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FIXPOINTS OF MODELS CONSTRUCTIONS

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Abstract
We start with Specker's result that NF is equiconsistent with $\mathsf{SCA}+\mathsf{Ext}+\mathsf{Amb}$ (SCA is the Simple (intensional) Type Theory). First, there is a model $\mathcal{M}_0$ of SCA (simple). We define 2 operations $\mathcal{A}_1$ and $\mathcal{A}_2$ acting on models of SCA ($\mathcal{A}_2$ will be parametrized by a finite list $\psi_1, \ldots, \psi_n$ of $\mathcal{L}_{\mathsf{TT}}$-statements, but this is enough by compactness):

(1) $\mathcal{M} \models \mathsf{SCA} \Longrightarrow \mathcal{A}_1(\mathcal{M}) \models \mathsf{SCA}+\mathsf{Ext}$;
(2) $\mathcal{M} \models \mathsf{SCA} \Longrightarrow \mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{M}) \models \mathsf{SCA}+\mathsf{Amb}(\psi_1,\ldots,\psi_n)$ (Jensen-Boffa's $\mathrm{Consis}(\mathsf{NFU})$ proof).

Denote $\mathcal{A}(\mathcal{M}) := \mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{A}_1(\mathcal{M}))$. If the operation $\mathcal{A}$ has a fixpoint (i.e. $\mathcal{M} \models \mathsf{SCA}$ s.t. $A(\mathcal{M}) = \mathcal{M}$), then this $\mathcal{M}$ is a model of $\mathsf{SCA}+\mathsf{Ext}+\mathsf{Amb}(\psi_1,\ldots,\psi_n)$.

For every $\mathcal{M} \models \mathsf{SCA}$ we define a "complexity measure" $\mathsf{J}(\mathcal{M})$ (which is a set) and show that (a) $\mathsf{J}(\mathcal{M}_0)$ is countable; (b) $\mathsf{J}(\mathcal{A}(\mathcal{M})) \subseteq \mathsf{J}(\mathcal{M})$. We also have $\mathsf{J}(\mathcal{A}(\mathcal{M})) = \mathsf{J}(\mathcal{M}) ==> \mathcal{A}(\mathcal{M}) = \mathcal{M}$. It could be tempting to think that $\mathcal{A}$ must have a fixpoint by cardinality argument (using existence of an uncountable ordinal), but in reality this is not clear.

The Axiom of Choice of ZFC is used for defining the operations $\mathcal{A}_1$ and $\mathcal{A}_2$.

To conclude, NF is consistent assuming that such a fixpoint always (i.e. for every finite list $\psi_1, \ldots, \psi_n$) exists.

1. *Background*

Our metatheory is ZFC. $\mathcal{L}_{\mathsf{TT}}$ is the language of Simple Type Theory.

Comprehension SCA is an axiom scheme

$$\mathsf{SCA}^n: \qquad \exists y^{n+1} \forall x^n \left(x \in y \leftrightarrow \varphi[x] \right),$$

with y^{n+1} not free in φ, for every n and for every formula $\varphi \in \mathcal{L}_{\mathsf{TT}}$.

Extensionality Ext is an axiom scheme

$$\mathsf{Ext}^n : \qquad \forall x^{n+1} \forall y^{n+1} \Big(\forall z^n (z \in x \leftrightarrow z \in y) \to x = y \Big),$$

for every n.

Given an $\mathcal{L}_{\mathsf{TT}}$-formula φ, by φ^+ we denote the result of raising all type indices in φ by 1. The *Ambiguity* scheme is

$$\mathsf{Amb} : \qquad \varphi \leftrightarrow \varphi^+,$$

for all statements $\varphi \in \mathcal{L}_{\mathsf{TT}}$.

Theorem 1.1: *(Specker)* NF *is equiconsistent with* $\mathsf{SCA} + \mathsf{Ext} + \mathsf{Amb}$.

Proof. This follows from Specker's result [6]. A different, proof-theoretic, proof of this fact can be found in [2]. □

Definition 1.2: *A* typed structure *is a set* $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i} \rangle \mid i \in \mathbb{N}\}$ *s.t.* $j_0 < \ldots < j_n < \ldots$ *is an increasing sequence of natural numbers,* $M^{j_0} \neq \emptyset$*, and* $\forall i \in \mathbb{N}\ \in^{j_i} \subseteq M^{j_i} \times M^{j_{i+1}}$.

Definition 1.3: *Assume that* $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i} \rangle \mid i \in \mathbb{N}\}$ *is a typed structure and* $i \in \mathbb{N}$.

(1) *For* $x \in M^{j_i}$, $y \in M^{j_i}$, $\mathcal{M} \models x = y$ *means* $x = y$.
(2) *For* $x \in M^{j_i}$, $y \in M^{j_{i+1}}$, $\mathcal{M} \models x \in y$ *means* $\langle x, y \rangle \in \in^{j_i}$.

For any $\varphi \in \mathcal{L}_{\mathsf{TT}}$, $\mathcal{M} \models \varphi$ *is now defined in the standard way.*

Lemma 1.4: *If* $\mathcal{M} \models \mathsf{SCA}$ *then* $\forall i \in \mathbb{N}\ M^{j_i} \neq \emptyset$.

