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CURRY-TYPE PARADOXES

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Abstract

Curry's paradox primarily concerns some versions of illative combinatory logic but, of course, also systems based on untyped λ-calculi. Typically, the paradox is studied with an eye toward compatibility of a logic with naive set theory. Our analysis emphasizes recursive equations together with the logical theorems and rules involved. We formulate some *new* paradoxes: one of them relies on *reductio*, another shows how to use the `if_then_else_` type constructor and the *double fixed point theorem* to prove q.

1. *Introduction*

H. B. Curry invented the argument that became known as "Curry's paradox." This paradox is based on few logical principles and rules, which have been questioned rarely. Roughly speaking, only the *contraction* axiom and *modus ponens* is needed. The paradox presented in various ways in the literature and it has been observed that various other logical principles lead similarly to inconsistency. Unfortunately, while Russell's paradox is famous and widely known, Curry's paradox seems to be frequently forgotten or overlooked despite its elegance and importance.

First, we recall some "variants" of the paradox from the literature. Then we introduce three new paradoxes that share certain properties with Curry's reasoning.

2. *Curry's paradox(es)*

2.1. *The implicational type of* W

The following natural deduction proof of Curry's paradox is reproduced nearly verbatim from one of the appendixes of [2]. (We changed only the letters to ps and qs.) Based on the fixed point lemma, there is a p such that

$p = p \to q$. This equality is used twice in the derivation, in the first and in the last but one step.[1]

Proof 1:

$$\dfrac{\dfrac{\dfrac{\dfrac{[p]^1 \qquad \dfrac{[p]^1}{p \to q}}{q}}{p \to q}\,{}^{1}}{p \quad p \to q}}{q}$$

A more transparent presentation of — essentially — the same proof is proof 2 in a Fitch-style natural deduction system.

Proof 2:

1.	$\vert\ p$	[assumption]
2.	$\vert\ p \to q$	[replacement according to $p = p \to q$]
3.	$\vert\ q$	[MP 1, 2]
4.	$p \to q$	[$\to$I [1], 3]
5.	p	[replacement according to $p = p \to q$]
6.	q	[MP 4, 5]

The Fitch-style system shows clearly which formulas depend on the assumptions; in particular, 3 doubly depends on 1, which is an analogue of contraction. However, the natural deduction systems hide the fact that the simple type of the identity combinator is used too.

The connection between the implicational fragment of intuitionistic logic ($H_\to$) and simple typed combinatory logic (or the λK-calculus) is well-known. (Perhaps, intuitionistic beliefs are the reason why the proof steps are not scrutinized in [2].) There is a connection between the implicational fragment of relevance logic ($R_\to$) and simple typed combinatory logic with base $\{\, \mathsf{B}, \mathsf{W}, \mathsf{C}, \mathsf{I} \,\}$ (or the λI-calculus) that is similar to the $H_\to$–λK relationship. Relevance logics were motivated by the aim of avoiding the so-called paradoxes of material implication including $p \wedge {\sim} p \to q$. Thus, it is not surprising that [9] considers Curry's paradox, which cannot be avoided by substituting $R_\to$ for $H_\to$, and then introduces a new conjunction–implication paradox, that is closely related to Curry's. (We recall these two arguments as proofs 3 and 6.) The impetus of [9] is an investigation of the possibility of

[1] The astute reader might detect a slight incoherence in proof 1 between steps 2 and 4.

naive set theory via limiting the logical system, hence, the proofs are phrased appealing to unrestricted comprehension rather that the fixed point lemma.

Proof 3:

1.	$C =_{df} \{ x\colon x \in x \to (x \in x \to q) \}$	[comprehension]
2.	$(p \to (p \to q)) \to (p \to q)$	[type of W, (axiom)]
3.	$C \in C \leftrightarrow (C \in C \to (C \in C \to q))$	[1, set membership]
4.	$(C \in C \to (C \in C \to q)) \to (C \in C \to q)$	[instance of 2]
5.	$C \in C \to (C \in C \to q)$	[replacement in 4 acc. 3]
6.	$C \in C$	[replacement in 5 acc. 3]
7.	$C \in C \to q$	[MP 6, 5]
8.	q	[MP 6, 7]

Note that the annotation is not unique, in the sense that other justifications could be given for some of the steps assuming classical, intuitionistic or relevance logic. For instance, 5 may be obtained from 3 and 4 by transitivity of implication. Similarly, instead of replacement modus ponens could yield 6 from 3 and 5. (Of course, the change of the annotation on line 5 would add a principle to those that have been used in the proof.)

