

FIXPOINTS OF MODELS CONSTRUCTIONS  
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*Abstract*

We start with Specker’s result that NF is equiconsistent with  $SCA + Ext + Amb$  (SCA is the Simple (intensional) Type Theory). First, there is a model  $\mathcal{M}_0$  of SCA (simple). We define 2 operations  $\mathcal{A}_1$  and  $\mathcal{A}_2$  acting on models of SCA ( $\mathcal{A}_2$  will be parametrized by a finite list  $\psi_1, \dots, \psi_n$  of  $\mathcal{L}_{\top\top}$ -statements, but this is enough by compactness):

- (1)  $\mathcal{M} \models SCA \implies \mathcal{A}_1(\mathcal{M}) \models SCA + Ext$ ;
- (2)  $\mathcal{M} \models SCA \implies \mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{M}) \models SCA + Amb(\psi_1, \dots, \psi_n)$   
 (Jensen-Boffa’s Consis(NFU) proof).

Denote  $\mathcal{A}(\mathcal{M}) := \mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{A}_1(\mathcal{M}))$ . If the operation  $\mathcal{A}$  has a fixpoint (i.e.  $\mathcal{M} \models SCA$  s.t.  $\mathcal{A}(\mathcal{M}) = \mathcal{M}$ ), then this  $\mathcal{M}$  is a model of  $SCA + Ext + Amb(\psi_1, \dots, \psi_n)$ .

For every  $\mathcal{M} \models SCA$  we define a "complexity measure"  $J(\mathcal{M})$  (which is a set) and show that (a)  $J(\mathcal{M}_0)$  is countable; (b)  $J(\mathcal{A}(\mathcal{M})) \subseteq J(\mathcal{M})$ . We also have  $J(\mathcal{A}(\mathcal{M})) = J(\mathcal{M}) \implies \mathcal{A}(\mathcal{M}) = \mathcal{M}$ . It could be tempting to think that  $\mathcal{A}$  must have a fixpoint by cardinality argument (using existence of an uncountable ordinal), but in reality this is not clear.

The Axiom of Choice of ZFC is used for defining the operations  $\mathcal{A}_1$  and  $\mathcal{A}_2$ .

To conclude, NF is consistent assuming that such a fixpoint always (i.e. for every finite list  $\psi_1, \dots, \psi_n$ ) exists.

1. *Background*

Our metatheory is ZFC.  $\mathcal{L}_{\top\top}$  is the language of Simple Type Theory.

*Comprehension* SCA is an axiom scheme

$$SCA^n : \quad \exists y^{n+1} \forall x^n (x \in y \leftrightarrow \varphi[x]),$$

with  $y^{n+1}$  not free in  $\varphi$ , for every  $n$  and for every formula  $\varphi \in \mathcal{L}_{\text{TT}}$ .

*Extensionality* Ext is an axiom scheme

$$\text{Ext}^n : \quad \forall x^{n+1} \forall y^{n+1} \left( \forall z^n (z \in x \leftrightarrow z \in y) \rightarrow x = y \right),$$

for every  $n$ .

Given an  $\mathcal{L}_{\text{TT}}$ -formula  $\varphi$ , by  $\varphi^+$  we denote the result of raising all type indices in  $\varphi$  by 1. The *Ambiguity* scheme is

$$\text{Amb} : \quad \varphi \leftrightarrow \varphi^+,$$

for all statements  $\varphi \in \mathcal{L}_{\text{TT}}$ .

*Theorem 1.1:* (Specker) NF is equiconsistent with SCA + Ext + Amb.

*Proof.* This follows from Specker's result [6]. A different, proof-theoretic, proof of this fact can be found in [2].  $\square$

*Definition 1.2:* A typed structure is a set  $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i} \rangle \mid i \in \mathbb{N}\}$  s.t.  $j_0 < \dots < j_n < \dots$  is an increasing sequence of natural numbers,  $M^{j_0} \neq \emptyset$ , and  $\forall i \in \mathbb{N} \in^{j_i} \subseteq M^{j_i} \times M^{j_{i+1}}$ .

*Definition 1.3:* Assume that  $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i} \rangle \mid i \in \mathbb{N}\}$  is a typed structure and  $i \in \mathbb{N}$ .

- (1) For  $x \in M^{j_i}, y \in M^{j_i}, \mathcal{M} \models x = y$  means  $x = y$ .
- (2) For  $x \in M^{j_i}, y \in M^{j_{i+1}}, \mathcal{M} \models x \in y$  means  $\langle x, y \rangle \in \in^{j_i}$ .

For any  $\varphi \in \mathcal{L}_{\text{TT}}, \mathcal{M} \models \varphi$  is now defined in the standard way.

*Lemma 1.4:* If  $\mathcal{M} \models \text{SCA}$  then  $\forall i \in \mathbb{N} M^{j_i} \neq \emptyset$ .

*Proof.* For  $i = 0$  the condition is given by Definition 1.2. For  $i + 1$  we use the fact

$$\mathcal{M} \models \exists y^{i+1} \forall x^i (x \in y \leftrightarrow x = x).$$

$\square$

*Definition 1.5:* Let  $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i} \rangle \mid i \in \mathbb{N}\}$  be a typed structure. The join  $J(\mathcal{M})$  of  $\mathcal{M}$  is defined by

$$J(\mathcal{M}) := \{\langle j_i, x \rangle \mid i \in \mathbb{N} \wedge x \in M^{j_i}\}.$$

*Theorem 1.6: There exists a model  $\mathcal{M}_0$  of SCA with countable join.*

*Proof.* Take  $\mathcal{M}_0 := \{\langle \mathcal{P}^{i+1}(\emptyset), \in \rangle \mid i \in \mathbb{N}\}$ . □

## 2. Operations $\mathcal{A}_1$ and $\mathcal{A}_2$

### 2.1. Operation $\mathcal{A}_1$ : securing Extensionality

In this subsection we are assuming that  $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i} \rangle \mid i \in \mathbb{N}\}$  is a model of SCA.

