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DIRECT DYNAMIC PROOFS FOR CLASSICAL COMPATIBILITY*

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1. *Introduction*

The scope of formal studies of reasoning has traditionally been restricted to deductive inference patterns. As a consequence, the contribution of formal logic to the understanding of actual reasoning processes has been relatively small. Indeed, both in the sciences and in everyday life, many reasoning processes are *ampliative* in nature: they lead to conclusions that ‘extend’ the information contained in the premises. Obvious examples include the use of induction to generate new generalizations and the use of abduction to generate new explanatory hypotheses.

It is only in recent years that computer scientists as well as logicians started paying attention to ampliative forms of reasoning. Important contributions to the formal study of ampliative reasoning are the (computer science) literature on non-monotonic logics and some recent results in the adaptive logics program (see, for instance, [5], [6] and [10]).

Two important insights resulted from these studies. The first is that compatibility is one of the basic concepts for the study of ampliative reasoning.[1] The reason is simple: a necessary requirement for an ampliative inference to be sound is that its conclusion is compatible with its premises. In the case of default reasoning, for instance, only those default rules can be applied that are compatible with one’s theory. Sometimes it is moreover required that the different conclusions are mutually compatible. For instance, in order for an inductive generalization to be sound, it should not only be compatible with the available data, but also with all other generalizations that can be generated from the same set of data (see [5]).

The second insight is that compatibility claims are not blind guesses, but are arrived at by *reasoning*. A central result in this respect is the logic

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[1] A sentence A is said to be compatible with a set of premises Γ, according to some logic L, iff $\Gamma \nvdash_{\mathsf{L}} \neg A$. What this comes to, semantically, is that A is true in *some* L-model of Γ.

COMPAT from [7]. This logic captures the concept of *classical compatibility*. Thus, where CL stands for Classical Logic, COMPAT leads from a set of sentences Γ to the set of sentences that are CL-compatible with Γ. If Γ is inconsistent, then, like CL, COMPAT leads to triviality.[2]

The importance of COMPAT is that it offers a *proof theory* for compatibility. It thus allows one to reason *from* a set of premises *to* the sentences that are compatible with them. A special feature of the proof theory is that it is *dynamic*. This is related to the fact that compatibility is *non-monotonic* (q is compatible with $\{p\}$, but not with $\{p, \neg q\}$), and that, moreover, at the predicative level, there is no positive test for it. As is shown in [7], the dynamic proof theory warrants that, even for undecidable fragments, one obtains a sensible and rational estimate of which sentences are compatible with the Γ under consideration. It is also shown that the dynamic proof theory is sound and complete with respect to a very nice and intuitive semantics.

As all forms of ampliative reasoning are ultimately based on specific kinds of compatibility relations, it seems warranted to expect that the results from [7] are a useful basis to design proof theories for different types of ampliative reasoning.

There is, however, one proviso: in [7], the logic COMPAT is characterized in an *indirect* way. More specifically, the definition of the logic is based, on the one hand, on a modal translation of the premises and the conclusion and, on the other hand, on a modal logic (based on S5, and called COM) that allows one to make inferences from this translation. The logic COMPAT is obtained by stipulating that A is a COMPAT-consequence of Γ iff the modal translation of A is a COM-consequence of the modal translation of Γ.

The modal translation is quite simple and intuitively appealing. Where $\Gamma^{\Box}$ stands for $\{\Box A \mid A \in \Gamma\}$, $\Gamma^{\Box} \vdash_{\mathsf{COM}} \Diamond A$ is taken to express that A is compatible with Γ, and $\Gamma^{\Box} \vdash_{\mathsf{COM}} \neg\Diamond A$ that A is incompatible with Γ. This is motivated by the fact that A is CL-compatible with Γ iff A is true in some CL-model of Γ, and hence, iff A is *possible* in view of Γ. As the members of Γ are true in *all* CL-models of Γ, it is easily observed that A is true in some CL-model of Γ iff $\Diamond A$ is true in some S5-model of $\Gamma^{\Box}$.

The advantage of this modal translation is that it leads in a very natural way to a semantics for compatibility (see the next section). The problem remains, however, that the proof theory of COM, precisely because it is formulated in

[2] In [7], a second logic of compatibility is presented, COMPAT*, that does not have this property. If Γ is inconsistent, then, according to COMPAT*, nothing is compatible with Γ, not even the members of Γ themselves. In [9], it is argued, however, that both COMPAT and COMPAT* are inadequate to handle inconsistent sets of premises. In the same paper, a logic of paraconsistent compatibility is presented that leads to the same results as COMPAT and COMPAT* for the consistent case, but that nevertheless allows for the sensible handling of inconsistent sets.

modal terms, falls short to explicate actual reasoning processes that involve compatibility considerations.