The proof's import is that given unrestricted comprehension, contraction, replacement and modus ponens the system becomes *inconsistent*. (Inconsistency, in the absence of negation, of course, means that any formula is provable, and we do not change the meaning of the term when negation is included.) Perhaps, it is useful to emphasize that just as comprehension is unrestricted, the set of theorems is supposed to be closed under substitution, that justifies the insertion of 4 into the proof. Substitution is a fundamental and desirable logical property and hardly avoidable at all.

[11] investigates which implicational formulas (like the type of W) lead to triviality in the context of naïve comprehension. Since in [11] the only connective is $\to$, our results in section 3 are complementary to the proofs of triviality in [11].

We recast proof 3 in an *extended untyped λ-calculus*, which also includes some constants, notably, K and Y.[2] K stands for *truth* and Y is the *fixed point combinator*. This proof quite closely resembles the one in [10], in particular, notationally.[3]

[2] In general, constants and λ-terms are not quite the same, but the differences do not enter into the present considerations, therefore, we use combinators as constants without further ado.

[3] We preserve the dual parenthesis notation of [10] that is very similar to the prefix notation in logic. While the latter allows complete freedom from parentheses, in the λ-calculus the grouping has to be indicated explicitly, because variables have no declared arity. ($M, N, P, Q, \ldots$ range over terms, as usual.)

Proof 4:

1.	$C =_{df} \lambda x.((\to)x)((\to)x)Q$	[abbreviation]
2.	$((\to)((\to)P)((\to)P)Q)((\to)P)Q =_\beta \mathsf{K}$	[type of W]
3.	$(\mathsf{Y})C =_\beta (C)(\mathsf{Y})C$	[Y's axiom]
4.	$(C)(\mathsf{Y})C =_\beta ((\to)(\mathsf{Y})C)((\to)(\mathsf{Y})C)Q$	[1, by β-conv.]
5.	$(\mathsf{Y})C =_\beta ((\to)(\mathsf{Y})C)((\to)(\mathsf{Y})C)Q$	[replacement 3, 4]
6.	$((\to)((\to)(\mathsf{Y})C)((\to)(\mathsf{Y})C)Q)$ $((\to)(\mathsf{Y})C)Q =_\beta \mathsf{K}$	[instance of 2]
7.	$(\mathsf{Y})C =_\beta ((\to)((\to)(\mathsf{Y})C)((\to)(\mathsf{Y})C)Q)$ $((\to)(\mathsf{Y})C)Q$	[replacement 5, 5]
8.	$(\mathsf{Y})C =_\beta \mathsf{K}$	[replacement 6, 7]
9.	$((\to)(\mathsf{Y})C)((\to)(\mathsf{Y})C)Q =_\beta \mathsf{K}$	[replacement 8, 5]
10.	$((\to)(\mathsf{Y})C)Q =_\beta \mathsf{K}$	[MP 8, 9]
11.	$Q =_\beta \mathsf{K}$	[MP 8, 10]

The main moves of the proof are the same as in the previous case, with the slight difference that the emphasis is on equational reasoning. (In particular, $=_\beta$ does not imply that the conjoined terms are sentences.) The last line of the proof means that for any term Q one can prove that it is β-equal to truth, which is a way to state inconsistency in the $\lambda\mathsf{K}$-calculus. A more obvious difference from the former proof is the explicit use of the fixed point combinator Y, which (roughly speaking) is an equivalent of the "self-application" of the set definition in proof 3.

The fixed point combinator *solves recursive equations* of the form $f = Nf$ (where f does not occur in N). The solution for f is $(\mathsf{Y})N$. Indeed, $(\mathsf{Y})N =_\beta (N)(\mathsf{Y})N$ — exactly, as in the concrete equality on line 3 above. One might wonder then what is the equation that is solved by $(\mathsf{Y})C$. The answer is, perhaps, shocking at the first sight, because the equation is

$$P =_\beta (\lambda x.((\to)x)((\to)x)Q)P.$$

With some rewriting into logical notation this is the same as

$$p \leftrightarrow p \to (p \to q).$$

The (classical two-valued) truth table for the formula is Table 1 (p. 231).

The formula $p \leftrightarrow p \to (p \to q)$ is true in exactly one of the four possible cases, and as the italic *1*s show, then q is true as well.