*Definition 2.1: Set*

$$\sim^{j_0} := \{\langle x, y \rangle \mid x \in M^{j_0} \wedge y \in M^{j_0}\}; \quad (1)$$

$$\begin{aligned} \sim^{j_{i+1}} &:= \{\langle x, y \rangle \in M^{j_{i+1}} \times M^{j_{i+1}} \mid \\ &\forall x' \in^{j_i} x \exists y' \in^{j_i} y \ x' \sim^{j_i} y' \bigwedge \forall y' \in^{j_i} y \exists x' \in^{j_i} x \ x' \sim^{j_i} y'\}; \end{aligned} \quad (2)$$

$$\tilde{\in}^{j_i} := \{\langle x, y \rangle \in M^{j_i} \times M^{j_{i+1}} \mid \exists z \in^{j_i} y \ x \sim^{j_i} z\}. \quad (3)$$

*Definition 2.2: A weak typed structure (wts) is a set*

$$\mathcal{N} = \{\langle N^{j_k}, \langle =^{j_k}, \varepsilon^{j_k} \rangle \rangle \mid k \in \mathbb{N}\}$$

*s.t.  $j_0 < \dots < j_n < \dots$  is an increasing sequence of natural numbers,  $M^{j_0} \neq \emptyset$ ,  $\forall k \in \mathbb{N} (\ =^{j_k} \subseteq N^{j_k} \times N^{j_k} \wedge \varepsilon^{j_k} \subseteq N^{j_k} \times N^{j_{k+1}})$ , and all equality axioms are satisfied, i.e. for all  $k \in \mathbb{N}$  and all  $x, y, z \in N^{j_k}$ ,  $u, v \in N^{j_{k+1}}$ , the following hold:*

$$x =^{j_k} x; \quad (4)$$

$$x =^{j_k} y \rightarrow y =^{j_k} x; \quad (5)$$

$$x =^{j_k} y \wedge y =^{j_k} z \rightarrow x =^{j_k} z; \quad (6)$$

$$x =^{j_k} y \wedge x \varepsilon^{j_k} u \rightarrow y \varepsilon^{j_k} u; \quad (7)$$

$$x \varepsilon^{j_k} u \wedge u =^{j_{k+1}} v \rightarrow x \varepsilon^{j_k} v. \quad (8)$$

*Definition 2.3: Assume that  $\mathcal{N} = \{\langle N^{j_k}, \langle =^{j_k}, \varepsilon^{j_k} \rangle \rangle \mid k \in \mathbb{N}\}$  is a wts and  $k \in \mathbb{N}$ .*

(1) For  $x \in N^{j_k}$ ,  $y \in N^{j_k}$ ,  $\mathcal{N} \models_w x = y$  means  $\langle x, y \rangle \in =^{j_k}$ .

(2) For  $x \in N^{j_k}, y \in N^{j_{k+1}}, \mathcal{N} \models_w x \in y$  means  $\langle x, y \rangle \in \varepsilon^{j_k}$ .

For any  $\varphi \in \mathcal{L}_{\text{TT}}, \mathcal{M} \models_w \varphi$  is now defined in the standard way.

*Lemma 2.4:*  $\mathcal{M}_w := \{ \langle M^{j_i}, \langle \sim^{j_i}, \tilde{\varepsilon}^{j_i} \rangle \mid i \in \mathbb{N} \}$  is a weak typed structure.

*Proof.* The requirement  $\forall i \in \mathbb{N} (\sim^{j_i} \subseteq M^{j_i} \times M^{j_i} \wedge \tilde{\varepsilon}^{j_i} \subseteq M^{j_i} \times M^{j_{i+1}})$  is immediate from (1)–(3).

Equality axioms (4)–(6) are proved by induction on  $i$ , using defining clauses (1)–(2):

(4): The Claim is obvious for  $i = 0$ . Then,

$$x \sim^{j_{i+1}} x \stackrel{(2)}{\iff} \forall x' \in^{j_i} x \exists y' \in^{j_i} x x' \sim^{j_i} y' \wedge \forall y' \in^{j_i} x \exists x' \in^{j_i} x x' \sim^{j_i} y',$$

and the RHS is true by IH.

(5): The Claim is obvious for  $i = 0$ . Assume  $x \sim^{j_{i+1}} y$ , i.e.

$$\forall x' \in^{j_i} x \exists y' \in^{j_i} y x' \sim^{j_i} y' \wedge \forall y' \in^{j_i} y \exists x' \in^{j_i} x x' \sim^{j_i} y'.$$

Using IH, this implies

$$\forall y' \in^{j_i} y \exists x' \in^{j_i} x y' \sim^{j_i} x' \wedge \forall x' \in^{j_i} x \exists y' \in^{j_i} y y' \sim^{j_i} x',$$

i.e.  $y \sim^{j_{i+1}} x$ .

(6): The Claim is obvious for  $i = 0$ . Assume  $x \sim^{j_{i+1}} y$  and  $y \sim^{j_{i+1}} z$ , i.e.

$$\forall x' \in^{j_i} x \exists y' \in^{j_i} y x' \sim^{j_i} y' \wedge \forall y' \in^{j_i} y \exists x' \in^{j_i} x x' \sim^{j_i} y'$$

and

$$\forall y' \in^{j_i} y \exists z' \in^{j_i} z y' \sim^{j_i} z' \wedge \forall z' \in^{j_i} z \exists y' \in^{j_i} y y' \sim^{j_i} z'.$$

Using IH, this implies

$$\forall x' \in^{j_i} x \exists z' \in^{j_i} z x' \sim^{j_i} z' \wedge \forall z' \in^{j_i} z \exists x' \in^{j_i} x x' \sim^{j_i} z',$$

i.e.  $x \sim^{j_{i+1}} z$ .