Proof. For $i = 0$ the condition is given by Definition 1.2. For $i + 1$ we use the fact

$$\mathcal{M} \models \exists y^{i+1} \forall x^i \Big(x \in y \leftrightarrow x = x \Big).$$

□

Definition 1.5: *Let* $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i} \rangle \mid i \in \mathbb{N}\}$ *be a typed stucture. The* join $\mathsf{J}(\mathcal{M})$ *of* $\mathcal{M}$ *is defined by*

$$\mathsf{J}(\mathcal{M}) := \{\langle j_i, x \rangle \mid i \in \mathbb{N} \wedge x \in M^{j_i}\}.$$

Theorem 1.6: *There exists a model $\mathcal{M}_0$ of* SCA *with countable join.*

Proof. Take $\mathcal{M}_0 := \{\langle \mathcal{P}^{i+1}(\emptyset), \in\rangle \mid i \in \mathbb{N}\}$. □

2. *Operations $\mathcal{A}_1$ and $\mathcal{A}_2$*

2.1. *Operation $\mathcal{A}_1$: securing Extensionality*

In this subsection we are assuming that $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i}\rangle \mid i \in \mathbb{N}\}$ is a model of SCA.

Definition 2.1: *Set*

$$\begin{aligned} \sim^{j_0} &:= \{\langle x, y\rangle \mid x \in M^{j_0} \wedge y \in M^{j_0}\}; && (1)\\ \sim^{j_{i+1}} &:= \{\langle x, y\rangle \in M^{j_{i+1}} \times M^{j_{i+1}} \mid \\ &\quad \forall x' \in^{j_i} x\, \exists y' \in^{j_i} y\, x' \sim^{j_i} y' \bigwedge \forall y' \in^{j_i} y\, \exists x' \in^{j_i} x\, x' \sim^{j_i} y'\}; && (2)\\ \tilde{\in}^{j_i} &:= \{\langle x, y\rangle \in M^{j_i} \times M^{j_{i+1}} \mid \exists z \in^{j_i} y\, x \sim^{j_i} z\}. && (3) \end{aligned}$$

Definition 2.2: *A* weak typed structure (wts) *is a set*

$$\mathcal{N} = \{\langle N^{j_k}, \langle =^{j_k}, \varepsilon^{j_k}\rangle\rangle \mid k \in \mathbb{N}\}$$

s.t. $j_0 < \ldots < j_n < \ldots$ *is an increasing sequence of natural numbers,* $M^{j_0} \neq \emptyset$, $\forall k \in \mathbb{N}\, (=^{j_k} \subseteq N^{j_k} \times N^{j_k} \wedge \varepsilon^{j_k} \subseteq N^{j_k} \times N^{j_{k+1}})$, *and all equality axioms are satisfied, i.e. for all* $k \in \mathbb{N}$ *and all* $x, y, z \in N^{j_k}$, $u, v \in N^{j_{k+1}}$, *the following hold:*

$$\begin{aligned} x &=^{j_k} x; && (4)\\ x =^{j_k} y &\to y =^{j_k} x; && (5)\\ x =^{j_k} y \wedge y =^{j_k} z &\to x =^{j_k} z; && (6)\\ x =^{j_k} y \wedge x\, \varepsilon^{j_k}\, u &\to y\, \varepsilon^{j_k}\, u; && (7)\\ x\, \varepsilon^{j_k}\, u \wedge u =^{j_{k+1}} v &\to x\, \varepsilon^{j_k}\, v. && (8) \end{aligned}$$

Definition 2.3: *Assume that* $\mathcal{N} = \{\langle N^{j_k}, \langle =^{j_k}, \varepsilon^{j_k}\rangle\rangle \mid k \in \mathbb{N}\}$ *is a wts and* $k \in \mathbb{N}$.

(1) *For* $x \in N^{j_k}$, $y \in N^{j_k}$, $\mathcal{N} \models_w x = y$ *means* $\langle x, y\rangle \in =^{j_k}$.

(2) *For* $x \in N^{j_k}$, $y \in N^{j_{k+1}}$, $\mathcal{N} \models_w x \in y$ *means* $\langle x, y\rangle \in \varepsilon^{j_k}$.

For any $\varphi \in \mathcal{L}_{\mathsf{TT}}$, $\mathcal{M} \models_w \varphi$ *is now defined in the standard way.*

Lemma 2.4: $\mathcal{M}_w := \{\langle M^{j_i}, \langle \sim^{j_i}, \tilde{\in}^{j_i}\rangle\rangle \mid i \in \mathbb{N}\}$ *is a weak typed structure.*

Proof. The requirement $\forall i \in \mathbb{N}\,(\sim^{j_i} \subseteq M^{j_i} \times M^{j_i} \wedge \tilde{\in}^{j_i} \subseteq M^{j_i} \times M^{j_{i+1}})$ is immediate from (1)–(3).

Equality axioms (4)–(6) are proved by induction on i, using defining clauses (1)–(2):

(4): The Claim is obvious for $i = 0$. Then,

$$x \sim^{j_{i+1}} x \overset{(2)}{\Longleftrightarrow} \forall x' \in^{j_i} x\, \exists y' \in^{j_i} x\, x' \sim^{j_i} y' \bigwedge \forall y' \in^{j_i} x\, \exists x' \in^{j_i} x\, x' \sim^{j_i} y',$$

and the RHS is true by IH.

(5): The Claim is obvious for $i = 0$. Assume $x \sim^{j_{i+1}} y$, i.e.

$$\forall x' \in^{j_i} x\, \exists y' \in^{j_i} y\, x' \sim^{j_i} y' \bigwedge \forall y' \in^{j_i} y\, \exists x' \in^{j_i} x\, x' \sim^{j_i} y'.$$

Using IH, this implies

$$\forall y' \in^{j_i} y\, \exists x' \in^{j_i} x\, y' \sim^{j_i} x' \bigwedge \forall x' \in^{j_i} x\, \exists y' \in^{j_i} y\, y' \sim^{j_i} x',$$

i.e. $y \sim^{j_{i+1}} x$.

