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PREFERENCE SEMANTICS FOR DEONTIC LOGIC
PART II – MULTIPLEX MODELS

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Abstract

Part I of this work demonstrated the adequacy of a simple preference semantics for deontic logic, both for standard deontic logic and for a logic that allows for conflicts of obligation, and for both as monadic deontic logics and also as dyadic logics for conditional obligation and for preferability. This part extends those results through the use of 'multiplex' models that apply multiple preference relations to represent pluralities of normative standards. This enables two dual general senses of 'ought' to be distinguished, and it allows for the ranking of normative standards. Logics for these notions are given and shown to be sound and complete with respect to the multiplex preference semantics.

In Part I of this work [3], I presented a basic preference semantics for deontic logic, in which possible worlds are ranked by a preference relation, P,[1] and 'ought' statements are interpreted so that OA is true just in case there is a possible world b where A is true which is such that for every world c that is just as good as b, i.e., where $c\mathrm{P}b$, A must hold at c as well. This basic idea can also be adapted to give models for conditional obligation and for the notion of preferability itself. If the preference relation P is required to be reflexive, transitive and connected on its field, the logic that is determined by this semantics is standard deontic logic in either its monadic or dyadic versions. If, however, the preference relation is allowed not to be connected, then the semantics determines a weaker logic that has the virtue of allowing for conflicts of obligation, cases in which both OA and $O\neg A$ are true. This too appears in both monadic and dyadic deontic versions, which I call P and DP respectively. In [3], as part of proving these latter systems to be complete with respect to the basic preference semantics, I introduced a generalization

[1] I use the letter 'P' to represent this relation, rather than, say, '$\geq$', to prevent some of the later proofs becoming optically oppressive.

of that semantics that utilized multiple preference relations. I call this the 'multiplex' preference semantics for deontic logic. This Part develops the multiplex semantics further and uses it to present two contrasting dual generalized senses of 'ought', which I call the 'indefinite' sense and the 'definite' or 'core' sense. Here, depending on the conditions on the multiple preference relations, the definite sense might follow standard deontic logic even while the indefinite follows the logic P, or both might follow P. It is noteworthy that these generalized senses of 'ought' can only be distinguished in the framework of multiplex models, or something similar. Moreover, while the logic in which both sorts of 'ought' follow P can be interpreted in the preference semantics, it does not have a corresponding Kripke-style multiple relational semantics (described in [2]; cf. also [5]) or even a corresponding neighborhood semantics.

In addition to providing an account of these dual senses of 'ought', the multiplex semantics also enables the characterization of a sense of comparative or ranked obligation that derives from an ordering relation imposed on the multiple preference relations themselves. This reflects the fact that in certain contexts some normative standards or authorities have priority over others. Such a notion of comparative obligation combines naturally with the other generalized oughts.

Section 1 below presents the multiplex preference semantics itself and shows how it leads to the two dual senses of 'ought'. Here we give logics for these operators that are sound and complete with respect to this semantics. These are considered both from the point of view of standard preference relations, which leads to an extension of standard deontic logic, SDL, and from the point of view of weaker preference relations, which leads to an extension of the logic P, described in [3], where conflicts of obligation can occur. Section 2 introduces the ordering of the multiple preference relations and the concomitant sense of comparative or ranked obligation. The logic for this in combination with the monadic 'ought' operators of Section 1 is given and shown to be sound and complete in the multiplex semantics. Again we see the two options of extending standard deontic logic and of extending P.[2] Section 3 considers conditional obligation and preferability within the multiplex framework, for it seems equally apt to distinguish generalized definite and indefinite senses of the dyadic connectives as for the monadic. The picture of the logics that emerges is, however, more complex than the preceding, and the question of their complete characterization remains open.

[2] In [2] I presented some of the present results and sketched their proofs. Here I give more direct proofs, and more details to establish the results.

As in [3], the results presented here are chiefly formal; I do not develop philosophical applications of the multiplex framework in any detail. (See [2] for more discussion along those lines.)

1. *Multiplex Models*

In [3], as part of the proof of completeness for P, I defined 'multiple preference frames'. These are structures $F = \langle W, \mathcal{P} \rangle$, in which W is a non-empty set of points or 'possible worlds', and $\mathcal{P}$ assigns to each $a \in W$ a non-empty set, $\mathcal{P}_a$, of binary relations on W, with the understanding that every relation $\mathrm{P} \in \mathcal{P}_a$ is non-empty. If, for every $a \in W$, every relation $\mathrm{P} \in \mathcal{P}_a$ is reflexive on its field and transitive, the frame is correspondingly called reflexive and transitive. If, moreover, P is also connected on its field, it is called 'standard', and if, for every $a \in W$, every $\mathrm{P} \in \mathcal{P}_a$ is standard, then the frame too will be called standard. In what follows, we will sometimes be interested in models on standard frames, and sometimes in a wider class of models. A 'multiple preference model', $M = \langle F, v \rangle$, is a model on a multiple preference frame F where, as usual, v is an function assigning sets of points in W to atomic formulas, i.e., $v(p) \subseteq W$. Truth-functional formulas are evaluated in the usual classical way.

The presence of multiple preference relations in a frame invites introducing multiple deontic operators. Thus, if one preference relation P_f represents the system of norms given by federal law, one might have an operator O_f by which $O_f A$ says that, according to federal law, it ought to be that A, and if another relation P_g represents the norms of a given game, one might have an operator O_g by which $O_g A$ says that, according to the rules of the game, it ought to be that A. Each of these would be evaluated in the semantics by the pattern of the basic preference semantics of [3], i.e.,

$M, a \models_{MP} O_f A$ iff there is a $b \in \mathcal{F}\mathrm{P}_f$ such that $M, b \models_{MP} A$ and, for all c, if $c\mathrm{P}_f b$ then $M, c \models_{MP} A$

$M, a \models_{MP} O_g A$ iff there is a $b \in \mathcal{F}\mathrm{P}_g$ such that $M, b \models_{MP} A$ and, for all c, if $c\mathrm{P}_g b$ then $M, c \models_{MP} A$

(The notation '$\mathcal{F}\mathrm{P}$' refers to the field of the relation P, i.e., to the set $\{b : \exists c(b\mathrm{P}c \text{ or } c\mathrm{P}b)\}$. The sign for the modelling relation, '$\models_{MP}$', has the subscript to indicate that the relation is defined for the present form of multiple preference models. Later we will consider such relations for other multiplex models; the notation helps reduce ambiguity.)

Under these rules, each separate deontic operator will then follow the logic of the monadic operator O as given in [3] for the simple preference semantics where only one relation has a role to play. This will be standard deontic logic (SDL) if the corresponding relation is standard, and P if it is not. Thus, nothing particularly new is introduced by going multi-modal in this way, and so we will not pursue this prospect further, although such multi-modal logics might be useful for a variety of purposes.

In [3] a genuinely multiple evaluation rule was given for formulas OA, namely

> $M, a \models_{MP} OA$ iff there is a relation $\mathrm{P} \in \mathcal{P}_a$ such that there is a $b \in \mathcal{F}\mathrm{P}$ such that $M, b \models_{MP} A$ and, for all c, if $c\mathrm{P}b$ then $M, c \models_{MP} A$

which applies the basic pattern for the evaluation of ought-statements in the preference semantics but now within the context of the quantification on relations in $\mathcal{P}_a$. In [3] it was shown how this determines the weak deontic logic P regardless of whether the several relations in $\mathcal{P}_a$ are reflexive or transitive or even standard. P is axiomatized by adding the rule and axioms

(RM)	If $\vdash A \to B$ then $\vdash OA \to OB$
(N)	$O\top$
(P)	$\neg O\bot$

to classical PC, with closure under *modus ponens*. If either of

(C)	$(OA \wedge OB) \to O(A \wedge B)$
(K)	$O(A \to B) \to (OA \to OB)$

is added to P the result is SDL, including the consistency principle (D), $OA \to \neg O\neg A$.

P was introduced as a logic that can accommodate conflicts of obligation. In the framework of the present multiple preference semantics it is easy to see how there might be such conflicts, for A might be obligatory with respect to one normative standard, one relation $\mathrm{P} \in \mathcal{P}_a$ while $\neg A$ is obligatory with respect to a different normative standard, a different relation $\mathrm{Q} \in \mathcal{P}_a$. Federal law might prescribe one thing while the rules of the game prescribe the opposite. To say merely that it ought to be that A might thus seem ambiguous or unspecific; it is to say that A is prescribed by some normative standard, but it does not indicate which. This is the sense of 'ought' I call 'indefinite'; it is reflected in the preceding rule.

There is, however, another way to look at the ambiguity or unspecificity of 'ought' in our ordinary discourse. This is to take the expression 'it ought to be that A' to say that A is prescribed by every normative standard, and thus no specification is necessary. This is the sense I call the 'core' or 'definite' sense of 'ought'. It is captured in the multiple preference semantics by the evaluation rule

$M, a \models_{MP} OA$ iff for every relation $\mathrm{P} \in \mathcal{P}_a$ there is a $b \in \mathcal{F}\mathrm{P}$ such that $M, b \models_{MP} A$ and, for all c, if $c\mathrm{P}b$ then $M, c \models_{MP} A$

Both senses can be treated together. This produces a bi-modal deontic logic, but in a rather different way than that suggested above.

Let us now make this more precise. Let the language L_{ae} of these deontic logics contain, in addition to all that is required for classical propositional logic, PC, two monadic operators O_e and O_a. These correspond to the indefinite and the definite senses of 'ought' respectively. The subscripts are intended to suggest the existential quantification and universal quantification inherent in their evaluation rules. Thus, for models on multiple preference frames, we stipulate that

(MP-O_e) $M, a \models_{MP} O_eA$ iff there is a relation $\mathrm{P} \in \mathcal{P}_a$ such that there is a $b \in \mathcal{F}\mathrm{P}$ such that $M, b \models_{MP} A$ and, for all c, if $c\mathrm{P}b$ then $M, c \models_{MP} A$

(MP-O_a) $M, a \models_{MP} O_aA$ iff for every relation $\mathrm{P} \in \mathcal{P}_a$ there is a $b \in \mathcal{F}\mathrm{P}$ such that $M, b \models_{MP} A$ and, for all c, if $c\mathrm{P}b$ then $M, c \models_{MP} A$

As in [3], it will be convenient to adopt the notation '$M, \mathrm{P} \models_{MP} A$' to abbreviate

there is a $b \in \mathcal{F}\mathrm{P}$ such that $M, b \models_{MP} A$ and, for all c, if $c\mathrm{P}b$ then $M, c \models_{MP} A$

So we might give these rules more succinctly as

$M, a \models_{MP} O_eA$ iff for some $\mathrm{P} \in \mathcal{P}_a$, $M, \mathrm{P} \models_{MP} A$

$M, a \models_{MP} O_aA$ iff for every $\mathrm{P} \in \mathcal{P}_a$, $M, \mathrm{P} \models_{MP} A$

O_e corresponds to the monadic O of [3] §2 in the proof of completeness for P. That is its logic here. O_a is new. Its logic depends on the properties of the relations $\mathrm{P} \in \mathcal{P}_a$ in a way that the logic of O_e does not. If all relations in the sets $\mathcal{P}_a$ are standard, then O_a will behave according to the principles of SDL even while O_e follows P.

Let the logic $\mathsf{SDL}_a\mathsf{P}_e$ be axiomatized by PC (with *modus ponens*) plus

(K$_a$) $O_a(A \to B) \to (O_aA \to O_aB)$
(D$_a$) $O_aA \to \neg O_a\neg A$
(RN$_a$) If $\vdash A$, then $\vdash O_aA$
(RM$_e$) If $\vdash A \to B$, then $\vdash O_eA \to O_eB$
(N$_e$) $O_e\top$
(P$_e$) $\neg O_e\bot$
(K$_{ae}$) $O_a(A \to B) \to (O_eA \to O_eB)$

where the first three postulates plainly give us SDL for O_a while the second three give us P for O_e. The last postulate links the two operators.

We note too, for future reference, that the principles

(RM$_a$)	If $\vdash A \rightarrow B$, then $\vdash O_aA \rightarrow O_aB$
(C$_a$)	$\vdash (O_aA \wedge O_aB) \rightarrow O_a(A \wedge B)$
(N$_a$)	$\vdash O_a\top$
(P$_a$)	$\vdash \neg O_a\bot$
(O_aO_e)	$\vdash O_aA \rightarrow O_eA$
(O_aP_e)	$\vdash O_aA \rightarrow \neg O_e\neg A$

are all derivable in $\mathsf{SDL}_a\mathsf{P}_e$. (Derivations are easy and so left to the reader.)

Theorem 1: $\mathsf{SDL}_a\mathsf{P}_e$ *is sound and complete with respect to the class of all standard multiple preference frames.*

Proof of this is contained in the proof of Theorem 3 below, and so we defer the details to Section 2.1.

If, on the other hand, not all the relations $\mathrm{P} \in \mathcal{P}_a$ are standard, especially if they are not all connected on their fields, then the core sense of 'ought', O_a, will follow the logic of P rather than SDL, as does the indefinite sense, O_e. Thus, without connectedness, the principles (K$_a$), (D$_a$) and also (K$_{ae}$) are no longer valid. Instead, the logic characterized by this wider class of frames, which I call $\mathsf{P}_a\mathsf{P}_e$, will be that axiomatized by the postulates (RM$_a$), (N$_a$), and (P$_a$), reflecting the P-character of O_a, as well as (RM$_e$), (N$_e$), and (P$_e$) for the P-character of O_e, and, in place of (K$_{ae}$), it has the weaker (O_aO_e) to connect the two. The rules (RN$_a$), if $\vdash A$ then $\vdash O_aA$, and (RN$_e$), if $\vdash A$ then $\vdash O_eA$, are both derivable given (N$_a$) and (RM$_a$) and (N$_e$) and (RM$_e$).