Remaining axioms (7)–(8) are proved by using defining clauses (2)–(3) and already established facts (5)–(6):

(7): Assume  $x \sim^{j_i} y$  and  $x \tilde{\epsilon}^{j_i} u$ . By (3) the latter means  $\exists z \in^{j_i} u \ x \sim^{j_i} z$ . For this  $z$ , by (5) and (6) we obtain  $y \sim^{j_i} z$ , i.e.  $y \tilde{\epsilon}^{j_i} u$ .

(8): Assume  $x \tilde{\epsilon}^{j_i} u$  and  $u \sim^{j_{i+1}} v$ . The former means

$$\exists z \in^{j_i} u \ x \sim^{j_i} z.$$

From  $u \sim^{j_{i+1}} v$ ,  $z \in^{j_i} u$  yields

$$\exists w \in^{j_i} v \ z \sim^{j_i} w.$$

Using transitivity (6), we obtain  $x \sim^{j_i} w$ , concluding  $x \tilde{\epsilon}^{j_i} v$ .

□

*Lemma 2.5:  $\mathcal{M}_w$  is a weak model of Extensionality, i.e.*

$$\forall i \in \mathbb{N} \forall x \in M^{j_{i+1}} \forall y \in M^{j_{i+1}} \left( \forall z \in M^{j_i} (z \tilde{\epsilon}^{j_i} x \leftrightarrow z \tilde{\epsilon}^{j_i} y) \longrightarrow x \sim^{j_{i+1}} y \right).$$

*Proof.* Assume

$$i \in \mathbb{N} \wedge x \in M^{j_{i+1}} \wedge y \in M^{j_{i+1}}$$

and

$$\forall z \in M^{j_i} (z \tilde{\epsilon}^{j_i} x \leftrightarrow z \tilde{\epsilon}^{j_i} y).$$

The latter is the same as

$$\forall z (z \tilde{\epsilon}^{j_i} x \leftrightarrow z \tilde{\epsilon}^{j_i} y),$$

which, using reflexivity (4), implies

$$\forall x' \in^{j_i} x \ \exists y' \in^{j_i} y \ x' \sim^{j_i} y' \ \wedge \ \forall y' \in^{j_i} y \ \exists x' \in^{j_i} x \ x' \sim^{j_i} y'.$$

By (2), this is the same as  $x \sim^{j_{i+1}} y$ .

□

*Definition 2.6: In Simple Type Theory, set*

$$\begin{aligned} x^0 \sim^0 y^0 & :\Leftrightarrow \top; \\ x^{i+1} \sim^{i+1} y^{i+1} & :\Leftrightarrow \forall u^i \in^i x^{i+1} \exists v^i \in^i y^{i+1} u \sim^i v \\ & \quad \bigwedge \forall v^i \in^i y^{i+1} \exists u^i \in^i x^{i+1} u \sim^i v; \\ x^i \tilde{\in}^i y^{i+1} & :\Leftrightarrow \exists z^i \in^i y^{i+1} x^i \sim^i z. \end{aligned}$$

Given an  $\mathcal{L}_{\text{TT}}$ -formula  $\varphi$ , the  $\mathcal{L}_{\text{TT}}$ -formula  $\tilde{\varphi}$  is defined by replacing every  $x^i = y^i$  by  $x^i \sim^i y^i$ , and every  $x^i \in y^{i+1}$  by  $x^i \tilde{\in}^i y^{i+1}$ .

*Lemma 2.7: For every  $\mathcal{L}_{\text{TT}}$ -formula  $\varphi$ ,*

$$\mathcal{M}_w \models_w \varphi \iff \mathcal{M} \models \tilde{\varphi}.$$

*Proof.* By induction on  $\varphi$ . The atomic case follows from Definitions 2.1 and 2.6.  $\square$

*Lemma 2.8: For every  $\mathcal{L}_{\text{TT}}$ -formula  $\varphi[x^n]$ ,*

$$\mathcal{M} \models \forall x_1^n \forall x_2^n (x_1 \sim^n x_2 \rightarrow (\tilde{\varphi}[x_1] \leftrightarrow \tilde{\varphi}[x_2])).$$

*Proof.* Since  $\mathcal{M}_w$  is a wts (Lemma 2.4), we have

$$\mathcal{M}_w \models_w \forall x_1^n \forall x_2^n (x_1 = x_2 \rightarrow (\varphi[x_1] \leftrightarrow \varphi[x_2])).$$

The Claim now follows from Lemma 2.7.  $\square$

*Lemma 2.9:  $\mathcal{M}_w$  is a weak model of Comprehension, i.e. for every  $\mathcal{L}_{\text{TT}}$ -formula  $\varphi[x^n]$ ,*

$$\mathcal{M}_w \models_w \exists y^{n+1} \forall x^n (x \in y \leftrightarrow \varphi[x]).$$

*Proof.* By Lemma 2.7 it's enough to prove

$$\mathcal{M} \models \exists y^{n+1} \forall x^n (x \tilde{\in}^n y \leftrightarrow \tilde{\varphi}[x]).$$

Since  $\mathcal{M} \models \text{SCA}$ , take  $y \in M^{j_{n+1}}$  so that

$$\mathcal{M} \models \forall x^n (x \in y \leftrightarrow \tilde{\varphi}[x]).$$