(6): The Claim is obvious for $i = 0$. Assume $x \sim^{j_{i+1}} y$ and $y \sim^{j_{i+1}} z$, i.e.

$$\forall x' \in^{j_i} x\, \exists y' \in^{j_i} y\, x' \sim^{j_i} y' \bigwedge \forall y' \in^{j_i} y\, \exists x' \in^{j_i} x\, x' \sim^{j_i} y'$$

and

$$\forall y' \in^{j_i} y\, \exists z' \in^{j_i} z\, y' \sim^{j_i} z' \bigwedge \forall z' \in^{j_i} z\, \exists y' \in^{j_i} y\, y' \sim^{j_i} z'.$$

Using IH, this implies

$$\forall x' \in^{j_i} x\, \exists z' \in^{j_i} z\, x' \sim^{j_i} z' \bigwedge \forall z' \in^{j_i} z\, \exists x' \in^{j_i} x\, x' \sim^{j_i} z',$$

i.e. $x \sim^{j_{i+1}} z$.

Remaining axioms (7)–(8) are proved by using defining clauses (2)–(3) and already established facts (5)–(6):

(7): Assume $x \sim^{j_i} y$ and $x \,\tilde{\in}^{j_i}\, u$. By (3) the latter means $\exists z \in^{j_i} u \, x \sim^{j_i} z$. For this z, by (5) and (6) we obtain $y \sim^{j_i} z$, i.e. $y \,\tilde{\in}^{j_i}\, u$.

(8): Assume $x \,\tilde{\in}^{j_i}\, u$ and $u \sim^{j_{i+1}} v$. The former means

$$\exists z \in^{j_i} u \, x \sim^{j_i} z.$$

From $u \sim^{j_{i+1}} v$, $z \in^{j_i} u$ yields

$$\exists w \in^{j_i} v \, z \sim^{j_i} w.$$

Using transitivity (6), we obtain $x \sim^{j_i} w$, concluding $x \,\tilde{\in}^{j_i} v$.

□

Lemma 2.5: *$\mathcal{M}_w$ is a weak model of Extensionality, i.e.*

$$\forall i \in \mathbb{N} \forall x \in M^{j_{i+1}} \forall y \in M^{j_{i+1}} \Big(\forall z \in M^{j_i} (z \,\tilde{\in}^{j_i}\, x \leftrightarrow z \,\tilde{\in}^{j_i}\, y) \longrightarrow x \sim^{j_{i+1}} y \Big).$$

Proof. Assume

$$i \in \mathbb{N} \wedge x \in M^{j_{i+1}} \wedge y \in M^{j_{i+1}}$$

and

$$\forall z \in M^{j_i} (z \,\tilde{\in}^{j_i}\, x \leftrightarrow z \,\tilde{\in}^{j_i}\, y).$$

The latter is the same as

$$\forall z \, (z \,\tilde{\in}^{j_i}\, x \leftrightarrow z \,\tilde{\in}^{j_i}\, y),$$

which, using reflexivity (4), implies

$$\forall x' \in^{j_i} x \, \exists y' \in^{j_i} y \, x' \sim^{j_i} y' \bigwedge \forall y' \in^{j_i} y \, \exists x' \in^{j_i} x \, x' \sim^{j_i} y'.$$

By (2), this is the same as $x \sim^{j_{i+1}} y$.

□

Definition 2.6: *In Simple Type Theory, set*

$$
\begin{array}{rcl}
x^0 \sim^0 y^0 & :\Leftrightarrow & \top; \\
x^{i+1} \sim^{i+1} y^{i+1} & :\Leftrightarrow & \forall u^i \in^i x^{i+1}\, \exists v^i \in^i y^{i+1}\, u \sim^i v \\
& & \bigwedge \forall v^i \in^i y^{i+1}\, \exists u^i \in^i x^{i+1}\, u \sim^i v; \\
x^i \,\tilde{\in}^i\, y^{i+1} & :\Leftrightarrow & \exists z^i \in^i y^{i+1}\, x^i \sim^i z.
\end{array}
$$

Given an $\mathcal{L}_{\mathsf{TT}}$-formula φ, the $\mathcal{L}_{\mathsf{TT}}$-formula $\tilde{\varphi}$ is defined by replacing every $x^i = y^i$ by $x^i \sim^i y^i$, and every $x^i \in y^{i+1}$ by $x^i \,\tilde{\in}^i\, y^{i+1}$.

Lemma 2.7: *For every $\mathcal{L}_{\mathsf{TT}}$-formula φ,*

$$\mathcal{M}_w \models_w \varphi \iff \mathcal{M} \models \tilde{\varphi}.$$

Proof. By induction on φ. The atomic case follows from Definitions 2.1 and 2.6. □

Lemma 2.8: *For every $\mathcal{L}_{\mathsf{TT}}$-formula $\varphi[x^n]$,*

$$\mathcal{M} \models \forall x_1^n \forall x_2^n \left(x_1 \sim^n x_2 \to (\tilde{\varphi}[x_1] \leftrightarrow \tilde{\varphi}[x_2]) \right).$$

Proof. Since $\mathcal{M}_w$ is a wts (Lemma 2.4), we have

$$\mathcal{M}_w \models_w \forall x_1^n \forall x_2^n \left(x_1 = x_2 \to (\varphi[x_1] \leftrightarrow \varphi[x_2]) \right).$$

The Claim now follows from Lemma 2.7. □

Lemma 2.9: *$\mathcal{M}_w$ is a weak model of Comprehension, i.e. for every $\mathcal{L}_{\mathsf{TT}}$-formula $\varphi[x^n]$,*

$$\mathcal{M}_w \models_w \exists y^{n+1} \forall x^n \left(x \in y \leftrightarrow \varphi[x] \right).$$