Of course, if instead of classical logic we assume intuitionistic logic, for instance, which is not finitely valued, then no simple truth table can be furnished for the formula; however, a possible worlds model can be given. Let us assume that $w \vDash p \to (p \to (p \to q))$ and $w \vDash (p \to (p \to q)) \to p$.

p	q	p	$\leftrightarrow$	$p \rightarrow$	$(p \rightarrow q)$
1	*1*	1	*1*	1	1
0	1	0	0	1	1
1	0	1	0	0	0
0	0	0	0	1	1

Table 1. Truth table for $p \leftrightarrow (p \rightarrow (p \rightarrow q))$

Using the reflexivity of the accessibility relation repeatedly, we get the successive conditions below.

$$
\begin{array}{l}
w \vDash p \rightarrow (p \rightarrow (p \rightarrow q)) \\
\forall w' \geq w (w' \vDash p \rightarrow w' \vDash p \rightarrow (p \rightarrow q)) \\
\qquad w \vDash p \rightarrow w \vDash p \rightarrow (p \rightarrow q) \\
\forall w' \geq w (w' \vDash p \rightarrow w' \vDash p \rightarrow q) \\
\qquad w \vDash p \rightarrow w \vDash p \rightarrow q \\
\forall w' \geq w (w' \vDash p \rightarrow w' \vDash q) \\
\qquad w \vDash p \rightarrow w \vDash q
\end{array}
$$

$$
\begin{array}{l}
w \vDash (p \rightarrow (p \rightarrow q)) \rightarrow q \\
\forall w' \geq w (w' \vDash p \rightarrow (p \rightarrow q) \rightarrow w' \vDash p) \\
\qquad w \vDash p \rightarrow (p \rightarrow q) \rightarrow w \vDash p \\
\forall w' \geq w (w' \vDash p \rightarrow w' \vDash p \rightarrow q) \\
\qquad w \vDash p \rightarrow w \vDash p \rightarrow q \\
\forall w' \geq w (w' \vDash p \rightarrow w' \vDash q) \\
\qquad w \vDash p \rightarrow w \vDash q
\end{array}
$$

The models show that when both implications are true at w, then the truth of p at w implies the truth of $p \rightarrow (p \rightarrow q)$ at w and vice versa, moreover, q is also true at w.

Before we include further type constructors we return to proofs 1 and 2, because the expression '$p = p \rightarrow q$' differs from '$p \leftrightarrow p \rightarrow (p \rightarrow q)$' by an extra antecedent. Indeed, Curry's original paradox depended on the theorem $p \rightarrow p$ too. Curry later changed the formula in the definition that allowed him to drop the type of I. Proof 5 reconstructs 1 and 2 with set notation.

Proof 5:

1. $C_1 =_{df} \{ x \colon x \in x \rightarrow q \}$ [comprehension]
2. $(p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)$ [W's type, (axiom)]
3. $C_1 \in C_1 \leftrightarrow (C_1 \in C_1 \rightarrow q)$ [1, set membership]

4.	$(C_1 \in C_1 \to (C_1 \in C_1 \to q)) \to (C_1 \in C_1 \to q)$	[instance of 2]
5.	$(C_1 \in C_1 \to C_1 \in C_1) \to (C_1 \in C_1 \to q)$	[replacement in 4 acc. 3]
6.	$p \to p$	[I's type, (axiom)]
7.	$C_1 \in C_1 \to C_1 \in C_1$	[instance of 6]
8.	$C_1 \in C_1 \to q$	[MP 5, 7]
9.	$C_1 \in C_1$	[replacement in 8 acc. 3]
10.	q	[MP 8, 9]

2.2. *Conjunction and fusion*

In the paper [9] it is shown that it is not necessary to appeal to the contraction axiom to collapse naive set theory. Instead idempotence of conjunction and a modus ponens axiom may be used. Proof 6 recalls this paradox.

Proof 6:

1.	$M =_{df} \{ x : x \in x \to q \}$	[comprehension]
2.	$((p \to q) \wedge p) \to q$	[modus ponens axiom]
3.	$(p \wedge p) \leftrightarrow p$	[idempotence of $\wedge$]
4.	$M \in M \leftrightarrow (M \in M \to q)$	[1, set membership]
5.	$((M \in M \to q) \wedge M \in M) \to q$	[instance of 2]
6.	$(M \in M \wedge M \in M) \to q$	[replacement in 5 acc. 4]
7.	$(M \in M \wedge M \in M) \leftrightarrow M \in M$	[instance of 3]
8.	$M \in M \to q$	[replacement in 6 acc. 7]
9.	$M \in M$	[replacement in 8 acc. 4]
10.	q	[MP 9, 8]

Some of the annotations could be varied in this proof too. Notably, 3 could be weakened to $p \to (p \wedge p)$ and then 8 could be obtained from 7 and 6 by transitivity of $\to$.

This paradox is not astonishing after the shift in the understanding of $\wedge$ as a type from pairing to intersection. It is well-known that the combinator WI (or W_*) has no purely implicational type. Obtaining 5 from 4 (in proof 5) might be viewed as a fortunate modification to an instance of W that allows a detachment of a self-implication. Indeed, the modus ponens axiom (2 in proof 6) is an instance of the intersection type of W_*.