We have to show

$$\mathcal{M} \models \forall x^n (x \in y \leftrightarrow x \tilde{\epsilon}^n y).$$

If  $x \in^{j_n} y$ , then  $x \tilde{\epsilon}^{j_n} y$  follows by reflexivity (4). Conversely, assuming  $x \tilde{\epsilon}^{j_n} y$ , we get  $\exists z \in^{j_n} y x \sim^{j_n} z$ , and then  $\mathcal{M} \models \tilde{\varphi}[x]$  and  $x \in^{j_n} y$  by Lemma 2.8. □

*Definition 2.10:* By Lemma 2.4, every  $M^{j_i}$  is divided by  $\sim^{j_i}$  into a set of non-empty equivalence classes. For every  $x \in M^{j_i}$ , we denote

$$[x] := \{x' \mid x' \sim^{j_i} x\}.$$

Define

$$[M^{j_i}] := \{[x] \mid x \in M^{j_i}\},$$

and, for  $[x] \in [M^{j_i}]$ ,  $[y] \in [M^{j_{i+1}}]$ ,

$$[x] [\tilde{\epsilon}^{j_i}] [y] \text{ iff } \forall x' \in [x] \forall y' \in [y] x' \tilde{\epsilon}^{j_i} y'.$$

The typed structure  $[\mathcal{M}]$  is now defined

$$[\mathcal{M}] := \{([M^{j_i}], [\tilde{\epsilon}^{j_i}]) \mid i \in \mathbb{N}\}.$$

*Lemma 2.11:* Let  $\varphi(x_1^{i_1}, \dots, x_k^{i_k})$  be an  $\mathcal{L}_{\text{TT}}$ -formula with all free variables shown, and  $x_1^{i_1} \in M^{j_{i_1}}, \dots, x_k^{i_k} \in M^{j_{i_k}}$ . Then:

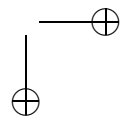
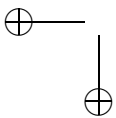
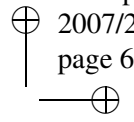
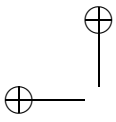
$$\mathcal{M}_w \models_w \varphi(x_1^{i_1}, \dots, x_k^{i_k}) \iff [\mathcal{M}] \models \varphi([x_1^{i_1}], \dots, [x_k^{i_k}]).$$

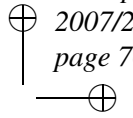
*Proof.* By induction on  $\varphi$ , using non-emptiness of  $[x]$  (Lemma 2.4). The atomic case follows from the equivalences

$$x \sim^{j_i} y \iff [x] = [y]$$

and

$$x \tilde{\epsilon}^{j_i} y \iff [x] [\tilde{\epsilon}^{j_i}] [y],$$





which also follow from Lemma 2.4. □

*Definition 2.12:* Let  $C_1$  be a choice function, picking one element from each equivalence class in  $[M^{j_i}]$ , for all  $i \in \mathbb{N}$ . Define

$$\mathcal{A}_1(M^{j_i}) := \{C_1[x] \mid [x] \in [M^{j_i}]\},$$

and, for  $x \in \mathcal{A}_1(M^{j_i})$ ,  $y \in \mathcal{A}_1(M^{j_{i+1}})$ ,

$$x \epsilon^{j_i} y \text{ iff } [x] [\tilde{C}^{j_i}] [y].$$

The typed structure  $\mathcal{A}_1(\mathcal{M})$  is now defined

$$\mathcal{A}_1(\mathcal{M}) := \{\langle \mathcal{A}_1(M^{j_i}), \epsilon^{j_i} \rangle \mid i \in \mathbb{N}\}.$$

*Lemma 2.13:* Let  $\varphi(x_1^{i_1}, \dots, x_k^{i_k})$  be an  $\mathcal{L}_{\top\top}$ -formula with all free variables shown, and  $x_1^{i_1} \in \mathcal{A}_1(M^{j_{i_1}}), \dots, x_k^{i_k} \in \mathcal{A}_1(M^{j_{i_k}})$ . Then:

$$\mathcal{A}_1(\mathcal{M}) \models \varphi(x_1^{i_1}, \dots, x_k^{i_k}) \iff [\mathcal{M}] \models \varphi([x_1^{i_1}], \dots, [x_k^{i_k}]).$$

*Proof.* By induction on  $\varphi$ . First remember that we always have  $[C_1[x]] = [x]$ . The atomic case

$$C_1[x] = C_1[y] \iff [x] = [y]$$

follows from the fact that  $C_1$  is a choice function, the atomic case

$$C_1[x] \epsilon^{j_i} C_1[y] \iff [x] [\tilde{C}^{j_i}] [y]$$

follows from the Definition 2.12 of  $\epsilon^{j_i}$ . □

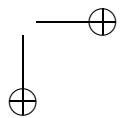
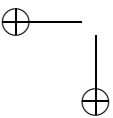
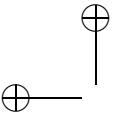
*Theorem 2.14:*  $\mathcal{A}_1(\mathcal{M})$  is a model of SCA + Ext.

*Proof.* Follows from Lemmata 2.5, 2.9, 2.11 and 2.13. □

*Theorem 2.15:*

$$J(\mathcal{A}_1(\mathcal{M})) \subseteq J(\mathcal{M}).$$

*Proof.* From the Definition 2.12 we have  $\mathcal{A}_1(M^{j_i}) \subseteq M^{j_i}$ , for every  $i \in \mathbb{N}$ . The Claim now follows from the Definition 1.5. □





## 2.2. Operation $\mathcal{A}_2$ : securing Ambiguity

*Definition 2.16:* Let  $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i} \rangle \mid i \in \mathbb{N}\}$  be a typed structure and  $i_0 < i_1 < \dots$  be an increasing sequence of natural numbers. We define a typed structure  $\mathcal{N} = \{\langle M^{j_{i_k}}, \in^{j_{i_k}} \rangle \mid k \in \mathbb{N}\}$  as follows: for  $x \in M^{j_{i_k}}$ ,  $y \in M^{j_{i_{k+1}}}$ :  $\mathcal{N} \models x \in y$  iff  $\mathcal{M} \models \exists z (x \in z \wedge \{\dots \{z} \dots\} = y)$ , where the singleton operation is iterated  $i_{k+1} - i_k - 1$  times.