Proof. By Lemma 2.7 it's enough to prove

$$\mathcal{M} \models \exists y^{n+1} \forall x^n \left(x \,\tilde{\in}^n\, y \leftrightarrow \tilde{\varphi}[x] \right).$$

Since $\mathcal{M} \models \mathsf{SCA}$, take $y \in M^{j_{n+1}}$ so that

$$\mathcal{M} \models \forall x^n \left(x \in y \leftrightarrow \tilde{\varphi}[x] \right).$$

We have to show

$$\mathcal{M} \models \forall x^n \left(x \in y \leftrightarrow x \mathrel{\tilde{\in}^n} y \right).$$

If $x \in^{j_n} y$, then $x \mathrel{\tilde{\in}^{j_n}} y$ follows by reflexivity (4). Conversely, assuming $x \mathrel{\tilde{\in}^{j_n}} y$, we get $\exists z \in^{j_n} y\, x \sim^{j_n} z$, and then $\mathcal{M} \models \tilde{\varphi}[x]$ and $x \in^{j_n} y$ by Lemma 2.8.

□

Definition 2.10: *By Lemma 2.4, every M^{j_i} is divided by $\sim^{j_i}$ into a set of non-empty equivalence classes. For every $x \in M^{j_i}$, we denote*

$$[x] := \{x' \mid x' \sim^{j_i} x\}.$$

Define

$$[M^{j_i}] := \{[x] \mid x \in M^{j_i}\},$$

and, for $[x] \in [M^{j_i}]$, $[y] \in [M^{j_{i+1}}]$,

$$[x]\, [\tilde{\in}^{j_i}]\, [y] \textit{ iff } \forall x' \in [x] \forall y' \in [y]\, x' \mathrel{\tilde{\in}^{j_i}} y'.$$

The typed structure $[\mathcal{M}]$ is now defined

$$[\mathcal{M}] := \{\langle [M^{j_i}], [\tilde{\in}^{j_i}] \rangle \mid i \in \mathbb{N}\}.$$

Lemma 2.11: *Let $\varphi(x_1^{i_1}, \ldots, x_k^{i_k})$ be an $\mathcal{L}_{\mathsf{TT}}$-formula with all free variables shown, and $x_1^{i_1} \in M^{j_{i_1}}, \ldots, x_k^{i_k} \in M^{j_{i_k}}$. Then:*

$$\mathcal{M}_w \models_w \varphi(x_1^{i_1}, \ldots, x_k^{i_k}) \iff [\mathcal{M}] \models \varphi([x_1^{i_1}], \ldots, [x_k^{i_k}]).$$

Proof. By induction on φ, using non-emptiness of $[x]$ (Lemma 2.4). The atomic case follows from the equivalences

$$x \sim^{j_i} y \iff [x] = [y]$$

and

$$x \mathrel{\tilde{\in}^{j_i}} y \iff [x]\, [\tilde{\in}^{j_i}]\, [y],$$

which also follow from Lemma 2.4. $\square$

Definition 2.12: *Let* C_1 *be a choice function, picking one element from each equivalence class in* $[M^{j_i}]$*, for all* $i \in \mathbb{N}$*. Define*

$$\mathcal{A}_1(M^{j_i}) := \{\mathsf{C}_1[x] \mid [x] \in [M^{j_i}]\},$$

and, for $x \in \mathcal{A}_1(M^{j_i})$*,* $y \in \mathcal{A}_1(M^{j_{i+1}})$*,*

$$x \,\epsilon^{j_i}\, y \textit{ iff } [x]\,[\tilde{\in}^{j_i}]\,[y].$$

The typed structure $\mathcal{A}_1(\mathcal{M})$ *is now defined*

$$\mathcal{A}_1(\mathcal{M}) := \{\langle \mathcal{A}_1(M^{j_i}), \epsilon^{j_i}\rangle \mid i \in \mathbb{N}\}.$$

Lemma 2.13: *Let* $\varphi(x_1^{i_1}, \dots, x_k^{i_k})$ *be an* $\mathcal{L}_{\mathsf{TT}}$*-formula with all free variables shown, and* $x_1^{i_1} \in \mathcal{A}_1(M^{j_{i_1}}), \dots, x_k^{i_k} \in \mathcal{A}_1(M^{j_{i_k}})$*. Then:*

$$\mathcal{A}_1(\mathcal{M}) \models \varphi(x_1^{i_1}, \dots, x_k^{i_k}) \iff [\mathcal{M}] \models \varphi([x_1^{i_1}], \dots, [x_k^{i_k}]).$$

Proof. By induction on φ. First remember that we always have $[\mathsf{C}_1[x]] = [x]$. The atomic case

$$\mathsf{C}_1[x] = \mathsf{C}_1[y] \Longleftrightarrow [x] = [y]$$

follows from the fact that C_1 is a choice function, the atomic case

$$\mathsf{C}_1[x]\,\epsilon^{j_i}\,\mathsf{C}_1[y] \Longleftrightarrow [x]\,[\tilde{\in}^{j_i}]\,[y]$$

follows from the Definition 2.12 of ϵ^{j_i}. $\square$

Theorem 2.14: $\mathcal{A}_1(\mathcal{M})$ *is a model of* $\mathsf{SCA} + \mathsf{Ext}$.