Theorem 2: $\mathsf{P}_a\mathsf{P}_e$ *is sound and complete with respect to the class of all multiple preference frames (or all reflexive or transitive multiple preference frames).*

Proof of this is contained in the proof of Theorem 8 below; see Section 2.2.

Plainly both the logics $\mathsf{SDL}_a\mathsf{P}_e$ and $\mathsf{P}_a\mathsf{P}_e$ require the multiplicity of the multiple preference frames, otherwise the dual senses of 'ought' would collapse into one. It is noteworthy, though, that $\mathsf{P}_a\mathsf{P}_e$ has only a multiple preference semantics. Although the basic logic P, here the component P_e, has a classical neighborhood semantics and can be given a corresponding multiple Kripke-type relational semantics (cf. [2]; also [5]), and the same can be said for all of $\mathsf{SDL}_a\mathsf{P}_e$, neither of those methods is applicable to $\mathsf{P}_a\mathsf{P}_e$. This is evident from the fact that any neighborhood model or multiple relational model, with a rule for evaluating formulas O_aA comparable to the rule (MP-O_a) with universal quantification on neighborhoods or relations, will automatically validate the principle (C_a), $(O_aA \wedge O_aB) \rightarrow O_a(A \wedge B)$, which is not provable in $\mathsf{P}_a\mathsf{P}_e$, while there are no conditions on such models that might now be loosened to avoid this consequence.

2. *Ranked Obligations*

Given the framework of multiple preference frames, it is natural to think that the relations in the sets $\mathcal{P}_a$ (for each $a \in W$) could themselves be ranked by an order of priority. This might correspond to the way some normative standards take precedence over others, as when one body of law is superior to another, or the instructions of one person in a hierarchy dominate those of another. To capture this notion, let us extend the previous multiple preference frames to include an ordering on their respective relations. A ranked multiple preference frame, or MP$\leqslant$-frame for short, is a triple $\langle W, \mathcal{P}, \leqslant\rangle$ where W and $\mathcal{P}$ are as before and $\leqslant$ is a function that assigns to every $a \in W$ a binary relation $\leqslant_a$ on $\mathcal{P}_a$ that is reflexive and transitive. (Connectedness for relations $\leqslant_a$ is optional. Generally speaking, whatever is said about systems without this form of connectedness applies *mutatis mutandis* to systems with this property. Frames in which every $\leqslant_a$ is connected over $\mathcal{P}_a$ will be called $\leqslant$-connected.)

To express this new ordering relation in the language of the logic, let the language L_{ae} of the preceding section be extended to $L_{ae\leq}$ by adding a binary connective '$\leq$', so that formulas $A \leq B$ are well-formed whenever A and B are. '$A \leq B$' could be read as saying that B is at least as obligatory as A. Given a model $M = \langle F, v\rangle$ on a MP$\leqslant$- frame, let these formulas be evaluated according to the rule

(MP$\leqslant$-$\leq$) $M, a \models_{MP\leqslant} A \leq B$ iff for every relation $\mathrm{P} \in \mathcal{P}_a$ such that $M, \mathrm{P} \models_{MP\leqslant} A$, there is a $\mathrm{Q} \in \mathcal{P}_a$ for which $M, \mathrm{Q} \models_{MP\leqslant} B$ and $\mathrm{P} \leqslant_a \mathrm{Q}$

Within this language formulas O_eA and O_aA are evaluated exactly as in Section 1.[3]

As we formulate the logic of this notion of ranked obligation, let us consider first the case when all the preference relations $\mathrm{P} \in \mathcal{P}_a$, for every $a \in W$, are standard, i.e., reflexive, transitive and connected on their fields. Then we shall examine the case when these relations need not be connected.

[3] It is important not to confuse the present notion of comparative or ranked obligation, $A \leq B$, with the notion of preferability, $A \geq B$, that was discussed in Part I of this work [3]. We return to that in Section 3 below.

2.1. *Standard Multiplex Models*

For purposes of this subsection, let us suppose that MP$\leqslant$-frames $F = \langle W, \mathcal{P}, \leqslant\rangle$ are such that every $\mathrm{P} \in \mathcal{P}_a$ is standard, i.e., reflexive on its field, transitive and connected on its field. The logic that is determined by this class of all standard MP$\leqslant$-frames extends $\mathsf{SDL}_a\mathsf{P}_e$ with the following principles containing $\leq$:

$(O_a \leq)$ $O_a(A \to B) \to (A \leq B)$
$(\neg O_e \leq)$ $\neg O_e A \to (A \leq B)$
$(\leq O_e)$ $(A \leq B) \to (O_e A \to O_e B)$
($\leq$-trans) $((A \leq B) \wedge (B \leq C)) \to (A \leq C)$

and so I call this logic $\mathsf{SDL}_a\mathsf{P}_e\leqslant$.[4] If all the relations $\leqslant_a$ are connected on $\mathcal{P}_a$ then

($\leq$-connex) $(A \leq B) \vee (B \leq A)$

will also be valid. Call the result of adding this to $\mathsf{SDL}_a\mathsf{P}_e\leqslant$, $\mathsf{SDL}_a\mathsf{P}_e\leqslant_c$ (and similarly for $\mathsf{P}_a\mathsf{P}_e\leqslant$ below).

Given the postulates of $\mathsf{SDL}_a\mathsf{P}_e$, other expected principles are derivable, e.g.,

($\leq$-reflex) $\vdash A \leq A$
(R$\leq$) If $\vdash A \to B$, then $\vdash A \leq B$

both of which come from (RN$_a$) and $(O_a \leq)$.

Theorem 3: $\mathsf{SDL}_a\mathsf{P}_e\leqslant$ *is sound and complete with respect to the class of all standard MP$\leqslant$-frames. (Likewise* $\mathsf{SDL}_a\mathsf{P}_e\leqslant_c$ *and all standard $\leqslant$-connected MP$\leqslant$-frames.)*

Proof: Soundness is, as usual, easy to demonstrate, and so is left to the reader. To prove completeness we follow familiar Henkin-style procedures. Set $F = \langle W, \mathcal{P}, \leqslant\rangle$ where W is the class of all maximal consistent extensions of $\mathsf{SDL}_a\mathsf{P}_e\leqslant$. To define relations in $\mathcal{P}_a$ we combine aspects from the proofs for Theorems 1 and 7 of [3]. That is, let us define, for each $a \in W$ and each formula $A \in L_{ae\leq}$, a binary relation

$$\mathrm{P}_a^A = \{\langle b, c\rangle : \text{if } c \in \Sigma_a A \text{ then } b \in \Sigma_a A\}$$

[4] $\mathsf{SDL}_a\mathsf{P}_e\leqslant$ is equivalent to Mark Brown's [1] system $\mathcal{CO}$ of comparative obligation extended with the axiom (N$_e$), an option Brown allows. Brown gave a neighborhood semantics for formulas corresponding to our $O_a A$ and $O_e A$, and a kind of hyper-neighborhood semantics for formulas $A \leq B$.

where

$$\Sigma_a A = \{b : (\text{if } O_e A \in a \text{ then } A \in b) \text{ and } (O_a^{-1} a \subseteq b)\}$$

when $O_a^{-1} a = \{B : O_a B \in a\}$. Then let

$$\mathcal{P}_a = \{\mathrm{P} : \exists A(\mathrm{P} = \mathrm{P}_a^A)\}$$

$\mathcal{P}$ in the frame F assigns $\mathcal{P}_a$ to a.

For $\leqslant$, define first, for each $a \in W$ and each formula $B \in L_{ae\leq}$:

$$\Psi_a B = \{\mathrm{P} \in \mathcal{P}_a : \forall C(\text{if } \mathrm{P}\ \varepsilon\ [C] \text{ then } B \leq C \in a)\}$$

where the notation '$[A]$' refers to $\{a : a \in W \text{ and } A \in a\}$, and '$\mathrm{P}\ \varepsilon\ [A]$' stands for

$$\exists b \in \mathcal{F}\mathrm{P}(A \in b \text{ and } \forall c(\text{if } c\mathrm{P}b \text{ then } A \in c))$$

(This corresponds syntactically to the semantical notation '$M, \mathrm{P} \models_{MP} A$' introduced earlier.) We then define, for $\mathrm{P}, \mathrm{Q} \in \mathcal{P}_a$,

$$\mathrm{P} \leqslant_a \mathrm{Q} \text{ iff } \forall B(\text{if } \mathrm{P} \in \Psi_a B, \text{ then } \exists C(\mathrm{Q} \in \Psi_a C \text{ and } B \leq C \in a))$$

$\leqslant$ in the frame F assigns each a its relation $\leqslant_a$. Finally, let $M = \langle F, v\rangle$ where, as usual,

$$v(p) = \{a \in W : p \in a\}$$

Lemma 4: *M is a model on a standard MP$\leqslant$-frame.*

Proof: $\mathcal{P}_a \neq \emptyset$ since there are formulas, A, and $\mathrm{P}_a^A \in \mathcal{P}_a$. It is easy to show that all the relations in $\mathcal{P}_a$ are reflexive on W (hence non-empty), transitive and connected on W; it is also easy to show that the relations $\leqslant_a$ are reflexive and transitive given ($\leq$-reflex) and ($\leq$-trans). These can be left to the reader. Further, for $\mathsf{SDL}_a\mathsf{P}_e\leqslant_c$, in the presence of ($\leq$-connex), $\leqslant_a$ will be connected; this too is easily shown.

Lemma 5: *(i) $O_a^{-1}a$ is consistent; (ii) If $O_a A \notin a$, then $O_a^{-1}a \cup \{\neg A\}$ is consistent; (iii) if $O_e A \in a$ then $O_a^{-1}a \cup \{A\}$ is consistent; (iv) if $O_e A \notin a$ then $O_a^{-1}a \cup \{\neg A\}$ is consistent.*

Proof: (i) and (ii) are standard in modal logic. For (ii), suppose $O_a A \notin a$ but that $O_a^{-1}a \cup \{\neg A\}$ is not consistent. $O_a^{-1}a \neq \emptyset$ (since $O_a\top \in a$). Hence, there are $C_1, \ldots, C_n \in O_a^{-1}a$ such that $\vdash (C_1 \wedge \ldots \wedge C_n) \rightarrow A$. For each C_i, $O_a C_i \in a$; hence $O_a C_1 \wedge \ldots \wedge O_a C_n \in a$. By ($\mathrm{C}_a$), $O_a(C_1 \wedge \ldots \wedge C_n) \in a$.

Since $\vdash (C_1 \wedge \ldots \wedge C_n) \rightarrow A$, $\vdash O_a(C_1 \wedge \ldots \wedge C_n) \rightarrow O_a A$ by (RM$_a$). Consequently, $O_a A \in a$, contrary to the opening hypothesis. Hence, if $O_a A \notin a$, then $O_a^{-1} a \cup \{\neg A\}$ must be consistent. The same argument applies to (i) given that $O_a \bot \notin a$ by virtue of (P$_a$). For (iii), suppose $O_e A \in a$ but that $O_a^{-1} a \cup \{A\}$ is not consistent. Then $O_a^{-1} a, A \vdash \bot$, and so there are $C_1, \ldots, C_n \in O_a^{-1} a$ $(0 \leq n)$ such that $\vdash (C_1 \wedge \ldots \wedge C_n) \rightarrow (A \rightarrow \bot)$ and so $\vdash O_a(C_1 \wedge \ldots \wedge C_n) \rightarrow O_a(A \rightarrow \bot)$, by (RM$_a$). Since $O_a C_1 \in a, \ldots, O_a C_n \in a$, $O_a(C_1 \wedge \ldots \wedge C_n) \in a$ and so $O_a(A \rightarrow \bot) \in a$. Since $\vdash O_a(A \rightarrow \bot) \rightarrow (O_e A \rightarrow O_e \bot)$, axiom (K_{ae}), $O_e A \rightarrow O_e \bot \in a$. Hence $O_e \bot \in a$, contrary to the consistency of a since $\neg O_e \bot \in a$ from (P$_e$), $\vdash \neg O_e \bot$. (iv) follows from (ii) since if $O_e A \notin a$ then $O_a A \notin a$, by principle $(O_a O_e)$.

Lemma 6: *(i) If* $\mathrm{P}_a^A \,\varepsilon\, [B]$ *then* $O_a(A \rightarrow B) \in a$ *(and hence* $A \leq B \in a$*); (ii) if* $O_e A \notin a$ *and* $\mathrm{P}_a^A \,\varepsilon\, [B]$ *then* $O_a B \in a$.

Proof: For (i), suppose $\mathrm{P}_a^A \,\varepsilon\, [B]$, so that there is a $b \in \mathcal{F}\mathrm{P}_a^A$ and $B \in b$ and for all c such that $c\mathrm{P}_a^A b$, $B \in c$. Suppose then that $O_a(A \rightarrow B) \notin a$. In that case, by Lemma 5.ii, $O_a^{-1} a \cup \{\neg(A \rightarrow B)\}$ is consistent, and so then is $O_a^{-1} a \cup \{A, \neg B\}$. Let c be a maximal consistent extension of that. $A \in c$; hence if $O_e A \in a$ then $A \in c$. Also, $O_a^{-1} a \subseteq c$; hence $c \in \Sigma_a A$, and so if $b \in \Sigma_a A$ then $c \in \Sigma_a A$. Thus $c\mathrm{P}_a^A b$. Therefore, $B \in c$, but also $\neg B \in c$, contrary to the consistency of c. Hence, $O_a(A \rightarrow B) \in a$, as required. (That $A \leq B \in a$ follows from axiom $(O_a \leq$.) For (ii) the argument is similar. Suppose $O_e A \notin a$ and $\mathrm{P}_a^A \,\varepsilon\, [B]$, so that there is a $b \in \mathcal{F}\mathrm{P}_a^A$ and $B \in b$ and for all c that $c\mathrm{P}_a^A b$, $B \in c$. And suppose that $O_a B \notin a$. Then $O_a^{-1} a \cup \{\neg B\}$ is consistent, by Lemma 5.ii, and so has a maximal consistent extension, c. Since $O_e A \notin a$, if $O_e A \in a$ then $A \in c$. Moreover, $O_a^{-1} a \subseteq c$, and so $c\mathrm{P}_a^A b$. Therefore, $B \in c$, contrary to its consistency. Hence $O_a B \in a$, as required.