Such a typed structure will be called extracted from  $\mathcal{M}$ , written  $\mathcal{N} \leq \mathcal{M}$ .

*Definition 2.17:* If  $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i} \rangle \mid i \in \mathbb{N}\}$  is a typed structure, then  $\mathcal{M}^+$  denotes the typed structure

$$\mathcal{M}^+ := \{\langle M^{j_{i+1}}, \in^{j_{i+1}} \rangle \mid i \in \mathbb{N}\}.$$

Obviously,  $\leq$  is reflexive and  $\mathcal{M}^+ \leq \mathcal{M}$ . For any  $x$ , we denote  $\{x\}_0 := x$ ,  $\{x\}_{n+1} := \{\{x\}_n\}$ ,  $n \in \mathbb{N}$ .

*Lemma 2.18:* Let  $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i} \rangle \mid i \in \mathbb{N}\}$  be a typed structure. If  $\mathcal{N} \leq \mathcal{M}$ ,  $x \in M^{j_{i_k}}$ ,  $y \in M^{j_{i_{k+1}}}$ , then  $\mathcal{N} \models \{x\} = y$  is equivalent to  $\mathcal{M} \models \{x\}_{i_{k+1}-i_k} = y$ .

*Proof.*

$$\begin{aligned} \mathcal{N} \models \{x\} = y &\iff \mathcal{N} \models x \in y \wedge \forall p (p \in y \rightarrow p = x) \\ &\iff \mathcal{M} \models \exists z (x \in z \wedge \{z\}_{i_{k+1}-i_k-1} = y) \\ &\quad \wedge \forall p (\exists z' (p \in z' \wedge \{z'\}_{i_{k+1}-i_k-1} = y) \rightarrow p = x) \\ &\iff \mathcal{M} \models \{x\}_{i_{k+1}-i_k} = y : \end{aligned}$$

Let's check the last  $\iff$ . Reason in  $\mathcal{M}$ . Assume LHS. For that  $z$ , we already have  $\{z\}_{i_{k+1}-i_k-1} = y$ , and it remains to show  $z = \{x\}$ .  $x \in z$  is given. Assuming  $e \in z$ , and taking in the second part of the conjunction  $p := e$  and  $z' := z$ , we obtain  $e = x$ , q.e.d. Conversely, assume RHS. For  $\exists z (x \in z \wedge \{z\}_{i_{k+1}-i_k-1} = y)$ , take  $z := \{x\}$ . For the second part, if  $\{z'\}_{i_{k+1}-i_k-1} = y$ , then  $z'$  must be  $\{x\}$ , and the only element of  $\{x\}$  is  $x$ , q.e.d. □

*Lemma 2.19:* Let  $\mathcal{M} = \{\langle M^{j_i}, \in^{j_i} \rangle \mid i \in \mathbb{N}\}$  be a typed structure. If  $\mathcal{N} \leq \mathcal{M}$ ,  $x \in M^{j_{i_k}}$ ,  $y \in M^{j_{i_{k+n}}}$ , then  $\mathcal{N} \models \{x\}_n = y$  is equivalent to  $\mathcal{M} \models \{x\}_{i_{k+n}-i_k} = y$ .

*Proof.* If  $n = 0$ , the assertion is obvious. For  $n > 0$ , apply the previous Lemma  $n$  times.  $\square$

*Lemma 2.20:*  $\leq$  is transitive.

*Proof.* Let  $\mathcal{N} \leq \mathcal{M}$  and  $\mathcal{O} \leq \mathcal{N}$ . We need to show  $\mathcal{O} \leq \mathcal{M}$ .

First, the domain of  $\mathcal{O}$  is  $\{M^{j_{i_{k_l}}} \mid l \in \mathbb{N}\}$ . Assume  $x \in M^{j_{i_{k_l}}}$ ,  $y \in M^{j_{i_{k_{l+1}}}}$ . Now compute:

$$\begin{aligned} \mathcal{O} \models x \in y &\stackrel{\mathcal{O} \leq \mathcal{N}}{\iff} \mathcal{N} \models \exists z (x \in z \wedge \{z\}_{k_{l+1}-k_l-1} = y) \\ &\stackrel{\mathcal{N} \leq \mathcal{M}, \text{L.2.19}}{\iff} \mathcal{M} \models \exists z (\exists z_1 (x \in z_1 \wedge \{z_1\}_{i_{k_{l+1}}-i_{k_l}-1} = z) \\ &\quad \wedge \{z\}_{i_{k_{l+1}}-i_{k_{l+1}}} = y) \\ &\iff \mathcal{M} \models \exists z_1 (x \in z_1 \wedge \{z_1\}_{i_{k_{l+1}}-i_{k_l}-1} = y), \end{aligned}$$

confirming that  $\mathcal{O} \leq \mathcal{M}$ .  $\square$

*Lemma 2.21:* If  $\mathcal{N} \leq \mathcal{M}$  and  $\mathcal{M} \models \text{SCA}$ , then  $\mathcal{N} \models \text{SCA}$ .