The proof in $\lambda\mathsf{K}$-calculus notation is as follows.

Proof 7:

1.	$M =_{df} \lambda x.((\to)x)Q$	[abbreviation]
2.	$((\to)((\wedge)P)((\to)P)Q)Q =_\beta \mathsf{K}$	[modus ponens axiom]

3.	$((\leftrightarrow)((\wedge)P)P)P =_\beta \mathsf{K}$	[idempotence of $\wedge$]
4.	$(\mathsf{Y})M =_\beta (M)(\mathsf{Y})M$	[Y's axiom]
5.	$(M)(\mathsf{Y})M =_\beta ((\to)(\mathsf{Y})M)Q$	[from 1 by β-conv.]
6.	$(\mathsf{Y})M =_\beta ((\to)(\mathsf{Y})M)Q$	[replacement in 4 acc. 5]
7.	$((\to)((\wedge)(\mathsf{Y})M)((\to)(\mathsf{Y})M)Q)Q =_\beta \mathsf{K}$	[instance of 2]
8.	$((\leftrightarrow)((\wedge)(\mathsf{Y})M)(\mathsf{Y})M)(\mathsf{Y})M =_\beta \mathsf{K}$	[instance of 3]
9.	$((\to)((\wedge)(\mathsf{Y})M)(\mathsf{Y})M)Q =_\beta \mathsf{K}$	[replacement in 7 acc. 6]
10.	$((\to)(\mathsf{Y})M)Q =_\beta \mathsf{K}$	[$\leftrightarrow$ replacement 9, 8]
11.	$(\mathsf{Y})M =_\beta \mathsf{K}$	[replacement in 10 acc. 6]
12.	$Q =_\beta \mathsf{K}$	[MP 11, 10]

In nonclassical logics sometimes an intensional conjunction $\circ$ is included in the language, which is called *fusion*. The analogy between $\wedge$ and $\circ$ goes beyond their "label," therefore, one might wonder if reasonable principles involving fusion may lead to inconsistency. Of course, the idea is that the intensional connectives — such as relevant implication and fusion — are stricter that their extensional cousins — such as the classical conditional $\supset$ or conjunction. Thus, it could happen, perhaps, that intensional connectives naturally lack properties that allow the derivation of an arbitrary proposition together with unrestricted comprehension. (For instance, $\circ$ is idempotent only in R-mingle, but not in some other well-known relevant logics as B, T, E, R or L.) We recall a proof from [12], that shows a "weak" relevant logic to be incompatible with naive comprehension. We include into the proof the axioms and rules used, however, we omit listing separately their instances.

Proof 8:

1.	$N =_{df} \{\, x\colon (x \in x \circ x \in x) \to q \,\}$	[comprehension]
2.	$N \in N \leftrightarrow ((N \in N \circ N \in N) \to q)$	[1, set membership]
3.	$p \to p$	[identity axiom]
4.	$(p \leftrightarrow q) \to (p \to q)$	[$\leftrightarrow$ simplification]
5.	$((p \circ q) \to r) \leftrightarrow (p \to (q \to r))$	[residuation axiom]
6.	$((p \to q) \wedge (q \to r)) \to (p \to r)$	[transitivity of $\to$]
7.	$p,\ p \to q \,//\, q$	[modus ponens]
8.	$p \to q,\ p \to r \,//\, p \to (q \wedge r)$	[$\wedge$ introduction]
9.	$p \to q,\ q \to r \,//\, p \to r$	[transitivity of $\to$]
10.	$N \in N \to ((N \in N \circ N \in N) \to q)$	[MP 2, 4]
11.	$N \in N \to (N \in N \to (N \in N \circ N \in N))$	[replacement in 3 acc. 5]
12.	$N \in N \to ((N \in N \to (N \in N \circ N \in N)) \wedge ((N \in N \circ N \in N) \to q))$	[$\wedge$ rule 10, 11]

13. $N \in N \to (N \in N \to q)$ [transitivity 11, 6]
14. $(N \in N \circ N \in N) \to q$ [MP 13, 5]
15. $N \in N$ [replacement in 14 acc. 2]
16. $N \in N \to q$ [MP 15, 13]
17. q [MP 15, 16]

3. *New Curry-type paradoxes*

3.1. *Reductio*

The beauty of Curry's paradox (see proofs 3 and 5) — we think — is that it uses one connective, one axiom and only two rules.[4] The other proofs (6 and 8) used further connectives or further — though undeniably plausible — theorems and rules involving those connectives.