Lemma 7: *For all* $A \in L_{ae\leq}$ *and all* $a \in W$, $A \in a$ *iff* $M, a \models_{MP\leqslant} A$.

Proof: By induction on A; I consider only the deontic cases where (a) $A = O_a B$, (b) $A = O_e B$, and (c) $A = B \leq C$, where we suppose the lemma to hold for B and C. First, observe that under the inductive hypothesis,

Observation 1: *For any relation* $\mathrm{P} \in \mathcal{P}_a$, $\mathrm{P} \,\varepsilon\, [B]$ *iff* $M, \mathrm{P} \models_{MP\leqslant} B$,

which is easily verified.

For case (a) of the lemma, (i) suppose $O_a B \in a$, and let P be some relation in $\mathcal{P}_a$. There is a formula C such that $\mathrm{P} = \mathrm{P}_a^C$. Either $O_e C \in a$ or $O_e C \notin a$.

Consider the first case first. If $O_eC \in a$, then $O_a^{-1}a \cup \{C\}$ is consistent, by Lemma 5.iii. Let b be a maximal consistent extension of that. We know that $b\mathrm{P}_a^C b$, by reflexivity (Lemma 4), and so $b \in \mathcal{F}\mathrm{P}_a^C$. Further, since $B \in O_a^{-1}a$, $B \in b$, and so $M, b \models_{MP\leqslant} B$, by the inductive hypothesis. Consider any c such that $c\mathrm{P}_a^C b$; then, by definition, it must be that if $b \in \Sigma_a C$ then $c \in \Sigma_a C$. But since $C \in b$ and $O_a^{-1}a \subseteq b$, $b \in \Sigma_a C$. Therefore, $c \in \Sigma_a C$ and so $O_a^{-1}a \subseteq c$, and then $B \in c$, and $M, c \models_{MP\leqslant} B$ by the inductive hypothesis. This suffices for $M, \mathrm{P} \models_{MP\leqslant} B$, and so for $M, a \models_{MP\leqslant} O_aB$. Suppose on the other hand, that $O_eC \notin a$. Then, since $O_a^{-1}a$ is consistent, Lemma 5.i, let b be a maximal consistent extension of that. As before, $B \in b$, and so $M, b \models_{MP\leqslant} B$, by the inductive hypothesis. And as before, if c is such that $c\mathrm{P}_a^C b$, then if $b \in \Sigma_a C$ then $c \in \Sigma_a C$. But since $O_eC \notin a$ and $O_a^{-1}a \subseteq b$, $b \in \Sigma_a C$. So $c \in \Sigma_a C$, and we reason as before that $B \in c$ and so, by the inductive hypothesis $M, c \models_{MP\leqslant} B$, which suffices for $M, \mathrm{P} \models_{MP\leqslant} B$, and so for $M, a \models_{MP\leqslant} O_aB$, as required.

(ii) Suppose $M, a \models_{MP\leqslant} O_aB$, so that for every $\mathrm{P} \in \mathcal{P}_a$, $M, \mathrm{P} \models_{MP\leqslant} B$, but suppose also that $O_aB \notin a$. Then, by Lemma 5.ii, $O_a^{-1}a \cup \{\neg B\}$ is consistent, and so has a maximal consistent extension, c. Consider now the relation $\mathrm{P}_a^\top \in \mathcal{P}_a$. $M, \mathrm{P}_a^\top \models_{MP\leqslant} B$, so there is a $b \in \mathcal{F}\mathrm{P}_a^\top$ and $M, b \models_{MP\leqslant} B$ and, for every c, if $c\mathrm{P}_a^\top b$, $M, c \models_{MP\leqslant} B$. We show that $c\mathrm{P}_a^\top b$, i.e., if $b \in \Sigma_a\top$ then $c \in \Sigma_a\top$. But since $\top \in c$, then automatically if $O_e\top \in a$ then $\top \in c$, and since $O_a^{-1}a \subseteq c$, it follows that $c \in \Sigma_a\top$, and so that if $b \in \Sigma_a\top$ then $c \in \Sigma_a\top$, and thus $c\mathrm{P}_a^\top b$. Therefore, $M, c \models_{MP\leqslant} B$. By the inductive hypothesis, $B \in c$, but $\neg B \in c$, contrary to its consistency. Hence, $O_aB \in a$, as required.

For case (b), (i) suppose $O_eB \in a$, and take the relation $\mathrm{P}_a^B \in \mathcal{P}_a$. By Lemma 5.iii, $O_a^{-1}a \cup \{B\}$ is consistent, and so has a maximal consistent extension, b. $b\mathrm{P}_a^B b$, by reflexivity and so $b \in \mathcal{F}\mathrm{P}_a^B$; also since $B \in b$, $M, b \models_{MP\leqslant} B$ by the inductive hypothesis. Let c be any point such that $c\mathrm{P}_a^B b$. Hence, if $b \in \Sigma_a B$, then $c \in \Sigma_a B$. Since $B \in b$ and $O_a^{-1}a \subseteq b$, $b \in \Sigma_a B$ by definition. Hence $c \in \Sigma_a B$, and so, if $O_eB \in a$, then $B \in c$. Since $O_eB \in a$, $B \in c$, and by the inductive hypothesis $M, c \models_{MP\leqslant} B$, which suffices for $M, \mathrm{P}_a^B \models_{MP\leqslant} B$, and thus for $M, a \models_{MP\leqslant} O_eB$.

(ii) Suppose then that $M, a \models_{MP\leqslant} O_eB$, so that there is a $\mathrm{P} \in \mathcal{P}_a$ such that $M, \mathrm{P} \models_{MP\leqslant} B$, but suppose that $O_eB \notin a$. $\mathrm{P} = \mathrm{P}_a^C$ for some formula C. By the observation above, $\mathrm{P}_a^C \,\varepsilon\, [B]$, and so $O_a(C \to B) \in a$, by Lemma 6.i. Either $O_eC \in a$ or $O_eC \notin a$. Consider the first. Then $O_eB \in a$, by axiom (K_{ae}), contrary to the supposition. Consider the second, if $O_eC \notin a$, then

$O_aB \in a$, by Lemma 6.ii, and so $O_eB \in a$, by (O_aO_e), again contrary to the supposition. Hence $O_eB \in a$, as required.

For case (c), (i) suppose $B \leq C \in a$, and suppose a $\mathrm{P} \in \mathcal{P}_a$ such that $M, \mathrm{P} \models_{MP\leqslant} B$. $\mathrm{P} = \mathrm{P}_a^D$ for some D, and by the observation above, $\mathrm{P}_a^D\ \varepsilon\ [B]$, whence by Lemma 6.i, $D \leq B \in a$, and thus $D \leq C \in a$, by transitivity. Let $\mathrm{Q} = \mathrm{P}_a^C$. Either $O_eC \in a$ or $O_eC \notin a$. The second is not a possible case, however, for if $O_eC \notin a$, then $O_eD \notin a$, by axiom ($\leq O_e$), whence $O_aB \in a$, by Lemma 6.ii, and then $O_eB \in a$, axiom (O_aO_e), and so $O_eC \in a$, by ($\leq O_e$), a contradiction. Therefore, $O_eC \in a$, and so $O_a^{-1}a \cup \{C\}$ is consistent, Lemma 5.iii; let c be a maximal consistent extension of that. As above, $c \in \Sigma_aC$. Consider any d such that $d\mathrm{P}_a^C c$. Since $c \in \Sigma_aC$, $d \in \Sigma_aC$. Hence if $O_eC \in a$, then $C \in d$. Thus $C \in d$, and $M, d \models_{MP\leqslant} C$, by the inductive hypothesis. This suffices for $M, \mathrm{P}_a^C \models_{MP\leqslant} C$. We now show that $\mathrm{P}_a^D \leqslant_a \mathrm{P}_a^C$, that is, for any E such that $\mathrm{P}_a^D \in \Psi_aE$, there is an F such that $\mathrm{P}_a^C \in \Psi_aF$ and $E \leq F \in a$. Suppose such an E; i.e., for all G, if $\mathrm{P}_a^D\ \varepsilon\ [G]$ then $E \leq G \in a$. Hence, $E \leq B \in a$ and so $E \leq C \in a$. $\mathrm{P}_a^C \in \Psi_aC$, for given any H such that $\mathrm{P}_a^C\ \varepsilon\ [H]$ we have $C \leq H \in a$, by Lemma 6.i. Therefore, there is an F that $\mathrm{P}_a^C\ \varepsilon\ [F]$ and $E \leq F \in a$, which suffices for $\mathrm{P}_a^D \leqslant_a \mathrm{P}_a^C$. And that completes the case for $M, a \models_{MP\leqslant} B \leq C$.

For the converse (ii), suppose $M, a \models_{MP\leqslant} B \leq C$. Either $O_eB \in a$ or $O_eB \notin a$. In the latter case, $\neg O_eB \in a$, and so $B \leq C \in a$ by axiom ($\neg O_e \leq$), and we are done. Suppose then that $O_eB \in a$. Then $O_a^{-1}a \cup \{B\}$ is consistent, Lemma 5.iii. Let b be a maximal consistent extension of that. Suppose for *reductio* that $B \leq C \notin a$. Consider the relation $\mathrm{P}_a^B \in \mathcal{P}_a$. By general reflexivity $b\mathrm{P}_a^B b$, hence $b \in \mathcal{F}\mathrm{P}_a^B$. Also, since $B \in b$, $M, b \models_{MP\leqslant} B$, by the inductive hypothesis. Consider any c such that $c\mathrm{P}_a^B b$; then if $b \in \Sigma_aB$ then $c \in \Sigma_aB$. As above, $b \in \Sigma_aB$ (since $B \in b$ and $O_a^{-1}a \subseteq b$), so $c \in \Sigma_aB$, which means that if $O_eB \in a$ then $B \in c$. So $B \in c$, and by the inductive hypothesis, $M, c \models_{MP\leqslant} B$. This suffices for $M, \mathrm{P}_a^B \models_{MP\leqslant} B$ and so, by the observation above, $\mathrm{P}_a^B\ \varepsilon\ [B]$. Since $M, a \models_{MP\leqslant} B \leq C$, there is a $\mathrm{Q} \in \mathcal{P}_a$ such that $M, \mathrm{Q} \models_{MP\leqslant} C$ and $\mathrm{P}_a^B \leqslant_a \mathrm{Q}$. $\mathrm{Q} = \mathrm{P}_a^D$, for some D. Since $M, \mathrm{P}_a^D \models_{MP\leqslant} C$, $\mathrm{P}_a^D\ \varepsilon\ [C]$, by the observation above. Since $\mathrm{P}_a^B \leqslant_a \mathrm{P}_a^D$, for any E such that $\mathrm{P}_a^B \in \Psi_aE$ there is an F such that $\mathrm{P}_a^D \in \Psi_aF$ and $E \leq F \in a$. $\mathrm{P}_a^B \in \Psi_aB$, since for any G if $\mathrm{P}_a^B\ \varepsilon\ [G]$, $B \leq G \in a$, by Lemma 6.i. Hence there is an F such that $\mathrm{P}_a^D \in \Psi_aF$ and $B \leq F \in a$. For such an F, for every H such that $\mathrm{P}_a^D\ \varepsilon\ [H]$, $F \leq H \in a$. Hence $F \leq C \in a$. So $B \leq C \in a$, by ($\leq$-trans), contrary to the assumption above. Therefore, $B \leq C \in a$, as required to complete the lemma.

Theorem 3 now follows from Lemmas 4 and 7 in the usual way. Suppose a formula A is not provable in $\mathsf{SDL}_a\mathsf{P}_e\leqslant$, then $\{\neg A\}$ is consistent and so has a maximal consistent extension, a. Since, by consistency, $A \notin a$, it is not the case that $M, a \models A$, by Lemma 7, and thus there is a model on a standard MP$\leq$-frame on which A does not hold, Lemma 4. By contraposition, if A is valid on all standard MP$\leq$-frames, A must be provable in $\mathsf{SDL}_a\mathsf{P}_e\leqslant$. Notice that by suppressing all reference to $\leqslant$ in the canonical model M and to formulas $A \leq B$ the proof of Theorem 3 would provide a completeness proof for $\mathsf{SDL}_a\mathsf{P}_e$ (Theorem 1).

It also follows that $\mathsf{SDL}_a\mathsf{P}_e\leqslant$ is a conservative extension of $\mathsf{SDL}_a\mathsf{P}_e$ ($\mathsf{SDL}_a\mathsf{P}_e\leqslant_c$ too) since if A is a formula of $L_{ae\leq}$ that contains no occurrences of $\leq$ (and thus would be a formula of L_{ae}) if A is not provable in $\mathsf{SDL}_a\mathsf{P}_e$, by completeness for that system, there is a model $M = \langle F, v\rangle$ on a multiple preference frame $F = \langle W, \mathcal{P}\rangle$ such that A does not hold on M. Take the model $M^* = \langle F^*, v\rangle$ on the MP$\leq$-frame $F^* = \langle W, \mathcal{P}, \leqslant\rangle$, where W and $\mathcal{P}$ are the same as for F, and $\leqslant$ assigns each $a \in W$ the universal relation on $\mathcal{P}_a$. It is a small matter to show that any formula B that contains no occurrences of $\leq$ and any $a \in W$, $M, a \models_{MP} B$ iff $M^*, a \models_{MP\leqslant} B$. Hence, the formula A that was not provable in $\mathsf{SDL}_a\mathsf{P}_e$ does not hold on M^*, and so, by the soundness of $\mathsf{SDL}_a\mathsf{P}_e\leqslant$, A is not provable in $\mathsf{SDL}_a\mathsf{P}_e\leqslant$. (In a similar vein, $\mathsf{SDL}_a\mathsf{P}_e\leqslant$ is a conservative extension of the fragment P_e itself, which is P of [3].)