*Proof.* Assume  $\mathcal{N} \leq \mathcal{M}$  and  $\mathcal{M} \models \text{SCA}$ . Let  $\varphi[x^k] \in \mathcal{L}_{\text{TT}}$ . We need to show

$$\mathcal{N} \models \exists y^{k+1} \forall x^k (x \in y \leftrightarrow \varphi[x]). \quad (9)$$

Let  $\varphi^{\mathcal{N}}$  be obtained from  $\varphi$  by replacing every variable  $x^l$  by  $x^{i_l}$ , and replacing every  $x^l \in y^{l+1}$  by  $\exists z^{i_{l+1}} (x^{i_l} \in z \wedge \{z\}_{i_{l+1}-i_l-1} = y)$ . Then  $\varphi^{\mathcal{N}} \in \mathcal{L}_{\text{TT}}$ . Rephrasing (9), we need to show

$$\mathcal{M} \models \exists y^{i_{k+1}} \forall x^{i_k} (\exists z^{i_{k+1}} (x \in z \wedge \{z\}_{i_{k+1}-i_k-1} = y) \leftrightarrow \varphi^{\mathcal{N}}[x]). \quad (10)$$

First, since  $\mathcal{M} \models \text{SCA}$ , we have

$$\mathcal{M} \models \exists y_1^{i_{k+1}} \forall x^{i_k} (x \in y_1 \leftrightarrow \varphi^{\mathcal{N}}[x]). \quad (11)$$

Take  $y^{i_{k+1}} := \{y_1\}_{i_{k+1}-i_k-1}$ . Observe

$$\mathcal{M} \models \forall x^{i_k} \left( x \in y_1 \leftrightarrow \exists z^{i_{k+1}} (x \in z \wedge \{z\}_{i_{k+1}-i_k-1} = y) \right) : (12)$$

Reason in  $\mathcal{M}$ . Assume  $x \in y_1$ . Then RHS is satisfied by taking  $z := y_1$ . Conversely, assume  $\exists z^{i_{k+1}} (x \in z \wedge \{z\}_{i_{k+1}-i_k-1} = y)$ . Then it must be  $z = y_1$  and  $x \in y_1$ . Q.E.D.

(12) and (11) now imply (10) and (9). □

*Definition 2.22:* For any  $\mathcal{M} \models \text{SCA}$  and any sentence  $\psi \in \mathcal{L}_{\text{TT}}$ , let us say that  $\mathcal{M}$  forces  $\psi$  when  $\psi$  is true in every typed structure extracted from  $\mathcal{M}$ , and that  $\mathcal{M}$  decides  $\psi$  when  $\mathcal{M}$  forces either  $\psi$  or  $\neg\psi$ .

*Remark 2.23:* If  $\mathcal{M}$  decides  $\psi$ , then  $\mathcal{M} \models \psi \leftrightarrow \psi^+$ .

*Proof.* Remember  $\mathcal{M}^+ \leq \mathcal{M}$ . □

*Lemma 2.24:* (Extraction Lemma, Boffa [1]) Given any  $\mathcal{M} \models \text{SCA}$  and any sentence  $\psi \in \mathcal{L}_{\text{TT}}$ , there is a model  $\mathcal{N} \models \text{SCA}$  with  $\mathcal{N} \leq \mathcal{M}$  which decides  $\psi$ .

*Proof.* Let  $k$  be greater than all type indices appearing in  $\psi$ . Define a partition  $G_1, G_2$  of  $[\mathbb{N}]^{k+1}$  as follows:

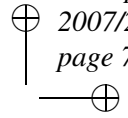
$$\begin{aligned} G_1 &:= \{i_0 < i_1 < \dots < i_k \mid \langle M^{j_{i_0}}, M^{j_{i_1}}, \dots, M^{j_{i_k}}, \dots \rangle \models \psi\}, \\ G_2 &:= \{i_0 < i_1 < \dots < i_k \mid \langle M^{j_{i_0}}, M^{j_{i_1}}, \dots, M^{j_{i_k}}, \dots \rangle \models \neg\psi\}. \end{aligned}$$

By Ramsey's theorem (cf. [5]), take an infinite set  $X$  of natural numbers  $i_0 < i_1 < \dots < i_n < \dots$  such that  $[X]^{k+1} \subseteq G_1$  or  $[X]^{k+1} \subseteq G_2$ , and set  $\text{dom}(\mathcal{N}) := \langle M^{j_{i_0}}, M^{j_{i_1}}, \dots, M^{j_{i_n}}, \dots \rangle$ . In the first case ( $[X]^{k+1} \subseteq G_1$ )  $\mathcal{N}$  forces  $\psi$ , and in the second case  $\mathcal{N}$  forces  $\neg\psi$ . □

*Lemma 2.25:* Given any  $\mathcal{M} \models \text{SCA}$  and any sentence  $\psi \in \mathcal{L}_{\text{TT}}$ , there is a model  $\mathcal{N} \models \text{SCA} + \psi \leftrightarrow \psi^+$  with  $\mathcal{N} \leq \mathcal{M}$ .

*Proof.* Corollary of Lemma 2.24 and Remark 2.23. □

*Lemma 2.26:* Given any  $\mathcal{M} \models \text{SCA}$  and any finite list of sentences  $\psi_1, \dots, \psi_n \in \mathcal{L}_{\text{TT}}$ , there is a model  $\mathcal{N} \models \text{SCA} + \bigwedge_{1 \leq i \leq n} \psi_i \leftrightarrow \psi_i^+$  with  $\mathcal{N} \leq \mathcal{M}$ .