The paradox we now formulate uses one axiom and three rules; however, *negation* is added to the set of connectives.

Proof 9:

1. $O =_{df} \{ x : x \in x \to x \notin x \}$ [comprehension]
2. $(\varphi \to \sim\varphi) \to \sim\varphi$ [reductio axiom]
3. $O \in O \leftrightarrow (O \in O \to O \notin O)$ [1, set membership]
4. $(O \in O \to O \notin O) \to O \notin O$ [instance of 2]
5. $O \in O \to O \notin O$ [replacement in 4 acc. 3]
6. $O \in O$ [replacement in 5 acc. 3]
7. $O \notin O$ [MP 5, 6]
8. $\varphi, \sim\varphi \,//\, \psi$ [contradiction rule]
9. q [rule 8, 5, 6]

First of all note that $(p \to \sim p) \to \sim p$ is valid classically, intuitionistically, and it is also a theorem of $T_{\overset{\sim}{\to}}$ (the implication negation fragment of "ticket entailment"), and all of its extensions, which include such well-known relevance logics as T itself, E (the logic of entailment), and R (the logic of relevant implication), but not L (linear logic). Of course, negation in classical, intuitionistic and relevance logics is three different sorts of negation — just as '$\to$' is not the same connective. As a result 8 is a rule of classical and intuitionistic logics, but only an admissible rule of T, E and R.

Further logics that validate reductio include such 3-valued logics as Heyting's, Bochvar's external logic and Post's system (with one designated value). All these logics assign a designated value to $(p \wedge \sim p) \to q$ as well.

[4] We do not count substitution and to emphasize this we use axiom schemes below.

The λK-calculus version of the proof is as follows.

Proof 10:

1.	$O =_{df} \lambda x.((\rightarrow)x)(\sim)x$	[abbreviation]
2.	$((\rightarrow)((\rightarrow)(\mathsf{Y})O)(\sim)(\mathsf{Y})O)$ $(\sim)(\mathsf{Y})O =_\beta \mathsf{K}$	[instance of reductio axiom]
3.	$(\mathsf{Y})O =_\beta (O)(\mathsf{Y})O$	[instance of Y's axiom]
4.	$(O)(\mathsf{Y})O =_\beta ((\rightarrow)(\mathsf{Y})O)(\sim)(\mathsf{Y})O$	[from 1 by β-conv.]
5.	$(\mathsf{Y})O =_\beta ((\rightarrow)(\mathsf{Y})O)(\sim)(\mathsf{Y})O$	[replacement in 3 acc. 4]
6.	$((\rightarrow)(\mathsf{Y})O)(\sim)(\mathsf{Y})O =_\beta \mathsf{K}$	[replacement in 6 acc. 5]
7.	$(\mathsf{Y})O =_\beta \mathsf{K}$	[replacement in 6 acc. 5]
8.	$\sim(\mathsf{Y})O =_\beta \mathsf{K}$	[MP 8, 7]
9.	$\Phi =_\beta \mathsf{K}, \sim\Phi =_\beta \mathsf{K} \,//\, \Psi =_\beta \mathsf{K}$	[contradiction rule]
10.	$Q =_\beta \mathsf{K}$	[by 10 from 8, 9]

Curry called Y the "paradoxical combinator." The naming seems to us somewhat misleading, because Y leads to *no paradoxes* in pure combinatory logic (or λK-calculus). A closed term — like O — could be more justly called so, although this illative term has to be combined with further quasi-equations to yield a "paradox" or inconsistency.

Perhaps, it is interesting to note that in proofs 9 and 10 the formula $p \leftrightarrow (p \rightarrow \sim p)$ is what corresponds to the recursive equation $f = (\lambda x.\ ((\rightarrow)x)(\sim)x)f$. $p \leftrightarrow (p \rightarrow \sim p)$ is always false classically. Intuitionistically, if $w \vDash p \leftrightarrow (p \rightarrow \sim p)$ then $w \vDash p$ if and only if $w \vDash p \rightarrow \sim p$ follows, however, no possible world satisfies the latter formula, and so $w \vDash p \leftrightarrow (p \rightarrow \sim p)$ implies $w \nvDash p$.