Finally, we end this subsection by noting, without proof, that $\mathsf{SDL}_a\mathsf{P}_e\leqslant$ has the finite model property, and so is decidable. Although common filtration methods do not lend themselves easily to proving this, nevertheless, the result is not difficult to establish using the method of 'mini-canonical models' of Hughes and Cresswell [4], p. 146, wherein one constructs a finite model that will falsify any non-theorem A from the composition of A itself. The argument then recapitualates the proof of Theorem 3. $\mathsf{SDL}_a\mathsf{P}_e\leqslant$ is therefore sound and complete with respect to the class of finite standard MP$\leqslant$-frames. This suffices to establish the finite model property for the system, and thus its decidability. The same holds for $\mathsf{SDL}_a\mathsf{P}_e$ (and P_e, i.e., P).

2.2. *More General Multiplex Models*

In this subsection we consider MP$\leqslant$-frames that are not required to be standard but allow assigned preference relations $\mathrm{P} \in \mathcal{P}_a$ that are not connected, and even not reflexive or transitive. (The latter turn out not to affect the resulting logic.) The logic determined by this broader class of models, which I call $\mathsf{P}_a\mathsf{P}_e\leqslant$, extends $\mathsf{P}_a\mathsf{P}_e$ of Section 1 with these principles for $\leq$:

$(\neg O_e \leq)$	$\neg O_e A \to (A \leq B)$
$(\leq O_e)$	$(A \leq B) \to (O_e A \to O_e B)$
($\leq$-trans)	$((A \leq B) \wedge (B \leq C)) \to (A \leq C)$
(R$\leq$)	If $\vdash A \to B$, then $\vdash A \leq B$

as given above, together with

$(O_a \leq)'$ $\quad O_a A \to B \leq A$

in place of $(O_a \leq)$ of $\mathsf{SDL}_a\mathsf{P}_e\leqslant$, which is no longer valid. (Note that (R$\leq$) must now be postulated separately, as it is no longer derivable from the other postulates. ($\leq$-reflex), $A \leq A$, is of course derivable given that rule. If all relations $\leqslant_a$ are connected, then ($\leq$-connex), $(A \leq B) \vee (B \leq A)$, is valid; call the system with that $\mathsf{P}_a\mathsf{P}_e\leqslant_c$.

Theorem 8: $\mathsf{P}_a\mathsf{P}_e\leqslant$ *is sound and complete with respect to the class of all MP$\leqslant$-frames (or the class of all reflexive or transitive MP$\leqslant$-frames). (Likewise* $\mathsf{P}_a\mathsf{P}_e\leqslant_c$ *and such $\leqslant$-connected MP$\leqslant$-frames.)*

Proof: Soundness as usual. To prove completeness is rather like Theorem 12 of [3] but somewhat more complicated in order to take both O_a and $\leq$ into account. We begin by taking a detour through a secondary semantics in which $\mathsf{P}_a\mathsf{P}_e\leqslant$ will be treated as a more conventional bi-modal logic, with its two monadic deontic operators evaluated in the same way but with respect to separate classes of relations. The binary operator $\leq$ will continue to be interpreted much as before.

To this end, consider a 2MP$\leqslant$-frame $F = \langle W, \mathcal{P}^a, \mathcal{P}^e, \leqslant\rangle$ with W as usual and each of $\mathcal{P}^a$ and $\mathcal{P}^e$ assigning each $a \in W$ a non-empty set, $\mathcal{P}^a_a$ and $\mathcal{P}^e_a$ respectively, of binary relations on W, just as in the ordinary multiple preference semantics, subject to the requirement that $\mathcal{P}^a_a \subseteq \mathcal{P}^e_a$. Relations in $\mathcal{P}^e_a$ may be unconstrained, or reflexive, transitive, or standard. As before, $\leqslant$ assigns each $a \in W$ a reflexive, transitive relation, $\leqslant_a$, on $\mathcal{P}^e_a$, subject to the further frame condition that if $\mathrm{P} \in \mathcal{P}^e_a$ and $\mathrm{Q} \in \mathcal{P}^a_a$ then $\mathrm{P} \leqslant_a \mathrm{Q}$. Models on such frames evaluate formulas as expected, with:

$M, a \models_{2MP\leqslant} O_a A$ iff there is a $\mathrm{P} \in \mathcal{P}^a_a$ such that $M, \mathrm{P} \models_{2MP\leqslant} A$;
$M, a \models_{2MP\leqslant} O_e A$ iff there is a $\mathrm{P} \in \mathcal{P}^e_a$ such that $M, \mathrm{P} \models_{2MP\leqslant} A$;
$M, a \models_{2MP\leqslant} A \leq B$ iff for all $\mathrm{P} \in \mathcal{P}^e_a$ if $M, \mathrm{P} \models_{2MP\leqslant} A$, then there is a $\mathrm{Q} \in \mathcal{P}^e_a$ such that $M, \mathrm{Q} \models_{2MP\leqslant} B$ and $\mathrm{P} \leqslant_a \mathrm{Q}$.

Lemma 9: $\mathsf{P}_a\mathsf{P}_e\leqslant$ *is sound and complete with respect to all such 2MP$\leqslant$ frames, and with respect to all that are reflexive, transitive or standard (for all relations in $\mathcal{P}^e_a$). (Similarly for* $\mathsf{P}_a\mathsf{P}_e\leqslant_c$ *and all such $\leqslant$-connected 2MP$\leqslant$ frames.)*

Proof: Soundness as usual. For completeness, define a canonical model thus: Let $F = \langle W, \mathcal{P}^a, \mathcal{P}^e, \leqslant\rangle$ with W the set of all maximal consistent extensions of $\mathsf{P}_a\mathsf{P}_e\leqslant$. For $\mathcal{P}^a$ and $\mathcal{P}^e$, first define, for each formula $A \in L_{ae\leq}$ and each $a \in W$, the binary relations

$$^a\mathrm{P}_a^A = \{\langle b, c\rangle : \text{ Either } O_aA \notin a \text{ or } A \in b \text{ or } A \notin c\}$$

$$^e\mathrm{P}_a^A = \{\langle b, c\rangle : \text{ Either } O_eA \notin a \text{ or } A \in b \text{ or } A \notin c\}$$

and let

$$\mathcal{P}_a^a = \{\mathrm{P} : \exists A(\mathrm{P} = {}^a\mathrm{P}_a^A)\}$$
$$\mathcal{P}_a^{e*} = \{\mathrm{P} : \exists A(\mathrm{P} = {}^e\mathrm{P}_a^A)\}$$

and then

$$\mathcal{P}_a^e = \mathcal{P}_a^{e*} \cup \mathcal{P}_a^a$$

$\mathcal{P}^a$ and $\mathcal{P}^e$ assign each $a \in W$ the sets $\mathcal{P}_a^a$ and $\mathcal{P}_a^e$ respectively. Define $\leqslant$ just as for Theorem 3, with the same definitions of Ψ_a and $\leqslant_a$ for all pairs of relations in $\mathcal{P}_a^e$. Let $M = \langle F, v\rangle$ with $v(p) = \{a \in W : p \in a\}$ as usual. Before establishing that M really is a model on a 2MP$\leqslant$-frame and that it is canonical, it is helpful to have two sublemmas.

Sublemma A: *For all formulas A and B, (i) if ${}^a\mathrm{P}_a^A\ \varepsilon\ [B]$ then $\vdash A \to B$; (ii) if ${}^e\mathrm{P}_a^A\ \varepsilon\ [B]$ then $\vdash A \to B$; (iii) if ${}^a\mathrm{P}_a^A\ \varepsilon\ [B]$ and $O_aA \notin a$ then $\vdash B$; (iv) if ${}^e\mathrm{P}_a^A\ \varepsilon\ [B]$ and $O_eA \notin a$ then $\vdash B$. (v) ${}^a\mathrm{P}_a^A\ \varepsilon\ [A]$ iff $a \in [O_aA]$; (vi) ${}^e\mathrm{P}_a^A\ \varepsilon\ [A]$ iff $a \in [O_eA]$.*

Proof: This is the same as Lemma 9 of [3] for the two deontic operators. I repeat that proof here. For (i), suppose ${}^a\mathrm{P}_a^A\ \varepsilon\ [B]$, so that there is a $b \in \mathcal{F}{}^a\mathrm{P}_a^A$ and $B \in b$ and for all c such that $c{}^a\mathrm{P}_a^Ab$, $B \in c$. Suppose that $\nvdash A \to B$. Then $\{A, \neg B\}$ is consistent and has a maximal consistent extension, c. Since $A \in c$, $c{}^a\mathrm{P}_a^Ab$, and so $B \in c$, contrary to its consistency. (ii) for ${}^e\mathrm{P}_a^A$ is the same. For (iii), suppose ${}^a\mathrm{P}_a^A\ \varepsilon\ [B]$ but that $O_aA \notin a$, and suppose $\nvdash B$. So $\{\neg B\}$ is consistent and has a maximal consistent extension, c. Since $O_aA \notin a$, automatically $c{}^a\mathrm{P}_a^Ab$, for the given b. So $B \in c$, contrary to its consistency. (iv) for ${}^e\mathrm{P}_a^A$ and O_eA is the same. For (v), (a) Suppose ${}^a\mathrm{P}_a^A\ \varepsilon\ [A]$, but that $a \notin [O_aA]$, i.e., $O_aA \notin a$. Then, by (iii), $\vdash A$, so $\vdash O_aA$ by (RN$_a$) and $O_aA \in a$, a contradiction. (b) Suppose then $O_aA \in a$. $\{A\}$ is consistent (else $\vdash A \to \bot$, and then by (RM$_a$) $\vdash O_aA \to O_a\bot$, and so $O_a\bot \in a$, but since by (P$_a$) $\vdash \neg O_a\bot$, $\neg O_a\bot \in a$, contrary to its consistency). Thus $\{A\}$ has a maximal consistent extension, b. $A \in b$, hence $b{}^a\mathrm{P}_a^Ab$ and $b \in \mathcal{F}{}^a\mathrm{P}_a^A$. Obviously, $b \in [A]$. Suppose any c such that $c{}^a\mathrm{P}_a^Ab$. Either $O_aA \notin a$ or $A \notin b$ or $A \in c$. The first two are ruled out, leaving the third, which suffices for ${}^a\mathrm{P}_a^A \; \varepsilon \; [A]$. (vi) for ${}^e\mathrm{P}_a^A$ and O_eA is the same.

Sublemma B: *For any A, (i)* ${}^a\mathrm{P}_a^A \in \Psi_a A$*; (ii)* ${}^e\mathrm{P}_a^A \in \Psi_a A$*; (iii) If* ${}^a\mathrm{P}_a^A \in \mathcal{P}_a^a$ *and* $O_aA \notin a$*, then* ${}^a\mathrm{P}_a^A \in \Psi_a\top$*; (iv) if* ${}^e\mathrm{P}_a^A \in \mathcal{P}_a^e$ *and* $O_eA \notin a$*, then* ${}^e\mathrm{P}_a^A \in \Psi_a\top$.

These are immediate from Sublemma A, for suppose, for (i), any B such that ${}^a\mathrm{P}_a^A \; \varepsilon \; [B]$, then by Sublemma A.i, $\vdash A \rightarrow B$, so $\vdash A \leq B$, by rule (R$\leq$), whence $A \leq B \in a$, which suffices for ${}^a\mathrm{P}_a^A \in \Psi_a A$. (ii) is the same. For (iii), suppose ${}^a\mathrm{P}_a^A \in \mathcal{P}_a^a$ and $O_aA \notin a$, and let B be any formula such that ${}^a\mathrm{P}_a^A \; \varepsilon \; [B]$, so, by Sublemma A.ii, $\vdash B$, and so $\vdash \top \rightarrow B$, whence $\vdash \top \leq B$ and $\top \leq B \in a$. That suffices for ${}^a\mathrm{P}_a^A \in \Psi_a\top$. (iv) is the same.

Sublemma C: M *is a model on a standard 2MP$\leqslant$ frame.*

Proof: That $\mathcal{P}_a^a$ and $\mathcal{P}_a^e$ are non-empty and that each relation in these sets is non-empty is obvious. That each relation in these sets is standard follows directly from the definitions. (Reflexivity figures in what follows, but not full standardness.) That $\mathcal{P}_a^a \subseteq \mathcal{P}_a^e$ is trivial. That each $\leqslant_a$ is reflexive and transitive is immediate from the definition and the presence of ($\leq$-reflex) and ($\leq$-trans). Finally, to show that if $\mathrm{P} \in \mathcal{P}_a^e$ and $\mathrm{Q} \in \mathcal{P}_a^a$ then $\mathrm{P} \leqslant_a \mathrm{Q}$, suppose some B such that $\mathrm{P} \in \Psi_a B$. There is an A that $\mathrm{Q} = {}^a\mathrm{P}_a^A$. By Sublemma B.i, ${}^a\mathrm{P}_a^A \in \Psi_a A$. Either $O_aA \in a$ or $O_aA \notin a$. In the first case, $B \leq A \in a$, by axiom $(O_a \leq)'$, and so there is a C such that $\mathrm{Q} \in \Psi_a C$ and $B \leq C \in a$, which suffices for $\mathrm{P} \leqslant_a \mathrm{Q}$. In the second case, if $O_aA \notin a$, then since ${}^a\mathrm{P}_a^A \in \mathcal{P}_a^a$, ${}^a\mathrm{P}_a^A \in \Psi\top$, by Sublemma B.iii. Since $\vdash B \rightarrow \top$, $\vdash B \leq \top$, so $B \leq \top \in a$, and again there is a C such that $\mathrm{Q} \in \Psi_a C$ and $B \leq C \in a$, which suffices for $\mathrm{P} \leqslant_a \mathrm{Q}$.