Proof. Follows from Lemmata 2.5, 2.9, 2.11 and 2.13. $\square$

Theorem 2.15:

$$\mathsf{J}(\mathcal{A}_1(\mathcal{M})) \subseteq \mathsf{J}(\mathcal{M}).$$

Proof. From the Definition 2.12 we have $\mathcal{A}_1(M^{j_i}) \subseteq M^{j_i}$, for every $i \in \mathbb{N}$. The Claim now follows from the Definition 1.5. $\square$

2.2. *Operation $\mathcal{A}_2$: securing Ambiguity*

Definition 2.16: *Let $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i}\rangle \mid i \in \mathbb{N}\}$ be a typed structure and $i_0 < i_1 < \ldots$ be an increasing sequence of natural numbers. We define a typed structure $\mathcal{N} = \{\langle M^{j_{i_k}}, \in'^{j_{i_k}}\rangle \mid k \in \mathbb{N}\}$ as follows: for $x \in M^{j_{i_k}}$, $y \in M^{j_{i_{k+1}}}$: $\mathcal{N} \models x \in y$ iff $\mathcal{M} \models \exists z\,(x \in z \wedge \{\ldots\{z\}\ldots\} = y)$, where the singleton operation is iterated $i_{k+1} - i_k - 1$ times.*
Such a typed structure will be called extracted *from $\mathcal{M}$, written $\mathcal{N} \leq \mathcal{M}$.*

Definition 2.17: *If $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i}\rangle \mid i \in \mathbb{N}\}$ is a typed structure, then $\mathcal{M}^+$ denotes the typed structure*

$$\mathcal{M}^+ := \{\langle M^{j_{i+1}}, \in^{j_{i+1}}\rangle \mid i \in \mathbb{N}\}.$$

Obviously, $\leq$ is reflexive and $\mathcal{M}^+ \leq \mathcal{M}$. For any x, we denote $\{x\}_0 := x$, $\{x\}_{n+1} := \{\{x\}_n\}$, $n \in \mathbb{N}$.

Lemma 2.18: *Let $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i}\rangle \mid i \in \mathbb{N}\}$ be a typed structure. If $\mathcal{N} \leq \mathcal{M}$, $x \in M^{j_{i_k}}$, $y \in M^{j_{i_{k+1}}}$, then $\mathcal{N} \models \{x\} = y$ is equivalent to $\mathcal{M} \models \{x\}_{i_{k+1}-i_k} = y$.*

Proof.

$$\begin{aligned}
\mathcal{N} \models \{x\} = y \iff\ & \mathcal{N} \models x \in y \wedge \forall p\,(p \in y \to p = x) \\
\iff\ & \mathcal{M} \models \exists z\,(x \in z \wedge \{z\}_{i_{k+1}-i_k-1} = y) \\
& \wedge \forall p\,(\exists z'\,(p \in z' \wedge \{z'\}_{i_{k+1}-i_k-1} = y) \to p = x) \\
\iff\ & \mathcal{M} \models \{x\}_{i_{k+1}-i_k} = y :
\end{aligned}$$

Let's check the last $\iff$. Reason in $\mathcal{M}$. Assume LHS. For that z, we already have $\{z\}_{i_{k+1}-i_k-1} = y$, and it remains to show $z = \{x\}$. $x \in z$ is given. Assuming $e \in z$, and taking in the second part of the conjunction $p := e$ and $z' := z$, we obtain $e = x$, q.e.d. Conversely, assume RHS. For $\exists z\,(x \in z \wedge \{z\}_{i_{k+1}-i_k-1} = y)$, take $z := \{x\}$. For the second part, if $\{z'\}_{i_{k+1}-i_k-1} = y$, then z' must be $\{x\}$, and the only element of $\{x\}$ is x, q.e.d.

□

Lemma 2.19: *Let $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i}\rangle \mid i \in \mathbb{N}\}$ be a typed structure. If $\mathcal{N} \leq \mathcal{M}$, $x \in M^{j_{i_k}}$, $y \in M^{j_{i_{k+n}}}$, then $\mathcal{N} \models \{x\}_n = y$ is equivalent to $\mathcal{M} \models \{x\}_{i_{k+n}-i_k} = y$.*

Proof. If $n = 0$, the assertion is obvious. For $n > 0$, apply the previous Lemma n times. □

Lemma 2.20: $\leq$ *is transitive.*

Proof. Let $\mathcal{N} \leq \mathcal{M}$ and $\mathcal{O} \leq \mathcal{N}$. We need to show $\mathcal{O} \leq \mathcal{M}$.

First, the domain of $\mathcal{O}$ is $\{M^{j_{i_{k_l}}} \mid l \in \mathbb{N}\}$. Assume $x \in M^{j_{i_{k_l}}}$, $y \in M^{j_{i_{k_{l+1}}}}$. Now compute:

$$\begin{aligned}
\mathcal{O} \models x \in y \quad &\overset{\mathcal{O} \leq \mathcal{N}}{\Longleftrightarrow} \quad \mathcal{N} \models \exists z \, (x \in z \wedge \{z\}_{k_{l+1}-k_l-1} = y) \\
&\overset{\mathcal{N} \leq \mathcal{M}, \text{L.2.19}}{\Longleftrightarrow} \quad \mathcal{M} \models \exists z \, (\exists z_1 \, (x \in z_1 \wedge \{z_1\}_{i_{k_l+1}-i_{k_l}-1} = z) \\
&\qquad\qquad \wedge \{z\}_{i_{k_{l+1}}-i_{k_l+1}} = y) \\
&\Longleftrightarrow \quad \mathcal{M} \models \exists z_1 \, (x \in z_1 \wedge \{z_1\}_{i_{k_{l+1}}-i_{k_l}-1} = y),
\end{aligned}$$

confirming that $\mathcal{O} \leq \mathcal{M}$.