Since the defining formula is similar to that of the Russell set, it is interesting to make a comparison with Russell's paradox.[5] By comprehension, $R =_{\text{df}} \{x \colon x \notin x\}$, and by instantiation $R \in R \leftrightarrow R \notin R$. In the λ-notation, $R =_{\text{df}} \lambda x.(\sim)x$, and the solution of $f = (\lambda x.(\sim)x)f$ by the fixed point combinator is $(\mathsf{Y})R$, which β-equals to $(\sim)(\mathsf{Y})R$. Sometimes no rules or axioms are made explicit, rather it is simply assumed that $p \leftrightarrow \sim p$ is a contradiction and leads to the provability of an arbitrary q. In other words, the derivation up to $(\mathsf{Y})R =_\beta (\sim)(\mathsf{Y})R$ does not establish inconsistency, rather, it proves that a particular set is both an element of itself and not an element of itself.[6]

[5] Even more so, because as it is pointed out in [6], the natural translation of the Russell paradox (more precisely, of $R \in R$) into map theory is $(\lambda x.(\sim)(x)x)\lambda x.(\sim)(x)x$.

[6] We intentionally do not use the term 'class' — which would be the appropriate term according to the NBG set theory — to emphasize the naive point of view. (Not that we think that that is the right one.)

3.2. *Peirce's law*

Now we show that using a variant of the reductio axiom and Peirce's law, together with naive comprehension, inconsistency results. The formulas and the proof contain only negation and implication and the rules of $\leftrightarrow$ weakening to $\rightarrow$, modus ponens and replacement are used.

Proof 11:

1.	$E =_{df} \{\, x\colon (q \rightarrow {\sim}\, x \in x) \rightarrow {\sim}\, x \in x \,\}$	[abbreviation]
2.	$(\varphi \rightarrow \psi) \leftrightarrow ((\psi \rightarrow {\sim}\,\varphi) \rightarrow {\sim}\,\varphi)$	[reductio's variant]
3.	$((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$	[Peirce's law]
4.	$\varphi \leftrightarrow \psi // \varphi \rightarrow \psi$	[$\leftrightarrow$ weakening]
5.	$E \in E \leftrightarrow ((\psi \rightarrow E \notin E) \rightarrow E \notin E)$	[1, set membership]
6.	$E \in E \leftrightarrow (E \in E \rightarrow \psi)$	[replacement in 5 acc. 2]
7.	$(E \in E \rightarrow \psi) \rightarrow E \in E$	[$\leftrightarrow$ weakening, 6]
8.	$E \in E$	[MP 7, 3]
9.	$E \in E \rightarrow (E \in E \rightarrow \psi)$	[$\leftrightarrow$ weakening, 6]
10.	$E \in E \rightarrow \psi$	[MP 8, 9]
11.	ψ	[MP 8, 10]

This proof is interesting, because unlike the previous proof it does not rely on the rule $\varphi, {\sim}\,\varphi \,//\, \psi$, nonetheless, proves inconsistency. Of course, Peirce's law is "very classical" in the sense that it is not a theorem of intuitionistic or relevance logics.[7]

The next proof formalizes the same reasoning in extended λK-calculus.

Proof 12:

1.	$E =_{df} \lambda x.((\rightarrow)((\rightarrow)Q)(\sim)x)(\sim)x$	[abbreviation]
2.	$((\leftrightarrow)((\rightarrow)\Phi)\Psi)$ $((\rightarrow)((\rightarrow)\Psi)(\sim)\Phi)(\sim)\Phi =_{\beta} \mathsf{K}$	[reductio's version]
3.	$((\rightarrow)((\rightarrow)((\rightarrow)\Phi)\Psi)\Phi)\Phi =_{\beta} \mathsf{K}$	[Peirce's law]
4.	$((\leftrightarrow)(\Phi))\Psi =_{\beta} \mathsf{K} \,//\, ((\rightarrow)(\Phi))\Psi =_{\beta} \mathsf{K}$	[$\leftrightarrow$ weakening]
5.	$(\mathsf{Y})E =_{\beta} (E)(\mathsf{Y})E$	[instance of Y's axiom]
6.	$(E)(\mathsf{Y})E =_{\beta}$ $((\rightarrow)((\rightarrow)Q)(\sim)(\mathsf{Y})E)(\sim)(\mathsf{Y})E$	[from 1 by β-conv.]
7.	$(\mathsf{Y})E =_{\beta}$ $((\rightarrow)((\rightarrow)Q)(\sim)(\mathsf{Y})E)(\sim)(\mathsf{Y})E$	[replacement in 5 acc. 6]
8.	$((\leftrightarrow)((\rightarrow)(\mathsf{Y})E)Q)(\mathsf{Y})E =_{\beta} \mathsf{K}$	[replacement in 2 acc. 7]

[7] As it is mentioned in [11], Curry's original "paradoxical set" together with Peirce's law and modus ponens has been shown before to lead to inconsistency.