*Proof.* Apply Lemma 2.25  $n$  times. Use transitivity of  $\leq$  (Lemma 2.20).  $\square$

*Definition 2.27:* Let a finite list of sentences  $\psi_1, \dots, \psi_n \in \mathcal{L}_{\text{TT}}$  be given. Let  $\mathcal{A}_2^{\psi_1, \dots, \psi_n}$  be a choice function such that

$$\text{if } \mathcal{M} \models \text{SCA} \text{ then } \mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{M}) \models \text{SCA}+$$

$$\bigwedge_{1 \leq i \leq n} \psi_i \leftrightarrow \psi_i^+ \text{ and } \mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{M}) \leq \mathcal{M}.$$

*Theorem 2.28:* Let a finite list of sentences  $\psi_1, \dots, \psi_n \in \mathcal{L}_{\text{TT}}$  be given. If  $\mathcal{M} \models \text{SCA}$  then

$$\mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{M}) \models \text{SCA} + \bigwedge_{1 \leq i \leq n} \psi_i \leftrightarrow \psi_i^+ \text{ and } \mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{M}) \leq \mathcal{M}.$$

*Proof.* Follows from Definition 2.27.  $\square$

*Lemma 2.29:* If  $\mathcal{N} \leq \mathcal{M}$  then  $J(\mathcal{N}) \subseteq J(\mathcal{M})$ .

*Proof.* By the Definition 2.16, the domain of  $\mathcal{N}$  is just a subsequence of the domain of  $\mathcal{M}$ .  $\square$

*Theorem 2.30:* For any finite list of sentences  $\psi_1, \dots, \psi_n \in \mathcal{L}_{\text{TT}}$ , if  $\mathcal{M} \models \text{SCA}$  then

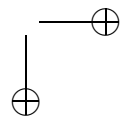
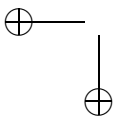
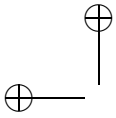
$$J(\mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{M})) \subseteq J(\mathcal{M}).$$

*Proof.* Follows from Theorem 2.28 and Lemma 2.29.  $\square$

### 3. Conclusion

*Definition 3.1:* Let a finite list of sentences  $\psi_1, \dots, \psi_n \in \mathcal{L}_{\text{TT}}$  be given. For  $\mathcal{M} \models \text{SCA}$  we define

$$\mathcal{A}^{\psi_1, \dots, \psi_n}(\mathcal{M}) := \mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{A}_1(\mathcal{M})).$$



*Definition 3.2:* Let a finite list of sentences  $\psi_1, \dots, \psi_n \in \mathcal{L}_{\text{TT}}$  be given.  $\mathcal{M} \models \text{SCA}$  is a fixpoint of  $\mathcal{A}^{\psi_1, \dots, \psi_n}$  iff

$$\mathcal{A}^{\psi_1, \dots, \psi_n}(\mathcal{M}) = \mathcal{M}.$$

*Lemma 3.3:* If  $\mathcal{M} \models \text{SCA}$  and  $J(\mathcal{A}_1(\mathcal{M})) = J(\mathcal{M})$  then  $\mathcal{A}_1(\mathcal{M}) = \mathcal{M}$ .

*Proof.* Assume  $J(\mathcal{A}_1(\mathcal{M})) = J(\mathcal{M})$ . By Definitions 1.5 and 2.12, this implies

$$\forall i \in \mathbb{N} \mathcal{A}_1(M^{j_i}) = M^{j_i},$$

which, using Definitions 2.12 and 2.10, further yields

$$\forall i \in \mathbb{N} \forall x \in M^{j_i} \forall x' \in M^{j_i} (x \sim^{j_i} x' \leftrightarrow x = x'). \quad (13)$$

By Definition 2.1 furthermore we have

$$\forall i \in \mathbb{N} \forall x \in M^{j_i} \forall y \in M^{j_{i+1}} (x \tilde{\epsilon}^{j_i} y \leftrightarrow x \in^{j_i} y). \quad (14)$$

(13) and (14) confirm that  $\mathcal{M}$  and  $\mathcal{A}_1(\mathcal{M})$  is the same set. □

*Lemma 3.4:* Let a finite list of sentences  $\psi_1, \dots, \psi_n \in \mathcal{L}_{\text{TT}}$  be given and  $\mathcal{M} \models \text{SCA}$ . If  $J(\mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{M})) = J(\mathcal{M})$  then  $\mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{M}) = \mathcal{M}$ .

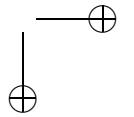
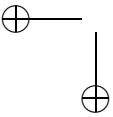
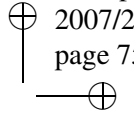
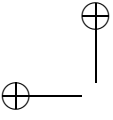
*Proof.* Assume that  $\mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{M})$  is given by an increasing sequence  $\{i_k \mid k \in \mathbb{N}\}$ . Assume  $J(\mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{M})) = J(\mathcal{M})$ , which is, by Definition 1.5,

$$\{\langle j_i, x \rangle \mid i \in \mathbb{N} \wedge x \in M^{j_i}\} = \{\langle j_{i_k}, x \rangle \mid k \in \mathbb{N} \wedge x \in M^{j_{i_k}}\}.$$

Claim.  $\forall k \in \mathbb{N} i_k = k$ .

/- By induction on  $k$ . First denote

$$\begin{aligned} A &:= \{\langle j_i, x \rangle \mid i \in \mathbb{N} \wedge x \in M^{j_i}\}, \\ B &:= \{\langle j_{i_k}, x \rangle \mid k \in \mathbb{N} \wedge x \in M^{j_{i_k}}\}. \end{aligned}$$



*Ind. base:* Take any  $x \in M^{j_0}$  (by Lemma 1.4 such an  $x$  exists). Then  $\langle j_0, x \rangle \in A$ , so we must have  $\langle j_0, x \rangle \in B$ . If  $i_0 > 0$ , then  $j_{i_0} > j_0$ , and  $j_{i_k} \geq j_{i_0} > j_0$  for every  $k$ . Therefore  $i_0 = 0$  must hold.