□

Lemma 2.21: *If* $\mathcal{N} \leq \mathcal{M}$ *and* $\mathcal{M} \models \mathsf{SCA}$, *then* $\mathcal{N} \models \mathsf{SCA}$.

Proof. Assume $\mathcal{N} \leq \mathcal{M}$ and $\mathcal{M} \models \mathsf{SCA}$. Let $\varphi[x^k] \in \mathcal{L}_{\mathsf{TT}}$. We need to show

$$\mathcal{N} \models \exists y^{k+1} \forall x^k \left(x \in y \leftrightarrow \varphi[x] \right). \tag{9}$$

Let $\varphi^{\mathcal{N}}$ be obtained from φ by replacing every variable x^l by x^{i_l}, and replacing every $x^l \in y^{l+1}$ by $\exists z^{i_l+1}(x^{i_l} \in z \wedge \{z\}_{i_{l+1}-i_l-1} = y)$. Then $\varphi^{\mathcal{N}} \in \mathcal{L}_{\mathsf{TT}}$. Rephrasing (9), we need to show

$$\mathcal{M} \models \exists y^{i_{k+1}} \forall x^{i_k} \left(\exists z^{i_k+1}(x \in z \wedge \{z\}_{i_{k+1}-i_k-1} = y) \leftrightarrow \varphi^{\mathcal{N}}[x] \right). \tag{10}$$

First, since $\mathcal{M} \models \mathsf{SCA}$, we have

$$\mathcal{M} \models \exists y_1^{i_k+1} \forall x^{i_k} \left(x \in y_1 \leftrightarrow \varphi^{\mathcal{N}}[x] \right). \tag{11}$$

Take $y^{i_{k+1}} := \{y_1\}_{i_{k+1}-i_k-1}$. Observe

$$\mathcal{M} \models \forall x^{i_k} \Big(x \in y_1 \leftrightarrow \exists z^{i_k+1}(x \in z \wedge \{z\}_{i_{k+1}-i_k-1} = y) \Big) : \quad (12)$$

Reason in $\mathcal{M}$. Assume $x \in y_1$. Then RHS is satisfied by taking $z := y_1$. Conversely, assume $\exists z^{i_k+1}(x \in z \wedge \{z\}_{i_{k+1}-i_k-1} = y)$. Then it must be $z = y_1$ and $x \in y_1$. Q.E.D.

(12) and (11) now imply (10) and (9).

□

Definition 2.22: *For any $\mathcal{M} \models \mathsf{SCA}$ and any sentence $\psi \in \mathcal{L}_{\mathsf{TT}}$, let us say that $\mathcal{M}$* forces *ψ when ψ is true in every typed structure extracted from $\mathcal{M}$, and that $\mathcal{M}$* decides *ψ when $\mathcal{M}$ forces either ψ or $\neg\psi$.*

Remark 2.23: *If $\mathcal{M}$ decides ψ, then $\mathcal{M} \models \psi \leftrightarrow \psi^+$.*

Proof. Remember $\mathcal{M}^+ \leq \mathcal{M}$. □

Lemma 2.24: *(*Extraction Lemma*, Boffa* [1]*) Given any $\mathcal{M} \models \mathsf{SCA}$ and any sentence $\psi \in \mathcal{L}_{\mathsf{TT}}$, there is a model $\mathcal{N} \models \mathsf{SCA}$ with $\mathcal{N} \leq \mathcal{M}$ which decides ψ.*

Proof. Let k be greater than all type indices appearing in ψ. Define a partition G_1, G_2 of $[\mathbb{N}]^{k+1}$ as follows:

$$\begin{aligned} G_1 &:= \{i_0 < i_1 < \ldots < i_k \mid \langle M^{j_{i_0}}, M^{j_{i_1}}, \ldots, M^{j_{i_k}}, \ldots\rangle \models \psi\}, \\ G_2 &:= \{i_0 < i_1 < \ldots < i_k \mid \langle M^{j_{i_0}}, M^{j_{i_1}}, \ldots, M^{j_{i_k}}, \ldots\rangle \models \neg\psi\}. \end{aligned}$$

By Ramsey's theorem (cf. [5]), take an infinite set X of natural numbers $i_0 < i_1 < \ldots < i_n < \ldots$ such that $[X]^{k+1} \subseteq G_1$ or $[X]^{k+1} \subseteq G_2$, and set $\mathrm{dom}(\mathcal{N}) := \langle M^{j_{i_0}}, M^{j_{i_1}}, \ldots, M^{j_{i_n}}, \ldots\rangle$. In the first case ($[X]^{k+1} \subseteq G_1$) $\mathcal{N}$ forces ψ, and in the second case $\mathcal{N}$ forces $\neg\psi$. □

Lemma 2.25: *Given any $\mathcal{M} \models \mathsf{SCA}$ and any sentence $\psi \in \mathcal{L}_{\mathsf{TT}}$, there is a model $\mathcal{N} \models \mathsf{SCA} + \psi \leftrightarrow \psi^+$ with $\mathcal{N} \leq \mathcal{M}$.*

Proof. Corollary of Lemma 2.24 and Remark 2.23. □

Lemma 2.26: *Given any $\mathcal{M} \models \mathsf{SCA}$ and any finite list of sentences $\psi_1, \ldots, \psi_n \in \mathcal{L}_{\mathsf{TT}}$, there is a model $\mathcal{N} \models \mathsf{SCA} + \bigwedge_{1 \leq i \leq n} \psi_i \leftrightarrow \psi_i^+$ with $\mathcal{N} \leq \mathcal{M}$.*