9.	$((\to)((\to)(\mathsf{Y})E)Q)(\mathsf{Y})E =_\beta \mathsf{K}$	[$\leftrightarrow$ weakening, 8]
10.	$(\mathsf{Y})E =_\beta \mathsf{K}$	[MP 9, 3]
11.	$((\to)(\mathsf{Y})E)Q =_\beta \mathsf{K}$	[replacement in 10 acc. 8]
12.	$Q =_\beta \mathsf{K}$	[MP 10, 11]

In this proof the formula corresponding to the recursive equation solved is $p \leftrightarrow ((q \to \sim p) \to \sim p)$. Again, the truth of this formula in classical logic implies the truth of q.

3.3. *Double fixed points*

Combinatory logic (or $\lambda\mathsf{K}$-calculus) has *double* and *multiple fixed points* — see [13]. The single fixed point theorem in combinatory logic states that $\forall M\, \exists N.\, N = MN$ (where the quantifiers belong to the metalanguage). The double fixed point theorem says that $\forall M_1, M_2\, \exists N_1, N_2$ such that both $N_1 = M_1 N_1 N_2$ and $N_2 = M_2 N_1 N_2$. (For instance, the two terms $\mathsf{Y}(\mathsf{W}(\mathsf{B}M_2(\mathsf{YB}(\mathsf{C}M_1))))$ and $\mathsf{Y}(\mathsf{C}M_1(\mathsf{Y}(\mathsf{W}(\mathsf{B}M_2(\mathsf{YB}(\mathsf{C}M_1))))))$ suffice for a proof, when the combinators in the terms are in the combinatory base.)

We show that the solvability of double recursive equations can also be used to construct a paradox. The sets contain pairs and for easy comprehension we use the '`if`__ `then`__ `else`__' type constructor, that is common in computer science and a basic connective in map theory (cf. [6]). The use of pairs corresponds to taking the fixed points of a pair of recursive equations both of which contain a binary function followed by two arguments (i.e., $((\lambda xy.\, M)N)Q$, where N, Q do not occur in M). The equations are as follows.

$$D_1 = ((\lambda xy.(((\mathtt{IF})x)((\wedge)((\sim)y)Q)(\sim)y)D_1)D_2$$
$$D_1 = ((\lambda xy.(((\mathtt{IF})y)((\wedge)((\sim)x)Q)(\sim)x)D_1)D_2,$$

or in a simpler form, $D_1 = (\lambda xy.\,\mathtt{if}\ x\ \mathtt{then}\ \sim y \wedge Q\ \mathtt{else}\ \sim y)D_1 D_2$ and $D_2 = (\lambda xy.\,\mathtt{if}\ y\ \mathtt{then}\ \sim x \wedge Q\ \mathtt{else}\ \sim x)D_1 D_2$.

Proof 13:

1.	$D_1 =_{df} \{\langle x,y\rangle\colon \mathtt{if}\,\langle x,y\rangle \in x\,\mathtt{then}\,\langle x,y\rangle \notin y \wedge q\,\mathtt{else}\,\langle x,y\rangle \notin y\,\}$	[comp.]
2.	$D_2 =_{df} \{\langle x,y\rangle\colon \mathtt{if}\,\langle x,y\rangle \in y\,\mathtt{then}\,\langle x,y\rangle \notin x \wedge q\,\mathtt{else}\,\langle x,y\rangle \notin x\,\}$	[comp.]
3.	$(\langle D_1,D_2\rangle \in D_1 \vee \langle D_1,D_2\rangle \notin D_1) \wedge (\langle D_1,D_2\rangle \in D_2 \vee \langle D_1,D_2\rangle \notin D_2)$	[excluded middle]

4. $(\langle D_1, D_2\rangle \in D_1 \wedge \langle D_1, D_2\rangle \in D_2)\vee$
$(\langle D_1, D_2\rangle \notin D_1 \wedge \langle D_1, D_2\rangle \in D_2)\vee$
$(\langle D_1, D_2\rangle \in D_1 \wedge \langle D_1, D_2\rangle \notin D_2)\vee$ [distributivity
$(\langle D_1, D_2\rangle \notin D_1 \wedge \langle D_1, D_2\rangle \notin D_2)$ of $\wedge$, $\vee$]

5. $(\langle D_1, D_2\rangle \in D_1 \wedge \langle D_1, D_2\rangle \in D_2)\vee$
$(\langle D_1, D_2\rangle \notin D_1 \wedge (((\langle D_1, D_2\rangle \notin D_1\wedge$
$\langle D_1, D_2\rangle \in D_2 \wedge q) \vee (\langle D_1, D_2\rangle \notin D_1\wedge$
$\langle D_1, D_2\rangle \notin D_2)) \vee (((\langle D_1, D_2\rangle \in D_1\wedge$
$\langle D_1, D_2\rangle \notin D_2 \wedge q) \vee (\langle D_1, D_2\rangle \notin D_1\wedge$
$\langle D_1, D_2\rangle \notin D_2)) \wedge \langle D_1, D_2\rangle \notin D_2\vee$ [replacement in
$\langle D_1, D_2\rangle \notin D_1 \wedge \langle D_1, D_2\rangle \notin D_2$ 4 acc. 1, 2]

