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THE SIGNIFICANCE OF YABLO'S PARADOX WITHOUT SELF-REFERENCE

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In [1993] Yablo reports a new paradox. Since it can be presented very briefly, I will recapitulate it here.

For each $i \in \mathbf{N}$, let S_i assert that for all $j > i$, S_j is untrue.

An (infinitary) proof of a contradiction is easily described. Let me start by identifying an approach that seems to me to lead nowhere. One could formalise this in the propositional fragment of $L_{\omega_1 \omega_1}$ as

$$\bigwedge_{i \in N} (p_i \longleftrightarrow (\bigwedge_{j > i} \neg p_j))$$

but all one gets from this is yet another example of a failure of compactness in infinitary languages. This approach does not seem to me to bring out the features of interest in Yablo's idea. For that we have to take S_0 not to be the conjunction of the negations of the later S_i but to be the assertion that they are all false.

There is a more-or-less received modern view of the paradoxes which one associates with the names of Russell and Tarski to the effect that paradox can be evaded if one recognises that every sentence belongs to a particular level in an infinite hierarchy of linguistic levels. The levels may be cumulative, but they must have a strict partial order on them.[1] The idea being that truth-predicates for languages lower down in the hierarchy are to be found only higher up.

It seems to me that the real significance of Yablo's observation is that this is not enough by itself. Evidently the sentences in Yablo's example can be *typed* in the appropriate sense: if S_i belongs to level $-i$ then the family of levels is indexed by the negative integers in such a way that each S_i makes assertions only about S's of strictly lower levels. Nevertheless we still have a paradox! What feature of the spirit of the Russell-Tarski analysis is not being observed to the letter? One obvious oddness is that the poset of types is not wellfounded.

[1] Of course the order is usually assumed to be a total order but that doesn't seem to matter.

Why might this matter? The first thing to notice is that the proof of the paradox is infinitely long. The concept of proof in a language with infinitary rules is not in principle problematic, and in practice humans can apprehend them. Not of course by executing them, but by reasoning in a (finitary) meta-language towards the conclusion that there must be such a (infinitary) proof in the object language. But what is the object language we are considering in this case? S_0 is a formula of a language $\mathcal{L}_0$ which contains *inter alia* a truth predicate for a language $\mathcal{L}_1$ (which contains *inter alia* a truth predicate for a language $\mathcal{L}_2$ ...). A little reflection will reveal to the reader that the subformula relation of each of these languages is *ill-founded.* Although semantics is possible for languages with infinitely long formulæ, this seems to hold only for languages whose infinitary nature arises from the presence of connectives with infinitely many arguments. Languages containing formulæ that are infinite because their subformula relation is illfounded do not in general have an intelligible semantics.

What has our reasoning in a finitary metalanguage actually established? The language in which are reasoning when we persuade ourselves of the paradoxical nature of Yablo's example is a fairly weak one that does not contain truth-predicates for any of the $\mathcal{L}_i$. It doesn't need to. All we are trying to do is show what will happen if these languages do in fact have intelligible semantics, not to reason *with* those semantics. There seem to be two possible conclusions. Either a contradiction is deducible in $\mathcal{L}_0$ (in a suitable weak infinitary sense) or the implicit assumptions of the availability of semantics for $\mathcal{L}_0$ are wrong. I suspect the second (in which case the only paradox we can recover is the failure of compactness minuted above), but it doesn't really matter: either way we conclude that the source of the problem is the illfounded nature of the hierarchy of language levels.

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