Proof. Apply Lemma 2.25 n times. Use transitivity of $\leq$ (Lemma 2.20). □

Definition 2.27: *Let a finite list of sentences $\psi_1, \ldots, \psi_n \in \mathcal{L}_{\mathsf{TT}}$ be given. Let $\mathcal{A}_2^{\psi_1,\ldots,\psi_n}$ be a choice function such that*

$$\textit{if } \mathcal{M} \models \mathsf{SCA} \textit{ then } \mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{M}) \models \mathsf{SCA}+$$

$$\bigwedge_{1\leq i\leq n} \psi_i \leftrightarrow \psi_i^+ \textit{ and } \mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{M}) \leq \mathcal{M}.$$

Theorem 2.28: *Let a finite list of sentences $\psi_1, \ldots, \psi_n \in \mathcal{L}_{\mathsf{TT}}$ be given. If $\mathcal{M} \models \mathsf{SCA}$ then*

$$\mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{M}) \models \mathsf{SCA} + \bigwedge_{1\leq i\leq n} \psi_i \leftrightarrow \psi_i^+ \textit{ and } \mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{M}) \leq \mathcal{M}.$$

Proof. Follows from Definition 2.27. □

Lemma 2.29: *If $\mathcal{N} \leq \mathcal{M}$ then $\mathsf{J}(\mathcal{N}) \subseteq \mathsf{J}(\mathcal{M})$.*

Proof. By the Definition 2.16, the domain of $\mathcal{N}$ is just a subsequence of the domain of $\mathcal{M}$. □

Theorem 2.30: *For any finite list of sentences $\psi_1, \ldots, \psi_n \in \mathcal{L}_{\mathsf{TT}}$, if $\mathcal{M} \models \mathsf{SCA}$ then*

$$\mathsf{J}(\mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{M})) \subseteq \mathsf{J}(\mathcal{M}).$$

Proof. Follows from Theorem 2.28 and Lemma 2.29. □

3. *Conclusion*

Definition 3.1: *Let a finite list of sentences $\psi_1, \ldots, \psi_n \in \mathcal{L}_{\mathsf{TT}}$ be given. For $\mathcal{M} \models \mathsf{SCA}$ we define*

$$\mathcal{A}^{\psi_1,\ldots,\psi_n}(\mathcal{M}) := \mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{A}_1(\mathcal{M})).$$

Definition 3.2: *Let a finite list of sentences* $\psi_1, \ldots, \psi_n \in \mathcal{L}_{\mathsf{TT}}$ *be given.* $\mathcal{M} \models \mathsf{SCA}$ *is a* fixpoint *of* $\mathcal{A}^{\psi_1,\ldots,\psi_n}$ *iff*

$$\mathcal{A}^{\psi_1,\ldots,\psi_n}(\mathcal{M}) = \mathcal{M}.$$

Lemma 3.3: *If* $\mathcal{M} \models \mathsf{SCA}$ *and* $\mathsf{J}(\mathcal{A}_1(\mathcal{M})) = \mathsf{J}(\mathcal{M})$ *then* $\mathcal{A}_1(\mathcal{M}) = \mathcal{M}$.

Proof. Assume $\mathsf{J}(\mathcal{A}_1(\mathcal{M})) = \mathsf{J}(\mathcal{M})$. By Definitions 1.5 and 2.12, this implies

$$\forall i \in \mathbb{N}\, \mathcal{A}_1(M^{j_i}) = M^{j_i},$$

which, using Definitions 2.12 and 2.10, further yields

$$\forall i \in \mathbb{N} \forall x \in M^{j_i} \forall x' \in M^{j_i}\, (x \sim^{j_i} x' \leftrightarrow x = x'). \tag{13}$$

By Definition 2.1 furthermore we have

$$\forall i \in \mathbb{N} \forall x \in M^{j_i} \forall y \in M^{j_{i+1}}\, (x \mathrel{\tilde{\in}^{j_i}} y \leftrightarrow x \in^{j_i} y). \tag{14}$$

(13) and (14) confirm that $\mathcal{M}$ and $\mathcal{A}_1(\mathcal{M})$ is the same set.

□

Lemma 3.4: *Let a finite list of sentences* $\psi_1, \ldots, \psi_n \in \mathcal{L}_{\mathsf{TT}}$ *be given and* $\mathcal{M} \models \mathsf{SCA}$. *If* $\mathsf{J}(\mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{M})) = \mathsf{J}(\mathcal{M})$ *then* $\mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{M}) = \mathcal{M}$.

Proof. Assume that $\mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{M})$ is given by an increasing sequence $\{i_k \mid k \in \mathbb{N}\}$. Assume $\mathsf{J}(\mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{M})) = \mathsf{J}(\mathcal{M})$, which is, by Definition 1.5,

$$\{\langle j_i, x\rangle \mid i \in \mathbb{N} \wedge x \in M^{j_i}\} = \{\langle j_{i_k}, x\rangle \mid k \in \mathbb{N} \wedge x \in M^{j_{i_k}}\}.$$

<u>*Claim.*</u> $\forall k \in \mathbb{N}\, i_k = k$.

/– By induction on k. First denote

$$\begin{aligned} A &:= \{\langle j_i, x\rangle \mid i \in \mathbb{N} \wedge x \in M^{j_i}\}, \\ B &:= \{\langle j_{i_k}, x\rangle \mid k \in \mathbb{N} \wedge x \in M^{j_{i_k}}\}. \end{aligned}$$

Ind. base: Take any $x \in M^{j_0}$ (by Lemma 1.4 such an x exists). Then $\langle j_0, x\rangle \in A$, so we must have $\langle j_0, x\rangle \in B$. If $i_0 > 0$, then $j_{i_0} > j_0$, and $j_{i_k} \geq j_{i_0} > j_0$ for every k. Therefore $i_0 = 0$ must hold.