6. $(\langle D_1, D_2\rangle \in D_1 \wedge \langle D_1, D_2\rangle \in D_2)\vee$
$(\langle D_1, D_2\rangle \notin D_1 \wedge \langle D_1, D_2\rangle \in D_2\wedge$
$q) \vee (\langle D_1, D_2\rangle \notin D_1 \wedge \langle D_1, D_2\rangle \notin D_2)\vee$
$(\langle D_1, D_2\rangle \in D_1 \wedge \langle D_1, D_2\rangle \notin D_2 \wedge q)\vee$ [distribution,
$(\langle D_1, D_2\rangle \notin D_1 \wedge \langle D_1, D_2\rangle \notin D_2)$ idempotence]

7. $(\langle D_1, D_2\rangle \in D_1 \wedge \langle D_1, D_2\rangle \in D_2)\vee$
$\langle D_1, D_2\rangle \in D_1 \vee \langle D_1, D_2\rangle \in D_2$ [replacement 6, 1, 2]

8. $\langle D_1, D_2\rangle \in D_1 \vee \langle D_1, D_2\rangle \in D_2$ [absorption]

9. $(\langle D_1, D_2\rangle \in D_1 \wedge \langle D_1, D_2\rangle \in D_1)\vee$
$(\langle D_1, D_2\rangle \in D_2 \wedge \langle D_1, D_2\rangle \in D_2)$ [idempotence of $\wedge$]

10. $(\langle D_1, D_2\rangle \in D_1 \wedge \langle D_1, D_2\rangle \in D_1\wedge$
$\langle D_1, D_2\rangle \notin D_2 \wedge q) \vee (\langle D_1, D_2\rangle \in D_1\wedge$
$\langle D_1, D_2\rangle \notin D_1 \wedge \langle D_1, D_2\rangle \in D_2)\vee$ [replacement 9, 1,
$(\langle D_1, D_2\rangle \in D_2 \wedge \langle D_1, D_2\rangle \in D_2)$ distributivity]

11. $(\langle D_1, D_2\rangle \in D_1 \wedge \langle D_1, D_2\rangle \notin D_2 \wedge q)\vee$
$(\langle D_1, D_2\rangle \in D_2 \wedge \langle D_1, D_2\rangle \in D_2)$ [bottom element]

12. $q \vee (\langle D_1, D_2\rangle \in D_2 \wedge \langle D_1, D_2\rangle \in D_2)$ [$\wedge$ elimination]

13. $q \vee (\langle D_1, D_2\rangle \in D_2 \wedge \langle D_1, D_2\rangle \notin D_1\wedge$
$q \wedge \langle D_1, D_2\rangle \in D_2) \vee (\langle D_1, D_2\rangle \notin D_1\wedge$ [rep. 12, 2,
$\langle D_1, D_2\rangle \notin D_2 \wedge \langle D_1, D_2\rangle \in D_2)$ distributivity]

14. $q \vee (\langle D_1, D_2\rangle \in D_2 \wedge \langle D_1, D_2\rangle \notin D_1 \wedge q)$ [bottom element]

15. $q \vee q$ [$\wedge$ elimination]

16. q [idempotence]

Although the proof might look complicated, the theorems justifying the steps are rather simple, for instance, absorption and idempotence of $\vee$. It seems to us that the possibility of arriving at inconsistency assuming the double fixed point theorem is interesting from the point of view of combinatory logic itself. Given a combinatorially complete base, the double fixed point combinators are obviously definable (e.g., from the two terms we gave). Considering the other direction, from the existence of double fixed points the existence of single fixed points follows. (One can find the fixed point of M from the double fixed point of $\mathsf{K}M$, because from $N = \mathsf{K}MNN$ it is immediate that the single fixed point of M is N.) However, it is not obvious — without the proviso of combinatorial completeness — that the fixed point theorems are equipotent.

4. *Conclusion*

We examined Curry's and related paradoxes from the point of view of *recursive equations*. This allowed us to formulate new paradoxes that are somewhat similar to Curry's, but involve *negation*. One of these paradoxes depends only on one axiom — reductio — somewhat similarly as Curry's paradox, at the same time it has a certain resemblance to Russell's paradox too. Lastly, we demonstrated that the existence of double fixed points is incompatible with unrestricted comprehension.

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