*Ind. step:* Take any  $x \in M^{j_{k+1}}$  (by Lemma 1.4 such an  $x$  exists). Then  $\langle j_{k+1}, x \rangle \in A$ , so we must have  $\langle j_{k+1}, x \rangle \in B$ . If  $i_{k+1} > k + 1$ , then  $j_{i_{k+1}} > j_{k+1}$ , and  $j_{i_{k'}} \geq j_{i_{k+1}} > j_{k+1}$  for every  $k' \geq k + 1$ . On the other hand, by IH we have  $i_{k'} = k'$  and  $j_{i_{k'}} = j_{k'} < j_{k+1}$  for every  $k' < k + 1$ . Since we always have  $i_{k+1} \geq k + 1$ , it remains to conclude that  $i_{k+1} = k + 1$ .

-/

Since  $\forall k \in \mathbb{N} i_k = k$ , by Definition 2.16 the  $\in$  relation is the same in  $\mathcal{M}$  and  $\mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{M})$ , so  $\mathcal{M}$  and  $\mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{M})$  is the same set.

□

*Theorem 3.5:* Let a finite list of sentences  $\psi_1, \dots, \psi_n \in \mathcal{L}_{\text{TT}}$  be given and  $\mathcal{M} \models \text{SCA}$ . If  $J(\mathcal{A}^{\psi_1, \dots, \psi_n}(\mathcal{M})) = J(\mathcal{M})$  then  $\mathcal{A}^{\psi_1, \dots, \psi_n}(\mathcal{M}) = \mathcal{M}$ .

*Proof.* Assume  $J(\mathcal{A}^{\psi_1, \dots, \psi_n}(\mathcal{M})) = J(\mathcal{M})$ , which is, by Definition 3.1,

$$J(\mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{A}_1(\mathcal{M}))) = J(\mathcal{M}). \quad (15)$$

By Theorems 2.30 and 2.15 we must have

$$J(\mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{A}_1(\mathcal{M}))) \subseteq J(\mathcal{A}_1(\mathcal{M})) \subseteq J(\mathcal{M}),$$

which, together with (15), implies

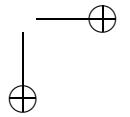
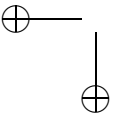
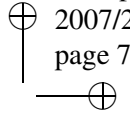
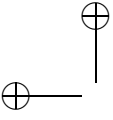
$$J(\mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{A}_1(\mathcal{M}))) = J(\mathcal{A}_1(\mathcal{M})) = J(\mathcal{M}).$$

The claim of the Theorem now follows from Lemmata 3.3 and 3.4.

□

*Theorem 3.6:* Let a finite list of sentences  $\psi_1, \dots, \psi_n \in \mathcal{L}_{\text{TT}}$  be given. If  $\mathcal{M}$  is a fixpoint of  $\mathcal{A}^{\psi_1, \dots, \psi_n}$  then

$$\mathcal{M} \models \text{SCA} + \text{Ext} + \bigwedge_{1 \leq i \leq n} \psi_i \leftrightarrow \psi_i^+.$$



*Proof.* Let  $\mathcal{M}$  be a fixpoint of  $\mathcal{A}^{\psi_1, \dots, \psi_n}$ . Then

$$J(\mathcal{A}^{\psi_1, \dots, \psi_n}(\mathcal{M})) = J(\mathcal{M}),$$

and, as in the proof of the previous Theorem,

$$J(\mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{A}_1(\mathcal{M}))) = J(\mathcal{A}_1(\mathcal{M})) = J(\mathcal{M})$$

and

$$\mathcal{A}_2^{\psi_1, \dots, \psi_n}(\mathcal{A}_1(\mathcal{M})) = \mathcal{A}_1(\mathcal{M}) = \mathcal{M}.$$

The Claim now follows from Theorems 2.14 and 2.28.  $\square$

*Definition 3.7:* Let  $\text{FIX}_{\mathcal{A}}$  be the following assumption:

*For every finite list of sentences  $\psi_1, \dots, \psi_n \in \mathcal{L}_{\text{TT}}$ , there exists a fixpoint of the operation  $\mathcal{A}^{\psi_1, \dots, \psi_n}$ .*

*Theorem 3.8:* NF is consistent relative to  $\text{ZFC} + \text{FIX}_{\mathcal{A}}$ .

*Proof.* Assume  $\text{FIX}_{\mathcal{A}}$ . By Theorem 3.6, there is a model of  $\text{SCA} + \text{Ext} + \bigwedge_{1 \leq i \leq n} \psi_i \leftrightarrow \psi_i^+$  for every finite list  $\psi_1, \dots, \psi_n$  of  $\mathcal{L}_{\text{TT}}$ -sentences. Then, by compactness, there is a model of  $\text{SCA} + \text{Ext} + \text{Amb}$ . By Specker's Theorem 1.1, there is a model of NF, i.e. NF is consistent.  $\square$

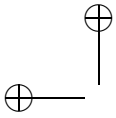
**Remark about  $\text{FIX}_{\mathcal{A}}$ .** It could be tempting to think that since the "value"  $J(\mathcal{M})$  is descending with the operation  $\mathcal{A}$  (Theorems 2.30 and 2.15), starting with a countable set (Theorem 1.6), it must have a fixpoint by cardinality argument (using existence of an uncountable ordinal). Unfortunately, the operation  $\mathcal{A}$  is defined on  $\mathcal{M}$ 's, not on  $J(\mathcal{M})$ 's, and the "evaluation"  $J$  is not one-to-one.

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