Ind. step: Take any $x \in M^{j_{k+1}}$ (by Lemma 1.4 such an x exists). Then $\langle j_{k+1}, x\rangle \in A$, so we must have $\langle j_{k+1}, x\rangle \in B$. If $i_{k+1} > k+1$, then $j_{i_{k+1}} > j_{k+1}$, and $j_{i_{k'}} \geq j_{i_{k+1}} > j_{k+1}$ for every $k' \geq k+1$. On the other hand, by IH we have $i_{k'} = k'$ and $j_{i_{k'}} = j_{k'} < j_{k+1}$ for every $k' < k+1$. Since we always have $i_{k+1} \geq k+1$, it remains to conclude that $i_{k+1} = k+1$.

⊣

Since $\forall k \in \mathbb{N}\, i_k = k$, by Definition 2.16 the $\in$ relation is the same in $\mathcal{M}$ and $\mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{M})$, so $\mathcal{M}$ and $\mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{M})$ is the same set.

□

Theorem 3.5: *Let a finite list of sentences $\psi_1, \ldots, \psi_n \in \mathcal{L}_{\mathsf{TT}}$ be given and $\mathcal{M} \models \mathsf{SCA}$. If $\mathsf{J}(\mathcal{A}^{\psi_1,\ldots,\psi_n}(\mathcal{M})) = \mathsf{J}(\mathcal{M})$ then $\mathcal{A}^{\psi_1,\ldots,\psi_n}(\mathcal{M}) = \mathcal{M}$.*

Proof. Assume $\mathsf{J}(\mathcal{A}^{\psi_1,\ldots,\psi_n}(\mathcal{M})) = \mathsf{J}(\mathcal{M})$, which is, by Definition 3.1,

$$\mathsf{J}(\mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{A}_1(\mathcal{M}))) = \mathsf{J}(\mathcal{M}). \tag{15}$$

By Theorems 2.30 and 2.15 we must have

$$\mathsf{J}(\mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{A}_1(\mathcal{M}))) \subseteq \mathsf{J}(\mathcal{A}_1(\mathcal{M})) \subseteq \mathsf{J}(\mathcal{M}),$$

which, together with (15), implies

$$\mathsf{J}(\mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{A}_1(\mathcal{M}))) = \mathsf{J}(\mathcal{A}_1(\mathcal{M})) = \mathsf{J}(\mathcal{M}).$$

The claim of the Theorem now follows from Lemmata 3.3 and 3.4.

□

Theorem 3.6: *Let a finite list of sentences $\psi_1, \ldots, \psi_n \in \mathcal{L}_{\mathsf{TT}}$ be given. If $\mathcal{M}$ is a fixpoint of $\mathcal{A}^{\psi_1,\ldots,\psi_n}$ then*

$$\mathcal{M} \models \mathsf{SCA} + \mathsf{Ext} + \bigwedge_{1\leq i\leq n} \psi_i \leftrightarrow \psi_i^+.$$

Proof. Let $\mathcal{M}$ be a fixpoint of $\mathcal{A}^{\psi_1,\ldots,\psi_n}$. Then

$$\mathsf{J}(\mathcal{A}^{\psi_1,\ldots,\psi_n}(\mathcal{M})) = \mathsf{J}(\mathcal{M}),$$

and, as in the proof of the previous Theorem,

$$\mathsf{J}(\mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{A}_1(\mathcal{M}))) = \mathsf{J}(\mathcal{A}_1(\mathcal{M})) = \mathsf{J}(\mathcal{M})$$

and

$$\mathcal{A}_2^{\psi_1,\ldots,\psi_n}(\mathcal{A}_1(\mathcal{M})) = \mathcal{A}_1(\mathcal{M}) = \mathcal{M}.$$

The Claim now follows from Theorems 2.14 and 2.28. □

Definition 3.7: *Let* $\mathsf{FIX}_{\mathcal{A}}$ *be the following assumption:*

For every finite list of sentences $\psi_1, \ldots, \psi_n \in \mathcal{L}_{\mathsf{TT}}$*, there exists a fixpoint of the operation* $\mathcal{A}^{\psi_1,\ldots,\psi_n}$.

Theorem 3.8: NF *is consistent relative to* $\mathsf{ZFC} + \mathsf{FIX}_{\mathcal{A}}$.

Proof. Assume $\mathsf{FIX}_{\mathcal{A}}$. By Theorem 3.6, there is a model of $\mathsf{SCA} + \mathsf{Ext} + \bigwedge_{1\leq i\leq n} \psi_i \leftrightarrow \psi_i^+$ for every finite list $\psi_1, \ldots, \psi_n$ of $\mathcal{L}_{\mathsf{TT}}$-sentences. Then, by compactness, there is a model of $\mathsf{SCA} + \mathsf{Ext} + \mathsf{Amb}$. By Specker's Theorem 1.1, there is a model of NF, i.e. NF is consisent. □

Remark about $\mathsf{FIX}_{\mathcal{A}}$**.** It could tempting to think that since the "value" $\mathsf{J}(\mathcal{M})$ is descending with the operation $\mathcal{A}$ (Theorems 2.30 and 2.15), starting with a countable set (Theorem 1.6), it must have a fixpoint by cardinality argument (using existence of an uncountable ordinal). Unfortunately, the operation $\mathcal{A}$ is defined on $\mathcal{M}$'s, not on $\mathsf{J}(\mathcal{M})$'s, and the "evaluation" J is not one-to-one.

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