (of both the informational data  $D$  and of the judgmental knowledge  $J$  formulated on the

basis of  $S_{inf}$ ) is irrelevant to the definition of ECDI. Information is an epistemic concept expressing conditions for knowledge to be acquired, without restrictions imposed by alethic values. In other words, these conditions are formulated by the agent in a way which might be completely independent from the actual alethic value of the contents at hand. The basic and necessary epistemic condition is formulated as the property of meaningfulness of data, to be interpreted in terms of a descriptive collection of defining properties of the concepts involved. To establish meaningfulness allows in the first instance the informal explanation of the concepts introduced, and consequently the formulation of the mentioned application and identity criteria, defining proper meaning.

### 5. *The formal representation of informational dynamics*

The structure of CTT lets us formalize in a clear and intuitive way the relations between informational and knowledge states, in terms of analytic judgments, conditions for hypothetical judgments and type-declarations. This general interpretation of the formal expressions for CTT is completed by analyzing the possible operations to be performed on an agent’s informational states. In the following, I will present the four basic cases of formal operations, and I will explain how information increases and decreases both syntactically and semantically according to them.

Let us start by describing the basic case of an agent stating a certain knowledge content on the basis of the relevant informational contents. The informational state expresses the meaningfulness of the concepts involved in the process and the conditions under which such knowledge can actually be asserted. This is simply formalized by the following dependent judgment:

$$\frac{\langle \alpha : type \rangle}{\frac{[x_1 : \alpha, \dots, x_n : \alpha]}{a : \alpha}}$$

By this formula one expresses that the knowledge-state represented by the judgemental content  $a : \alpha$  — to be  $\alpha$  a concept verified by a proper instance  $a$  — holds under the assumptions contained in the related context (within square brackets); the informational-state  $S_{inf}$  containing the context of assumptions is completed by the set of presuppositions introducing the meaningful terms in that context, in this case for the unique concept  $\alpha$  (within

angled brackets). Let me now consider the operations on the informational state and their possible meanings<sup>14</sup>.

- a. The first operation is the extension of the context by a new assumption which uses a concept already present in the agent's  $S_{inf}$ :

$$\langle \alpha : type \rangle \\ [x_1 : \alpha, \dots, x_n : \alpha] \leftarrow x_{n+1} : \alpha.$$

Under our interpretation, the informational content of this epistemic state does not increase relatively to meaningful data relevant to the knowledge process, rather only syntactically.

- b. The second case extends the first one; the previous situation is now modified in terms of a larger set of presuppositions (meaningful data) and the substitution of specific informational data by proper values:

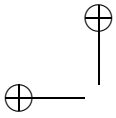
$$\langle \alpha_1 : type, \dots, \alpha_n : type \rangle \\ [x_1 : \alpha_1, \dots, x_n : \alpha_n] \leftarrow x_n = a : \alpha_n.$$

In this case, information decreases relatively to the amount of syntactic data relevant to the determination of meaningful concepts. It follows that the agent's knowledge state grows by determining part of the informational content (i.e. something is now *known* and it no longer needs to be assumed).

- c. The third case develops both the previous situations. The starting  $S_{inf}$  is extended by (a set of) assumption(s) based on related presupposition(s):

$$\langle \alpha_1 : type, \dots, \alpha_n : type \rangle \leftarrow \beta : type \\ [x_1 : \alpha_1, \dots, x_n : \alpha_n] \leftarrow x : \beta.$$

<sup>14</sup>I will use the formal expression  $\Gamma \leftarrow \Delta$  (with  $\Gamma$  and  $\Delta$  type-theoretical contexts, eventually each or one of them containing a single type-theoretical expression) to represent the standard type-theoretical update of  $\Gamma$  by  $\Delta$  (see [29] and [31]). The type theoretical dynamics here mentioned needs obviously to be interpreted with respect to more standard forms of dynamic reasoning. For what follows, one might consider some intuitive equivalences: the contextual updates produced by the introduction or the evaluation of an hypothesis correspond to the standard expansion operation (+) from the AGM-paradigm; on the other hand, the formulation of a new type-declaration is equivalent to the update operation ( $\circ$ ) from [20]. The extensions of type-theoretical informational states have been considered in relation to the AGM-paradigm in [27]. A more detailed analysis, presented in [30], extends this comparison to a general understanding of the basic operations of revision, update and merging for the operations mentioned in the following.