Sublemma D: *For all* A *and all* $a \in W$*,* $A \in a$ *iff* $M, a \models_{2MP\leqslant} A$.

Proof: As usual, by induction on A. The interesting cases are the deontic ones, (a) when $A = O_aB$, (b) when $A = O_eB$, and (c) when $A = B \leq C$. The arguments for (a) and (b) reprise the argument for Lemma 10 in [3] with Sublemma A in place of Lemma 9 there. Here is the argument for (a); (b) is similar. Suppose the sublemma holds for B and C.

(a.i) Suppose $O_aB \in a$. Then by Sublemma A.v, ${}^a\mathrm{P}_a^B \; \varepsilon \; [B]$, i.e., there is a $b \in \mathcal{F}{}^a\mathrm{P}_a^B$ such that $B \in b$ and for every c if $c\,{}^a\mathrm{P}_a^B\, b$ then $B \in c$, but, with the inductive hypothesis, this yields $M, a \models_{MP} O_aB$ directly since ${}^a\mathrm{P}_a^B \in \mathcal{P}_a^a$. (a.ii) Suppose $M, a \models_{MP} O_aB$, i.e., there is a $\mathrm{P} \in \mathcal{P}_a^a$ and $M, \mathrm{P} \models_{MP} B$.

$\mathrm{P} = {}^a\mathrm{P}_a^C$ for some C. Since $M, {}^a\mathrm{P}_a^C \models_{MP} B$, the inductive hypothesis yields ${}^a\mathrm{P}_a^C \,\varepsilon\, [B]$, and so by Sublemma A.i, $\vdash C \to B$, whence, by (RM$_a$) $\vdash O_aC \to O_aB$. Suppose, however, $O_aB \notin a$. Then $O_aC \notin a$. But then, by Sublemma A.iii, $\vdash B$, in which case $\vdash O_aB$, by (RN$_a$), and so $O_aB \in a$, a contradiction. Therefore, if $M, a \models_{MP} O_aB$, then $O_aB \in a$.

The argument for case (c) is a reprise of the argument for this case under Lemma 7 above in Subsection 2.1, but with Sublemma A in place of Lemma 6. The reader can fill in the details.

The present Lemma 9 now follows in the usual way. As a corollary we have that $\mathsf{P}_a\mathsf{P}_e\leqslant$ is sound and complete with respect to all 2MP$\leqslant$ frames that are (a) reflexive, (b) reflexive and transitive, (c) standard. These results apply *mutatis mutandis* to $\mathsf{P}_a\mathsf{P}_e\leqslant_c$ since the relations $\leqslant_a$ will be connected in the presence of ($\leq$-connex).

We are now back in line to prove Theorem 8, that $\mathsf{P}_a\mathsf{P}_e\leqslant$ ($\mathsf{P}_a\mathsf{P}_e\leqslant_c$) is complete with respect to the class of all original, non-bi-modal ($\leqslant$-connected) MP$\leqslant$ frames. The argument proceeds from Lemma 9 much as Theorem 12 of [3] for the simple logic P followed from Theorem 7 there.[5]

Given a model $M = \langle F, v\rangle$ on a 2MP$\leqslant$ frame $F = \langle W, \mathcal{P}^a, \mathcal{P}^e, \leqslant\rangle$ in which, for every $a \in W$, every relation in $\mathrm{P} \in \mathcal{P}_a^e$ is reflexive on its field, let $F^* = \langle W^*, \mathcal{P}^*, \leqslant^*\rangle$, be defined thus: Let each relation $\mathrm{P} \subseteq W \times W$ bear a distinct index i and let I be the set of these indexes. Let $W^* = \{\langle a, i\rangle : a \in W$ and $i \in I\}$. For each relation $\mathrm{P}^j \subseteq W \times W$ let

$$\mathrm{P}^{\times j} = \{\langle b^*, c^*\rangle : \text{there are } b, c \in W \text{ such that } b^* = \langle b, j\rangle \;\&\; c^* = \langle c, j\rangle \;\&\; b\mathrm{P}^j c\}$$

(We note that that if $j \neq k$, then $\mathrm{P}^{\times j}$ and $\mathrm{P}^{\times k}$ are disjoint, and that if $b^* \in \mathcal{F}\mathrm{P}^{\times j}$ then $b^* = \langle b, j\rangle$, for some $b \in W$.)

For each $\langle a, i\rangle \in W^*$, let

$$\Omega_{\langle a,i\rangle} = \bigcup\{\mathrm{P}^{\times j} : \mathrm{P}^j \in \mathcal{P}_a^a\}$$

and let

$$\mathrm{P}^{*j}_{\langle a,i\rangle} = \mathrm{P}^{\times j} \cup \Omega_{\langle a,i\rangle}$$

then

$$\mathcal{P}^*_{\langle a,i\rangle} = \{\mathrm{P}^* : \exists j \in I(\mathrm{P}^* = \mathrm{P}^{*j}_{\langle a,i\rangle} \;\&\; \mathrm{P}^j \in \mathcal{P}_a^e)\}$$

$\mathcal{P}^*$ of F^* assigns $\mathcal{P}^*_{\langle a,i\rangle}$ to $\langle a, i\rangle$.

[5] The proof for this result that was briefly described in [2] p. 131 contained an error in its definition of relations corresponding to $\leqslant^*_{\langle a,i\rangle}$ below; that is repaired here.

Next, for all $\mathrm{P}^*, \mathrm{Q}^* \in \mathcal{P}^*_{\langle a,i\rangle}$, let

$$\mathrm{P}^* \leqslant^*_{\langle a,i\rangle} \mathrm{Q}^* \text{ iff } \forall j \in I(\text{ if } \mathrm{P}^* = \mathrm{P}^{*j}_{\langle a,i\rangle} \text{ then } \exists k \in I(\mathrm{Q}^* = \mathrm{P}^{*k}_{\langle a,i\rangle} \text{ and } \mathrm{P}^j \leqslant_a \mathrm{P}^k))$$

$\leqslant^*$ of F^* assigns $\leqslant^*_{\langle a,i\rangle}$ to $\langle a,i\rangle$.

Finally, let $M^* = \langle F^*, v^*\rangle$ where

$$v^*(p) = \{\langle a,i\rangle : a \in v(p) \ \&\ i \in I\}$$

Lemma 10: *M^* is a model on a reflexive MP$\leqslant$-frame (and if F is transitive, then so is F^* and if F is $\leqslant$-connected, then so is F^*).*

Proof: That each $\mathcal{P}^*_{\langle a,i\rangle}$ is non-empty follows from $\mathcal{P}^e_a$ being non-empty, and that each $\mathrm{P}^{*j}_{\langle a,i\rangle} \in \mathcal{P}^*_{\langle a,i\rangle}$ is non-empty likewise follows from the non-emptiness of the $\mathrm{P}^j \in \mathcal{P}^e_a$. That each $\leqslant^*_{\langle a,i\rangle}$ is reflexive and transitive follows from the definition and the same properties of $\leqslant_a$. (Further, for $\mathsf{P}_a\mathsf{P}_e{\leqslant}_c$, if $\leqslant_a$ is connected then so too is $\leqslant^*_{\langle a,i\rangle}$.)

Lemma 11: *For all formulas A, for all $a \in W$ and $i \in I$, $M, a \models_{2MP\leqslant} A$ iff $M^*, \langle a,i\rangle \models_{MP\leqslant} A$*

Proof: As usual, by induction on A. We consider the cases (a) $A = O_aB$, (b) $A = O_eB$, and (c) $A = B \leq C$, supposing the lemma to hold for B and C, for any $a \in W$ and $i \in I$. It expedites these to have

Sublemma E: *Under the inductive hypothesis, given $\langle a,i\rangle \in W^*$, for any $j \in I$, $M^*, \mathrm{P}^{*j}_{\langle a,i\rangle} \models_{MP\leqslant} B$ iff $M, \mathrm{P}^j \models_{2MP\leqslant} B$ or there is a $k \in I$ such that $\mathrm{P}^k \in \mathcal{P}^a_a$ and $M, \mathrm{P}^k \models_{2MP\leqslant} B$.*

Proof: Given $\langle a,i\rangle \in W^*$, (i) suppose $M^*, \mathrm{P}^{*j}_{\langle a,i\rangle} \models_{MP\leqslant} B$, so that there is a $b^* \in \mathcal{F}\mathrm{P}^{*j}_{\langle a,i\rangle}$ such that $M^*, b^* \models_{MP\leqslant} B$ and for all c^* such that $c^*\mathrm{P}^{*j}_{\langle a,i\rangle}b^*$, $M^*, c^* \models_{MP\leqslant} B$. Since $b^* \in \mathcal{F}\mathrm{P}^{*j}_{\langle a,i\rangle}$, either $b^* \in \mathcal{F}\mathrm{P}^{\times j}$ or $b^* \in \mathcal{F}\Omega_{\langle a,i\rangle}$. In the first case, $b^* = \langle b,j\rangle$ and $b \in \mathcal{F}\mathrm{P}^j$. Also $M, b \models_{2MP\leqslant} B$ by the inductive hypothesis. Consider any c such that $c\mathrm{P}^j b$. $\langle c,j\rangle \mathrm{P}^{\times j}\langle b,j\rangle$, and so $\langle c,j\rangle \mathrm{P}^{*j}_{\langle a,i\rangle}\langle b,j\rangle$ in which case $M^*, \langle c,j\rangle \models_{MP\leqslant} B$, and then $M, c \models_{2MP\leqslant} B$

by the inductive hypothesis, which suffices for $M, \mathrm{P}^j \models_{2MP\leqslant} B$. If, on the other hand, $b^* \in \mathcal{F}\Omega_{\langle a,i\rangle}$, then there is a $k \in I$ such that $\mathrm{P}^k \in \mathcal{P}^a_a$ and $b^* \in \mathcal{F}\mathrm{P}^{\times k}$. In that case, $b^* = \langle b, k\rangle$ and $b \in \mathcal{F}\mathrm{P}^k$. Moreover, $M, b \models_{2MP\leqslant} B$ by the inductive hypothesis. Consider any c such that $c\mathrm{P}^k b$. $\langle c, k\rangle \mathrm{P}^{\times k}\langle b, k\rangle$ so $\langle c, k\rangle\Omega_{\langle a,i\rangle}\langle b, k\rangle$ and then $\langle c, k\rangle \mathrm{P}^{*j}_{\langle a,i\rangle}\langle b, k\rangle$, whence $M^*, \langle c, k\rangle \models_{MP\leqslant} B$. So, $M, c \models_{2MP\leqslant} B$ by the inductive hypothesis. That suffices for $M, \mathrm{P}^k \models_{2MP\leqslant} B$. Therefore, under either case, either $M, \mathrm{P}^j \models_{2MP\leqslant} B$ or there is a $k \in I$ such that $\mathrm{P}^k \in \mathcal{P}^a_a$ and $M, \mathrm{P}^k \models_{2MP\leqslant} B$.

For the converse, (ii) suppose $M, \mathrm{P}^j \models_{2MP\leqslant} B$ or there is a $k \in I$ such that $\mathrm{P}^k \in \mathcal{P}^a_a$ and $M, \mathrm{P}^k \models_{2MP\leqslant} B$. In the first case, there is a $b \in \mathcal{F}\mathrm{P}^j$ such that $M, b \models_{2MP\leqslant} B$ and for all c that $c\mathrm{P}^j b$, $M, c \models_{2MP\leqslant} B$. $b\mathrm{P}^j b$, so $\langle b, j\rangle \mathrm{P}^{\times j}\langle b, j\rangle$ and so $\langle b, j\rangle \mathrm{P}^{*j}_{\langle a,i\rangle}\langle b, j\rangle$ and $\langle b, j\rangle \in \mathcal{F}\mathrm{P}^{*j}_{\langle a,i\rangle}$. Further, $M^*, \langle b, j\rangle \models_{MP\leqslant} B$ by the inductive hypothesis. Consider any c^* such that $c^*\mathrm{P}^{*j}_{\langle a,i\rangle}\langle b, j\rangle$. Either $c^*\mathrm{P}^{\times j}\langle b, j\rangle$ or $c^*\Omega_{\langle a,i\rangle}\langle b, j\rangle$. In the first case, $c^* = \langle c, j\rangle$ and $c\mathrm{P}^j b$, and then $M, c \models_{2MP\leqslant} B$ and so $M^*, \langle c, j\rangle \models_{MP\leqslant} B$ by the inductive hypothesis. If $c^*\Omega_{\langle a,i\rangle}\langle b, j\rangle$, there is a $k \in I$ such that $\mathrm{P}^k \in \mathcal{P}^a_a$ and $c^*\mathrm{P}^{\times k}\langle b, j\rangle$, but then $k = j$, and so $c^*\mathrm{P}^{\times j}\langle b, j\rangle$; also $c^* = \langle c, j\rangle$, so $\langle c, j\rangle \mathrm{P}^{\times j}\langle b, j\rangle$. Hence, as before, $c\mathrm{P}^j b$, and then $M, c \models_{2MP\leqslant} B$, so $M^*, \langle c, j\rangle \models_{MP\leqslant} B$ by the inductive hypothesis. That suffices for $M^*, \mathrm{P}^{*j}_{\langle a,i\rangle} \models_{MP\leqslant} B$. For the second case, in which there is a $k \in I$ such that $\mathrm{P}^k \in \mathcal{P}^a_a$ and $M, \mathrm{P}^k \models_{2MP\leqslant} B$, the argument is similar.