This kind of extension provides for each new assumption the related meaningful presupposition, and therefore now information increases both relatively to meaningful data considered relevant to the knowledge process, i.e. semantically, and syntactically.

- d. The fourth and final case completes the entire picture, the extension on the informational state provides a restriction at the semantic level. It is obtained by setting a new hypothesis and the related new presupposition:

$$\langle \alpha_1 : type, \dots, \alpha_n : type \rangle \leftarrow \beta : type$$

$$[x_1 : \alpha_1, \dots, x_n : \alpha_n] \leftarrow x_{n+1} : \beta (\beta = \perp).$$

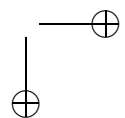
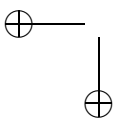
In this case the extension is contradictory, i.e. it is based on a semantic extension which supports the introduction of new syntactic data, with a semantic value not coherent with the previous state. This amounts to say that information decreases relatively to meaningful data considered relevant to the knowledge process, and the agent is no longer able to furnish a coherent base to her set of informational contents<sup>15</sup>.

The four cases, introducing the possible operations on informational states for CTT, allow us to extract the related general properties for the epistemic constructive notion of information:

- I. Information can be semantically stable, while it increases syntactically;
- II. Information can decrease syntactically;
- III. Information can increase semantically;
- IV. Information can decrease semantically.

In all of these cases, “syntactically” means with respect to the set of simple data, i.e. within the given set of concepts; “semantically” means in terms of an update operation on the set of meaningful concepts.

<sup>15</sup>This example, treated explicitly as a case of error in [27], and falling under the explanation provided by the therein introduced *Principle of restricted Monotonicity I/II*, shows how one could give up in a constructive frame with the property of persistency mentioned for infon logics.



## 6. *Final epistemic and ethical remarks*

The notion of information presented in this paper is partially different from its various understandings in the field of the Philosophy of Information, and it is also a novel formulation with respect to other agent-based non-classical formalizations. Our interpretation aims at recovering some features proper to a common and intuitive understanding of the notion of information. The difference with the usual definition is given essentially by the rejection of the thesis according to which any possible declarative, well-formed and semantically determined content provides information. One stresses here instead the idea according to which information defines a specific epistemic state, dependent from the nature of data, from the source and from the receiver of those data. The resulting notion corresponds roughly to "being in the condition of judging that", and it is distinct from the proper (justified) state of knowing. Under this view, an agent's informational state is strictly dependent on the origin of the content received and on the starting epistemic situation of the receiver. Hence, this approach maintains the common understanding of information as a flow of data for processing systems, but it seems to simulate better a group of rational systems able to ascribe in different ways well-formedness and meaningfulness to configurations of data, possibly in order to draw decisions and to make choices on their basis. Human beings seem to treat information at this level of understanding. A further step for the development of a general epistemic definition of information shall be represented by a corresponding formalization for a multi-agent model, with a notion of event collecting single agents' states.

There is at least one important ethical consequence following from the present definition of information. It concerns the correct treatment of the information flow and the due epistemic reactions by a rational agent as the receiver of such information. To start with, our definition presents information as an epistemic content, to be considered meaningful and true-to-the-agent (i.e. not provably true). This is confirmed by the relational nature of flowing data: an informational content is thought to be a flow coming from a specific source and directed to a certain receiver. On the one hand, this means essentially that the flow always conveys some meaning; on the other hand, it means also that such content is meaningful for the receiver for which it is produced, but it can be meaningless for any other agent whose starting informational state is incomplete or irrelevant with respect to the contents conveyed. One could say that the information flow is essentially defined by its transmission channel. This explains the weak epistemic nature here attributed to information, and the rigid distinction drawn with any content accepted and recognized as true by any agent. The latter epistemic notion has been called "knowable content" (or simply "knowledge"). In this sense, the possibility of rejecting informational data as "misinformation" is preserved,

whereas it is maintained a strong epistemic basis for knowledge. Obviously, this also clarifies the possibility of contradictory sources and different epistemic states in different receivers of the same informational content. This latter point is particularly relevant for an analysis of the information society and the consequences of those knowledge processes based on information flows received by media systems.

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