It also helps to know that, although in general $\mathrm{P}^{*j}_{\langle a,i\rangle} = \mathrm{P}^{*k}_{\langle a,i\rangle}$ does not imply that $\mathrm{P}^j = \mathrm{P}^k$, nevertheless this is true:

Sublemma F: *For any* $\langle a, i\rangle \in W^*$*, for all* $\mathrm{P}^* \in \mathcal{P}^*_{\langle a,i\rangle}$*, if* $\mathrm{P}^* = \mathrm{P}^{*j}_{\langle a,i\rangle}$*, then either (a) for all* $\mathrm{P}^k \in \mathcal{P}^e_a$*, if* $\mathrm{P}^* = \mathrm{P}^{*k}_{\langle a,i\rangle}$ *then* $\mathrm{P}^j = \mathrm{P}^k$*, or (b)* $\mathrm{P}^j \in \mathcal{P}^a_a$*, or (c) there is a* $\mathrm{P}^k \in \mathcal{P}^a_a$ *such that* $\mathrm{P}^* = \mathrm{P}^{*k}_{\langle a,i\rangle}$ *and* $\mathrm{P}^j \subseteq \mathrm{P}^k$.

Proof: Suppose that $\mathrm{P}^* = \mathrm{P}^{*j}_{\langle a,i\rangle}$, but that (a) is not the case, i.e., there is a $k \in I$ such that $\mathrm{P}^* = \mathrm{P}^{*k}_{\langle a,i\rangle}$ but $\mathrm{P}^j \neq \mathrm{P}^k$. So there will be a pair of points $b, c \in W$ such that $b\mathrm{P}^j c$ but not $b\mathrm{P}^k c$ or else a pair of points $b, c \in W$ such that $b\mathrm{P}^k c$ but not $b\mathrm{P}^j c$. In the first case, since $b\mathrm{P}^j c$, $\langle b, j\rangle \mathrm{P}^{\times j}\langle c, j\rangle$; hence $\langle b, j\rangle \mathrm{P}^{*j}_{\langle a,i\rangle}\langle c, j\rangle$. So $\langle b, j\rangle \mathrm{P}^{*k}_{\langle a,i\rangle}\langle c, j\rangle$, and thus either $\langle b, j\rangle \mathrm{P}^{\times k}\langle c, j\rangle$ or else $\langle b, j\rangle\Omega_{\langle a,i\rangle}\langle c, j\rangle$. But if $\langle b, j\rangle \mathrm{P}^{\times k}\langle c, j\rangle$, $j = k$, by definition, and then $\mathrm{P}^j =$

P^k, contrary to the supposition. Hence $\langle b, j\rangle\Omega_{\langle a,i\rangle}\langle c, j\rangle$. Then there is an $l \in I$ such that $\langle b, j\rangle\mathrm{P}^{\times l}\langle c, j\rangle$ and $\mathrm{P}^l \in \mathcal{P}^a_a$. As noted above, $\langle b, j\rangle \in \mathcal{F}\mathrm{P}^{\times l}$ only if $j = l$; hence, $\mathrm{P}^j = \mathrm{P}^l$ and $\mathrm{P}^j \in \mathcal{P}^a_a$, and (b) must hold. In the second case, with $b\mathrm{P}^k c$ but not $b\mathrm{P}^j c$, argue similarly that $\mathrm{P}^k \in \mathcal{P}^a_a$. Now suppose that $\mathrm{P}^j \not\subseteq \mathrm{P}^k$. So there is a pair of points $d, e \in W$ such that $d\mathrm{P}^j e$ but not $d\mathrm{P}^k e$. Argue as before that $\mathrm{P}^j \in \mathcal{P}^a_a$ Hence, if neither (a) nor (c), then (b).

Returning now to the cases of the lemma, given $a \in W$ and $\langle a, i\rangle \in W^*$, for case (a), (i) suppose $M, a \models_{2MP\leqslant} O_a B$, i.e., there is a $\mathrm{P} \in \mathcal{P}^a_a$ such that $M, \mathrm{P} \models_{2MP\leqslant} B$. $\mathrm{P} = \mathrm{P}^j$ for some $j \in I$. Consider any $\mathrm{P}^* \in \mathcal{P}^*_{\langle a,i\rangle}$. $\mathrm{P}^* = \mathrm{P}^{*k}_{\langle a,i\rangle}$ for some $k \in I$ and $\mathrm{P}^k \in \mathcal{P}^e_a$. There is a $b \in \mathcal{F}\mathrm{P}^j$ such that $M, b \models_{2MP\leqslant} B$ and for all c such that $c\mathrm{P}^j b$, $M, c \models_{2MP\leqslant} B$. Since $\mathrm{P}^j \in \mathcal{P}^a_a$ and $b\mathrm{P}^j b$, $\langle b, j\rangle\mathrm{P}^{\times j}\langle b, j\rangle$ and hence, $\langle b, j\rangle\Omega_{\langle a,i\rangle}\langle b, j\rangle$ so that $\langle b, j\rangle\mathrm{P}^{*k}_{\langle a,i\rangle}\langle b, j\rangle$ and $\langle b, j\rangle \in \mathcal{F}\mathrm{P}^{*k}_{\langle a,i\rangle}$. $M^*, \langle b, j\rangle \models_{MP\leqslant} B$ by the inductive hypothesis. Consider any c^* such that $c^*\mathrm{P}^{*k}_{\langle a,i\rangle}\langle b, j\rangle$. Either $c^*\mathrm{P}^{\times k}\langle b, j\rangle$ or $c^*\Omega_{\langle a,i\rangle}\langle b, j\rangle$, in which case there is an $l \in I$ such that $c^*\mathrm{P}^{\times l}\langle b, j\rangle$ and $\mathrm{P}^l \in \mathcal{P}^a_a$. In either case, $\mathrm{P}^k = \mathrm{P}^j$ or $\mathrm{P}^l = \mathrm{P}^j$ and $c^* = \langle c, j\rangle$. Hence $c\mathrm{P}^j b$, and so $M, c \models_{2MP\leqslant} B$, whence $M^*, c^* \models_{MP\leqslant} B$ by the inductive hypothesis, which suffices for $M^*, \langle a, i\rangle \models_{MP\leqslant} O_a B$. For the converse, (ii) suppose $M^*, \langle a, i\rangle \models_{MP\leqslant} O_a B$, i.e., for all $\mathrm{P}^* \in \mathcal{P}^*_{\langle a,i\rangle}$, $M^*, \mathrm{P}^* \models_{MP\leqslant} B$. There is a $\mathrm{P}^j \in \mathcal{P}^a_a$, hence $\mathrm{P}^{*j}_{\langle a,i\rangle} \in \mathcal{P}^*_{\langle a,i\rangle}$, and $M^*, \mathrm{P}^{*j}_{\langle a,i\rangle} \models_{MP\leqslant} B$. By Sublemma E, either $M, \mathrm{P}^j \models_{2MP\leqslant} B$ or there is a $k \in I$ such that $\mathrm{P}^k \in \mathcal{P}^a_a$ and $M, \mathrm{P}^k \models_{2MP\leqslant} B$. Either case suffices for $M, a \models_{2MP\leqslant} O_a B$.

Case (b), when $A = O_e B$, is even more immediate from Sublemma E; it is left to the reader.

For case (c), when $A = B \leq C$, (i) suppose $M, a \models_{2MP\leqslant} B \leq C$ so that for every $\mathrm{P} \in \mathcal{P}^e_a$ such that $M, \mathrm{P} \models_{2MP\leqslant} B$ there is a $\mathrm{Q} \in \mathcal{P}^e_a$ for which $M, \mathrm{Q} \models_{2MP\leqslant} C$ and $\mathrm{P} \leqslant_a \mathrm{Q}$. Now consider an arbitrary $\mathrm{P}^* \in \mathcal{P}^*_{\langle a,i\rangle}$ such that $M^*, \mathrm{P}^* \models_{MP\leqslant} B$. $\mathrm{P}^* = \mathrm{P}^{*j}_{\langle a,i\rangle}$ for some $j \in I$. By Sublemma F there are three cases: (i.a) for all $k \in I$ if $\mathrm{P}^{*j}_{\langle a,i\rangle} = \mathrm{P}^{*k}_{\langle a,i\rangle}$ then $\mathrm{P}^j = \mathrm{P}^k$; (i.b) $\mathrm{P}^j \in \mathcal{P}^a_a$; (i.c) there is an $k \in I$ such that $\mathrm{P}^{*j}_{\langle a,i\rangle} = \mathrm{P}^{*k}_{\langle a,i\rangle}$ and $\mathrm{P}^k \in \mathcal{P}^a_a$ and $\mathrm{P}^j \subseteq \mathrm{P}^k$. We treat (i.a) and (i.b) together. In these cases, by Sublemma E either $M, \mathrm{P}^j \models_{2MP\leqslant} B$ or there is a $k \in I$ such that $\mathrm{P}^k \in \mathcal{P}^a_a$ and $M, \mathrm{P}^k \models_{2MP\leqslant} B$. In either case, there is a $\mathrm{Q} \in \mathcal{P}^e_a$ for which $M, \mathrm{Q} \models_{2MP\leqslant} C$ and either $\mathrm{P}^j \leqslant_a \mathrm{Q}$ or, for such a k, $\mathrm{P}^k \leqslant_a \mathrm{Q}$. But in the latter case, since $\mathrm{P}^k \in \mathcal{P}_a^a$, $\mathrm{P}^j \leqslant_a \mathrm{P}^k$ by the frame condition, and so $\mathrm{P}^j \leqslant_a \mathrm{Q}$ by transitivity. So $\mathrm{P}^j \leqslant_a \mathrm{Q}$. $\mathrm{Q} = \mathrm{P}^m$ for some $m \in I$ and $\mathrm{P}^j \leqslant_a \mathrm{P}^m$. Since $M, \mathrm{P}^m \models_{2MP\leqslant} C$, $M^*, \mathrm{P}^{*m}_{\langle a,i\rangle} \models_{MP\leqslant} C$ by Sublemma E. We now show that $\mathrm{P}^{*j}_{\langle a,i\rangle} \leqslant^*_{\langle a,i\rangle} \mathrm{P}^{*m}_{\langle a,i\rangle}$. Consider any $x \in I$ such that $\mathrm{P}^{*j}_{\langle a,i\rangle} = \mathrm{P}^{*x}_{\langle a,i\rangle}$. Under (i.a), $\mathrm{P}^j = \mathrm{P}^x$ and so $\mathrm{P}^x \leqslant_a \mathrm{P}^m$, which suffices for $\mathrm{P}^{*j}_{\langle a,i\rangle} \leqslant^*_{\langle a,i\rangle} \mathrm{P}^{*m}_{\langle a,i\rangle}$. Under (i.b), $\mathrm{P}^x \leqslant_a \mathrm{P}^j$ by the frame condition, so $\mathrm{P}^x \leqslant_a \mathrm{P}^m$ by transitivity, which suffices for $\mathrm{P}^{*j}_{\langle a,i\rangle} \leqslant^*_{\langle a,i\rangle} \mathrm{P}^{*m}_{\langle a,i\rangle}$. Hence under these two cases there is a $\mathrm{Q}^* \in \mathcal{P}^*_{\langle a,i\rangle}$, namely $\mathrm{P}^{*m}_{\langle a,i\rangle}$, such that $M^*, \mathrm{Q}^* \models_{MP\leqslant} C$ and $\mathrm{P}^* \leqslant^*_{\langle a,i\rangle} \mathrm{Q}^*$, which suffices for $M^*, \langle a,i\rangle \models_{MP\leqslant} B \leq C$. For case (i.c), where there is an $k \in I$ such that $\mathrm{P}^{*j}_{\langle a,i\rangle} = \mathrm{P}^{*k}_{\langle a,i\rangle}$ and $\mathrm{P}^k \in \mathcal{P}_a^a$ and $\mathrm{P}^j \subseteq \mathrm{P}^k$, since $M^*, \mathrm{P}^{*k}_{\langle a,i\rangle} \models_{MP\leqslant} B$, either $M, \mathrm{P}^k \models_{2MP\leqslant} B$ or there is a $l \in I$ such that $\mathrm{P}^l \in \mathcal{P}_a^a$ and $M, \mathrm{P}^l \models_{2MP\leqslant} B$, by Sublemma E, and as before, in either case, there is a $\mathrm{Q} \in \mathcal{P}_a^e$ for which $M, \mathrm{Q} \models_{2MP\leqslant} C$ and either $\mathrm{P}^k \leqslant_a \mathrm{Q}$ or for such an l, $\mathrm{P}^l \leqslant_a \mathrm{Q}$, and again since $\mathrm{P}^k \leqslant_a \mathrm{P}^l$ by the frame condition, $\mathrm{P}^k \leqslant_a \mathrm{Q}$. As before, $\mathrm{Q} = \mathrm{P}^m$ for some $m \in I$ and thus $\mathrm{P}^k \leqslant_a \mathrm{P}^m$. Since $M, \mathrm{P}^m \models_{2MP\leqslant} C$, $M^*, \mathrm{P}^{*m}_{\langle a,i\rangle} \models_{MP\leqslant} C$ by Sublemma E. To show that $\mathrm{P}^{*j}_{\langle a,i\rangle} \leqslant^*_{\langle a,i\rangle} \mathrm{P}^{*m}_{\langle a,i\rangle}$, consider any $x \in I$ such that $\mathrm{P}^{*j}_{\langle a,i\rangle} = \mathrm{P}^{*x}_{\langle a,i\rangle}$. $\mathrm{P}^x \leqslant_a \mathrm{P}^k$ by the frame condition; hence $\mathrm{P}^x \leqslant_a \mathrm{P}^m$, which suffices for $\mathrm{P}^{*j}_{\langle a,i\rangle} \leqslant^*_{\langle a,i\rangle} \mathrm{P}^{*m}_{\langle a,i\rangle}$. Since $M^*, \mathrm{P}^{*m}_{\langle a,i\rangle} \models_{MP\leqslant} C$, this in turn suffices for $M^*, \langle a,i\rangle \models_{MP\leqslant} B \leq C$, as required to complete case (c.i).

For (c.ii), suppose $M^*, \langle a,i\rangle \models_{MP\leqslant} B \leq C$, so that for every $\mathrm{P}^* \in \mathcal{P}^*_{\langle a,i\rangle}$ such that $M^*, \mathrm{P}^* \models_{MP\leqslant} B$ there is a $\mathrm{Q}^* \in \mathcal{P}^*_{\langle a,i\rangle}$ such that $M^*, \mathrm{Q}^* \models_{MP\leqslant} C$ and $\mathrm{P}^* \leqslant^*_{\langle a,i\rangle} \mathrm{Q}^*$. Consider any relation $\mathrm{P}^j \in \mathcal{P}_a^e$ such that $M, \mathrm{P}^j \models_{2MP\leqslant} B$, and find a $\mathrm{Q} \in \mathcal{P}_a^e$ such that $M, \mathrm{Q} \models_{2MP\leqslant} C$ and $\mathrm{P}^j \leqslant_a \mathrm{Q}$. By Sublemma E, $M^*, \mathrm{P}^{*j}_{\langle a,i\rangle} \models_{MP\leqslant} B$. Thus there is a $\mathrm{Q}^* \in \mathcal{P}^*_{\langle a,i\rangle}$ such that $M^*, \mathrm{Q}^* \models_{MP\leqslant} C$ and $\mathrm{P}^{*j}_{\langle a,i\rangle} \leqslant^*_{\langle a,i\rangle} \mathrm{Q}^*$. $\mathrm{Q}^* = \mathrm{P}^{*k}_{\langle a,i\rangle}$ for some $k \in I$ and $\mathrm{P}^k \in \mathcal{P}_a^e$. Hence, by Sublemma E either $M, \mathrm{P}^k \models_{2MP\leqslant} C$ or there is an $l \in I$ such that $\mathrm{P}^l \in \mathcal{P}_a^a$ and $M, \mathrm{P}^l \models_{2MP\leqslant} C$. If the latter, then for such an l, $\mathrm{P}^j \leqslant_a \mathrm{P}^l$, by the frame condition, and so there is a $\mathrm{Q} \in \mathcal{P}_a^e$ that $M, \mathrm{Q} \models_{2MP\leqslant} C$ and $\mathrm{P}^j \leqslant_a \mathrm{Q}$, which suffices for $M, a \models_{2MP\leqslant} B \leq C$. Suppose then that $M, \mathrm{P}^k \models_{2MP\leqslant} C$. Since $\mathrm{P}^{*j}_{\langle a,i\rangle} \leqslant^*_{\langle a,i\rangle} \mathrm{P}^{*k}_{\langle a,i\rangle}$, there is an $x \in I$ such that $\mathrm{P}^{*k}_{\langle a,i\rangle} = \mathrm{P}^{*x}_{\langle a,i\rangle}$ and $\mathrm{P}^j \leqslant_a \mathrm{P}^x$. As before, under Sublemma F, there are three subcases, either (ii.a) for every $m \in I$ if $\mathrm{P}^{*k}_{\langle a,i\rangle} = \mathrm{P}^{*m}_{\langle a,i\rangle}$ then $\mathrm{P}^k = \mathrm{P}^m$, or (ii.b) $\mathrm{P}^k \in \mathcal{P}_a^a$ or (ii.c) there is an $m \in I$ such that $\mathrm{P}^{*k}_{\langle a,i\rangle} = \mathrm{P}^{*m}_{\langle a,i\rangle}$ and $\mathrm{P}^m \in \mathcal{P}^a_a$ and $\mathrm{P}^k \subseteq \mathrm{P}^m$. Under (ii.a), take $\mathrm{Q} = \mathrm{P}^k$, for $\mathrm{P}^k = \mathrm{P}^x$, so $\mathrm{P}^j \leqslant_a \mathrm{P}^k$. Under (ii.b) take $\mathrm{Q} = \mathrm{P}^k$, for then $\mathrm{P}^j \leqslant_a \mathrm{P}^k$ by the frame condition. Under (ii.c), if $m \in I$ is such that $\mathrm{P}^{*k}_{\langle a,i\rangle} = \mathrm{P}^{*m}_{\langle a,i\rangle}$ and $\mathrm{P}^m \in \mathcal{P}^a_a$ and $\mathrm{P}^k \subseteq \mathrm{P}^m$, then $M^*, \mathrm{P}^{*m}_{\langle a,i\rangle} \models_{MP\leqslant} C$ and $\mathrm{P}^{*j}_{\langle a,i\rangle} \leqslant^*_{\langle a,i\rangle} \mathrm{P}^{*m}_{\langle a,i\rangle}$. By Sublemma E either $M, \mathrm{P}^m \models_{2MP\leqslant} C$ or there is an $n \in I$ such that $\mathrm{P}^n \in \mathcal{P}^a_a$ and $M, \mathrm{P}^n \models_{MP\leqslant} C$. Take Q to be P^m or P^n as appropriate. In either case $\mathrm{P}^j \leqslant_a \mathrm{Q}$ by the frame condition. Hence, under all three subcases, there is a $\mathrm{Q} \in \mathcal{P}^e_a$ such that $M, \mathrm{Q} \models_{2MP\leqslant} C$ and $\mathrm{P}^j \leqslant_a \mathrm{Q}$, which suffices for $M, a \models_{2MP\leqslant} B \leq C$, to complete case (ii), and so the lemma.

Theorem 8 now follows as expected. That is, suppose a formula A is not provable in $\mathsf{P}_a\mathsf{P}_e\!\leqslant$ ($\mathsf{P}_a\mathsf{P}_e\!\leqslant_c$). Then, by Lemma 9, there is a model on a reflexive (transitive; $\leqslant$-connected) 2MP$\leqslant$ frame that falsifies A, and so, by Lemmas 10 and 11 there is a model on a conventional reflexive (transitive; $\leqslant$-connected) MP$\leqslant$ frame that falsifies A. Thus, if A is valid in all MP$\leqslant$ frames, then A must be provable in $\mathsf{P}_a\mathsf{P}_e\!\leqslant$ ($\mathsf{P}_a\mathsf{P}_e\!\leqslant_c$).

As with Theorem 3, the proof of Theorem 8 contains a proof of Theorem 2, the completeness of $\mathsf{P}_a\mathsf{P}_e$, by suppressing reference to $\leqslant$ in models and to formulas $B \leq C$. So also $\mathsf{P}_a\mathsf{P}_e\!\leqslant$ is a conservative extensions of $\mathsf{P}_a\mathsf{P}_e$, by the argument given after Theorem 3. Likewise, just as $\mathsf{SDL}_a\mathsf{P}_e\!\leqslant$ and $\mathsf{SDL}_a\mathsf{P}_e$ possess the finite model property and are decidable, the same holds for $\mathsf{P}_a\mathsf{P}_e\!\leqslant$ and $\mathsf{P}_a\mathsf{P}_e$. To establish this, it is easiest to apply the method of mini-canonical models in the framework of the bi-modal 2MP$\leqslant$ frames, recapitulating the proof of Lemma 9, and so establish an analogous completeness result with respect to finite frames.

3. *Dyadic Connectives*

On the face of it, the results of the previous sections should extend naturally to the dyadic deontic connectives of conditional obligation and its correlative for preference. In Part I of this work, [3] §4, multiple preference models for conditional obligation and preference were introduced as part of proof of the completeness of the logics DP of conditional obligation and PPref of preference. There those logics were proved complete with respect to the class of all reflexive, transitive multiple preference frames, Theorem 28 of that Part, which led to the proof of their completeness with respect to the class of all reflexive and transitive simple preference frames, Theorem 33 of that Part. The use of multiple preference models there was largely a technical device to obtain the primary result of Theorem 33. From a philosophical point of view, however, it makes as much sense to develop such multiple preference models for conditional obligation as for simple obligation, as in Section 1 here. For just as it is plausible to allow for conflicts of (simple) obligation that might arise from different normative standards, so it is plausible for there to be conflicts of conditional obligation that arise in the same way. It might be, for example, that, according to the laws of the land, Jones should be executed given that he robbed a bank, while, according to the norms of humane justice, Jones should not be executed, given that he robbed a bank. Furthermore, once the framework of multiple preference relations is in place, it seems just as appropriate to distinguish an indefinite sense of conditional obligation from a definite or core sense as for the monadic case. Nevertheless, although this extension should be routine, there are complications, and significant questions remain open.

Let the propositional language L_{cae} contain operators for an indefinite and a definite sense of conditional obligation, $O_e(-/-)$ and $O_a(-/-)$, respectively. These will be treated analogously to the monadic operators of Section 1, as well as to conditional obligation introduced in [3]. (For convenience we shall not now consider ranked obligations from Section 2; this is solely to avoid an extra dimension of complication.) To evaluate this language in the multiplex semantics, we consider now only multiple preference frames in which, for every $a \in W$, every relation $\mathrm{P} \in \mathcal{P}_a$ is reflexive on its field and transitive. For some purposes we may also require that every such relation is standard, i.e., also connected on its field.

Formulas $O_e(B/A)$ and $O_a(B/A)$ are to be evaluated in models M on multiple preference frames $F = \langle W, \mathcal{P} \rangle$ according to the rules:

(MP-CO_e) $M, a \models_{MP} O_e(B/A)$ iff for some $\mathrm{P} \in \mathcal{P}_a$, there is a $b \in \mathcal{F}\mathrm{P}$ such that $M, b \models_{MP} A \wedge B$ and, for all c, if $c\mathrm{P}b$ and $M, c \models_{MP} A$ then $M, c \models_{MP} B$

(MP-CO_a) $M, a \models_{MP} O_a(B/A)$ iff for every $\mathrm{P} \in \mathcal{P}_a$, there is a $b \in \mathcal{F}\mathrm{P}$ such that $M, b \models_{MP} A \wedge B$ and, for all c, if $c\mathrm{P}b$ and $M, c \models_{MP} A$ then $M, c \models_{MP} B$

which follow the standard pattern for conditional obligation within the context of the quantification on relations $\mathrm{P} \in \mathcal{P}_a$. Indeed, $O_e(-/-)$ is nothing but the conditional obligation of [3] §4, as interpreted in the multiple preference semantics that was introduced in passing there. Now it comes to the fore, and has its dual companion to accompany it.

Each of these connectives allows the definition of a connective for preference, following the pattern described in [3] §3. Thus:

$$A \geq_a B =_{df} \neg O_e(\neg A/A \vee B)$$

$$A \geq_e B =_{df} \neg O_a(\neg A/A \vee B)$$

Given these definitions these rules of evaluation are derivable:

(MP-$\geq_a$) $M, a \models_{MP} A \geq_a B$ iff for all P $\in \mathcal{P}_a$, for every $c \in \mathcal{F}$P, if $M, c \models_{MP} B$ then there is a b such that bPc and $M, b \models_{MP} A$

(MP-$\geq_e$) $M, a \models_{MP} A \geq_e B$ iff there is a P $\in \mathcal{P}_a$, such that, for every $c \in \mathcal{F}$P, if $M, c \models_{MP} B$ then there is a b such that bPc and $M, b \models_{MP} A$

(Note that the subscripts 'a' and 'e' are merely mnemonics for the quantification in the evaluation rules. $O_e(-/-)$ corresponds to $\geq_a$ and not to $\geq_e$, and similarly for $O_a(-/-)$ and $\geq_e$.)

Alternatively, as we saw in [3], the preference connectives could be introduced as primitives and conditional obligation defined in terms of them. Thus, in the language $L_{\geq ae}$ containing the connectives $\geq_a$ and $\geq_e$ evaluated according to the rules (MP-$\geq_a$) and (MP-$\geq_e$) stipulated, one could define $O_e(-/-)$ and $O_a(-/-)$ thus

$$O_e(B/A) =_{df} \neg((A \wedge \neg B) \geq_a (A \wedge B))$$

$$O_a(B/A) =_{df} \neg((A \wedge \neg B) \geq_e (A \wedge B))$$

and then the evaluation rules (MP-CO_e) and (MP-CO_a) are derivable. Because of these reciprocal equivalences, we shall treat the two forms of connective, conditional obligation and preference, simultaneously, and for the most part not worry about which is primitive.

The logic of the indefinite conditional obligation $O_e(-/-)$ follows the logic DP that was described in [3] §4 since $O_e(-/-)$ is effectively the same as the conditional obligation given there. DP is axiomatized by adding to PC with *modus ponens*:

(RCE$_e$) If $\vdash A \leftrightarrow A'$ then $\vdash O_e(B/A) \leftrightarrow O_e(B/A')$
(RCM$_e$) If $\vdash B \rightarrow C$ then $\vdash O_e(B/A) \rightarrow O_e(C/A)$
(CN$_e$) $O_e(\top/\top)$
(CP$_e$) $\neg O_e(\bot/A)$
(C$O_e\wedge$) $O_e(B/A) \rightarrow O_e(A \wedge B/A)$
(trans$_a$) $((A \geq_a B) \wedge (B \geq_a C)) \rightarrow (A \geq_a C)$
($O_e\vee$) $O_e(A/B \vee C) \rightarrow (O_e(A/B) \vee O_e(A/C))$

Similarly, $\geq_a$ follows the logic of PPref defined in the same place. It is given by:

(R.1$_a$) If $\vdash A \to B$ then $\vdash B \geq_a A$
(trans$_a$) $((A \geq_a B) \wedge (B \geq_a C)) \to (A \geq_a C)$
($\geq_a \vee$) $((A \geq_a B) \wedge (A \geq_a C)) \to (A \geq_a (B \vee C))$
(poss$_a$) $\neg(\bot \geq_a \top)$

Of course, (reflex$_a$), $A \geq_a A$, is derivable directly from (R.1).

Both logics DP and PPref — or as we might now say, DP$_e$ and PPref$_a$ — are characterized by the class of all reflexive transitive multiple preference frames, and also the class of all standard multiple preference frames. (Cf. Theorem 28 of [3].)

By analogy with the results of Section 1 above, one would expect that the logic of the definite $O_a(-/-)$ would be standard dyadic deontic logic (SDDL) when all relations $\mathrm{P} \in \mathcal{P}_a$ are standard, and that it would be DP when they are not required to be standard. Similarly, one would expect $\geq_e$ to behave according to SPref when all preference relations are standard, and according to PPref when they are not. This is almost, but not quite, the case.

The following from SDDL are valid in all standard multiple preference frames:

(RCE$_a$) If $\vdash A \leftrightarrow A'$ then $\vdash O_a(B/A) \leftrightarrow O_a(B/A')$
(RCM$_a$) If $\vdash B \to C$ then $\vdash O_a(B/A) \to O_a(C/A)$
(CK$_a$) $O_a(B \to C/A) \to (O_a(B/A) \to O_a(C/A))$
(CD$_a$) $O_a(B/A) \to \neg O_a(\neg B/A)$
(CN$_a$) $O_a(\top/\top)$
(C$O_a\wedge$) $O_a(B/A) \to O_a(A \wedge B/A)$

Also valid is

($\vee \geq_e$) $((A \vee B) \geq_e C) \to ((A \geq_e C) \vee (B \geq_e C))$

And as one would expect, this mixed principle, which connects the two conditional obligation operators, is also valid in all standard multiple preference frames,

(CK$_{ae}$) $O_a(B \to C/A) \to (O_e(B/A) \to O_e(C/A))$

These further mixed principles are also valid in all standard multiple preference frames

(trans$_{ae}$) $((A \geq_a B) \wedge (B \geq_e C)) \to (A \geq_e C)$
(trans$_{ea}$) $((A \geq_e B) \wedge (B \geq_a C)) \to (A \geq_e C)$
(connex$_{ae}$) $(A \geq_a B) \vee (B \geq_e A)$
($\geq_{ae} \vee$) $((A \geq_a B) \wedge (A \geq_e C)) \to (A \geq_e (B \vee C))$
(R$\vee \geq_{ae}$) If $\vdash A \to (B \vee C)$, then $\vdash (B \geq_a A) \vee (C \geq_e A)$

$\geq_e$ may be thought to be defined in terms of $O_a(-/-)$, or not. If it is, then

$$O_a(B/A) \leftrightarrow \neg((A \wedge \neg B) \geq_e (A \wedge B))$$

is derivable from the preceding. Similarly, if $\geq_e$ is taken as primitive and $O_a(-/-)$ defined in terms of it, then

$$(A \geq_e B) \leftrightarrow \neg O_a(\neg A/A \vee B)$$

is derivable.

With $\geq_e$ defined, then from the preceding, together with the principles of DP_e or PPref_a, the following validities, familiar from SDDL, can likewise be derived:

(D.1_a)	$O_a(\top/A) \leftrightarrow O_a(A/A)$
(D.$1\mathrm{a}_a$)	$O_a(B/A) \to O_a(A/A)$
(D.2_a)	$\neg O_a(\neg A/A)$
(D.3_a)	$(O_a(A/C) \wedge O_a(B/C)) \to O_a(A \wedge B/C)$
(D.4_a)	$(O_a(A/B) \wedge O_a(A/C)) \to O_a(A/B \vee C)$
(D.5_e)	$A \geq_e A$
(D.6_e)	$(A \geq_e B) \vee (B \geq_e A)$
(D.7_e)	$(A \geq_e (A \vee B)) \vee (B \geq_e (A \vee B))$
(D.8_e)	$\neg(\bot \geq_e \top)$
(DR.1_e)	If $\vdash A \to B$ then $\vdash B \geq_e A$
(DR.2_e)	If $\vdash A \to (B \vee C)$ then $\vdash (B \geq_e A) \vee (C \geq_e A)$
(DR.$2\mathrm{gen}_e$)	If $\vdash A \to (B_1 \vee \cdots \vee B_n)$ then $\vdash (B_1 \geq_e A) \vee \cdots \vee (B_n \geq_e A)$
(DR.3_a)	If $\vdash A \to B$ then $\vdash A \geq_e C \to B \geq_e C$
(DR.4_a)	If $\vdash A \to B$ then $\vdash C \geq_e B \to C \geq_e A$
(D.9_{ae})	$O_a(B/A) \to O_e(B/A)$
(D.10_{ae})	$(A \geq_a B) \to (A \geq_e B)$

If one begins with $\geq_e$ primitive, and $O_a(-/-)$ defined, then all of the preceding can be derived from (DR.1_e), (reflex$_e$) = (D.5_e), (connex$_{ae}$), (poss$_e$) = (D.8_a), and ($\vee \geq_e$), along with (trans$_{ae}$), (trans$_{ea}$), ($\geq_{ae} \vee$), and (R$\vee \geq_{ae}$), and the principles of DP_e or PPref_a. (I leave these derivations as a diversion for the reader.)

Of the preceding, (CK$_a$), (CD$_a$), and (CK$_{ae}$) depend for their validity on relations $\mathrm{P} \in \mathcal{P}_a$ being connected on their fields. So do (D.3_a), (D.4_a), (D.6_a), (D.7_a), (DR.2_a) and its generalization (DR.2gen$_a$), ($\vee \geq_e$), (connex$_{ae}$), ($\geq_{ae} \vee$), and (R$\vee \geq_{ae}$). All the rest remain valid when the requirement of connectedness is given up.

Conspicuous for its absence from the preceding lists is the principle of transitivity for $\geq_e$, namely

(trans$_e$) $((A \geq_e B) \wedge (B \geq_e C)) \to (A \geq_e C)$

This is not valid under the rule (MP-$\geq_e$). That it is not is easily seen as one considers that a state of affairs A might be at least as good as another B under one set of norms while B is at least as good as a third C under some other set of norms, but A might not be as good as C under any norms. The possibility that transitivity might fail for a preference connection is analogous to the possibility of conflicts of obligation that arise from distinct normative standards. Hence, it seems quite reasonable to have a sense of preference, '$\geq_e$', for which transitivity, (trans$_e$), would not be valid.[6]

Although transitivity fails for $\geq_e$, a strict counterpart to $\geq_e$ is transitive. That is, define $A >_e B =_{df} \neg(B \geq_e A)$, (equivalent to $(A \geq_e B) \wedge \neg(B \geq_e A)$ in the presence of (D.6$_e$), connectedness for $\geq_e$), then

(s-trans$_e$) $((A >_e B) \wedge (B >_e C)) \rightarrow (A >_e C)$

is valid. Plainly too $>_e$ is irreflexive (given (reflex$_e$)), and hence asymmetric. By contrast, the strict counterpart of $\geq_a$ — $A >_a B =_{df} \neg(B \geq_a A)$ (or even $(A \geq_a B) \wedge \neg(B \geq_a A)$) — is not transitive. In a similar vein, we see that just as transitivity fails for $\geq_e$ but holds for $\geq_a$, connectedness fails for $\geq_a$ and holds for $\geq_e$ (within standard models). Thus these two properties are divided between the indefinite and the definite preference connectives. This is due entirely to the quantification on relations $\mathrm{P} \in \mathcal{P}_a$.

So far this all seems quite reasonable and natural. Nevertheless, the failure of transitivity for $\geq_e$, or perhaps we should say, the combined failure of transitivity for $\geq_e$ and of connectedness for $\geq_a$, plays havoc with the effort to form a complete axiomatization for the logic of $\geq_e$ and with it $O_a(-/-)$ in combination with $\geq_a$ and $O_e(-/-)$.

By analogy with $\mathsf{SDL}_a\mathsf{P}_e$, one might define a logic $\mathsf{SDDL}_a\mathsf{DP}_e$ that combines the principles of SDDL for $O_a(-/-)$ and DP for $O_e(-/-)$. That is, its axioms would be the initial formulas listed above as valid (with respect to all standard frames), along with the postulates for DP_e, and including the several mixed principles that were mentioned, and even including (s-trans$_e$). One might similarly define an equivalent logic $\mathsf{SPref}_e\mathsf{PPref}_a$ that takes the preference connectives as primitive and combines principles from SPref for $\geq_e$ and PPref for $\geq_a$. Plainly these logics would be sound with respect to the class of all standard multiple preference frames. Completeness, however, is another matter.

[6] One might also consider the possibility of a failure of transitivity due to a single preference relation not being transitive; this would be like conflicts of obligation that arise with respect to a single preference relation due to a sort of internal incoherence of the normative standard it represents. This is what led to the logic P and away from standard deontic logic SDL. Although this seems to be a reasonable possibility, it is not included in the present framework for dyadic deontic systems, for it is here required that all preference relations $\mathrm{P} \in \mathcal{P}_a$ be transitive. Without that, transitivity of $\geq_a$ would also fail.

One can see the difficulty as one considers constructing a canonical model along the lines described in Section 2.1 above for $\mathsf{SDL}_a\mathsf{P}_e$ (ignoring ranked obligations) but adapted to suit SDDL as described in [3] §3, and DP as in [3] §4. Needed are a set of relations, P_a^A, that are reflexive, transitive and connected on their fields. Combining the lessons of [3] §§3, 4, these might be defined so that, for each formula A,

$$\mathrm{P}_a^A = \{\langle b, c\rangle : \text{if } c \in \Theta_a A \text{ then } (b \in \Theta_a A \text{ and } b\mathrm{X}_a c)\}$$

where

$$\Theta_a A = \{b : \forall B(\text{if } B \in b \text{ then } \neg A \geq_a B \notin a)\}$$

which is designed to accommodate formulas $O_e(B/C)$ and $B \geq_a C$ and so to take care of the DP_e part of the logic, and X_a is a relation corresponding to the relation P_a that was given in the completeness proof for SDDL in [3] §3. So long as this relation X_a is reflexive, transitive and connected on its field, then so will be P_a^A, as required.

In [3] §3, P_a was defined so that, given $\Pi_a A = \{b : \forall B \in b, B \geq A \in a\}$,

$$\mathrm{P}_a = \{\langle b, c\rangle : \Diamond_a b \text{ and } \Diamond_a c \text{ and } \forall C(c \in \Pi_a C \Rightarrow \exists B(b \in \Pi_a B \text{ and } B \geq C \in a))\}$$

In the present framework, however, this is ambiguous, depending on which connective, $\geq_a$ or $\geq_e$, is applied in the definition. To define X_a, if the clause $B \geq C$ is read $B \geq_e C$, then the relation will not be transitive, because transitivity fails for $\geq_e$. If, on the other hand, the clause is read $B \geq_a C$ then the relation will not be connected on its field, because connectedness fails for $\geq_a$. I see no way out of this dilemma.

The situation is even more dire if one tries to adapt the completeness proof for DP and PPref from [3] to apply to logics $\mathsf{DP}_a\mathsf{DP}_e$ and $\mathsf{PPref}_e\mathsf{PPref}_a$ that result from combining the principles of DP, respectively PPref, for both $O_a(-/-)$ and $(O_e(-/-))$, or $\geq_e$ and $\geq_a$, e.g., taking as axioms and rules those validities listed above that do not require connectedness for all relations $\mathrm{P} \in \mathcal{P}_a$.

As we saw in Section 2.2 above, it seems easiest to prove completeness for P-like systems (not requiring connectedness) by taking a detour through a framework of bi-modal 2MP-frames, $F = \langle W, \mathcal{P}^a, \mathcal{P}^e, \leqslant\rangle$, where the operators O_a and O_e were evaluated in the same way but with respect to the two classes of relations $\mathcal{P}_a^a$ and $\mathcal{P}_a^e$ assigned to $a \in W$. The same could be done for the conditional operators $O_a(-/-)$ and $O_e(-/-)$ and preference connectives $\geq_e$ and $\geq_a$. There is a problem, however. In order for that framework to be equivalent to the basic multiple preference framework, all the relations in $\mathcal{P}_a^e$ must be transitive in order that (trans$_a$) be valid. At the same time, it must not be the case that all the relations in $\mathcal{P}_a^a$ are transitive, lest (trans$_e$) be valid. But it is also required that $\mathcal{P}_a^a \subseteq \mathcal{P}_a^e$ in order to validate (D.9$_{ae}$) or its counterpart for preference, (D.10$_{ae}$), which would be axioms of the weaker systems. These are incompatible requirements. I see no way out of this dilemma either.

Thus the techniques used to demonstrate completeness for SDDL or SPref and SDL$_a$P$_e$, or for DP or PPref and P$_a$P$_e$, do not lend themselves to a demonstration for SDDL$_a$DP$_e$ or SPref$_e$PPref$_a$ or for DP$_a$DP$_e$ or PPref$_e$ PPref$_a$, even if these systems are enriched with hitherto unnoticed valid postulates. This does not mean that these systems are not complete with respect to the multiple preference semantics as described. For all we know, it is possible that other techniques would be more succesful in establishing that. It is also possible, however, that the set of valid principles for conditional obligation and preference in the multiple preference semantics is not axiomatizable. The question remains open.[7]

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