The aim of this paper is to present a *direct* proof theory for COMPAT (that proceeds entirely in the language of CL) and to show that it is equivalent to the indirect proof theory from [7]. Note that our claim is not that the direct characterization of COMPAT should replace the indirect one. The logic COM remains important from a meta-theoretical point of view—it is, for instance, far from evident that it is possible to design a direct semantics for compatibility. We do claim, however, that the direct proof theory is better suited to explicate actual reasoning processes.

2. *The Logic* COM

In this section, we present a brief overview of the logic COM. For more details, we refer the reader to [7].

The logic COM is an adaptive logic. The first logic in this family was designed by Diderik Batens around 1980 (see [1]) and was meant to interpret inconsistent sets of premises 'as consistently as possible'. Later, the notion of an adaptive logic was generalized not only to include other types of logical abnormalities (such as negation-incompleteness) but also to include ampliative forms of reasoning—see [3] and [4] for recent introductions to the topic.

The basic idea behind adaptive logics is that they interpret sets of premises 'as normally as possible'. What this comes to is that a sentence A is supposed to behave 'normally' with respect to a set of premises Γ *unless* Γ explicitly prevents so. Depending on how 'normal' is specified, one obtains a different adaptive logic. Inconsistency-adaptive logics, for instance, interpret sets of premises as consistently as possible; ambiguity-adaptive logics interpret them as unambiguously as possible.

In the case of the logic COM, it is the incompatibility of A with Γ that counts as an abnormality.[3] Thus, the plot behind COM is to assume that a sentence A is compatible with Γ unless this is prevented by Γ—that is, unless A is incompatible with Γ or, what comes to the same, unless $\Gamma^{\Box} \vDash_{\mathsf{S5}} \neg\Diamond A$.

Semantically, this is realized by making a *selection* of the S5-models of $\Gamma^{\Box}$. Intuitively, those S5-models of $\Gamma^{\Box}$ are selected that verify a formula of the form $\neg\Diamond A$ iff it is 'unavoidable' in view of $\Gamma^{\Box}$ (that is, iff it is true in *all* S5-models of $\Gamma^{\Box}$). For example, some S5-models of $\{\Box p\}$ verify $\neg\Diamond q$ and

[3] Note that "abnormality" does not refer to some standard of deduction, say CL. It refers to presuppositions that, in a particular application context, are regarded as desirable, but that may be overruled.

others verify $\neg\Diamond\neg q$—this is the reason why neither $\Diamond q$ nor $\Diamond\neg q$ is an S5-consequence of $\{\Box p\}$. However, as neither $\neg\Diamond q$ nor $\neg\Diamond\neg q$ are unavoidable in view of $\Gamma^{\Box}$, S5-models that verify one of them, are not included in the selection. As a consequence, all selected models verify $\Diamond q$ as well as $\Diamond\neg q$, which is exactly what we want.

In order to formulate the semantics of COM in a more precise way, we first need some definitions.[4]

Let $\mathcal{L}$ be the standard language of CL (including $\bot$, syntactically defined by $\bot \supset A$) and let Ω be the set $\{\neg\Diamond A \mid A \text{ is a wff of } \mathcal{L}\}$. Henceforth, members of Ω will be called "abnormalities".

The abnormalities that are *unavoidable* in view of $\Gamma^{\Box}$ are defined as:

Definition 1: $Ab(\Gamma^{\Box}) = \{A \in \Omega \mid \Gamma^{\Box} \models_{\mathsf{S5}} A\}$.

and the "abnormal part" of an S5-model $\mathcal{M}$ as:

Definition 2: $Ab(\mathcal{M}) = \{A \in \Omega \mid \mathcal{M}$ *verifies* $A\}$.

For a given set of premises $\Gamma^{\Box}$, the selection of the COM-models is now defined as follows:

Definition 3: *An* S5*-model* $\mathcal{M}$ *is a* COM*-model of* $\Gamma^{\Box}$ *iff* $Ab(\mathcal{M}) = Ab(\Gamma^{\Box})$.

Definition 3 warrants that, for any $\Gamma^{\Box}$, the selected models verify no other abnormalities than those that are unavoidable in view of $\Gamma^{\Box}$. Note especially that, in view of this definition, and as is usual for adaptive logics, it does not make sense to say that some S5-model is a COM-model, but only that it is a COM-model *of* some set of premises $\Gamma^{\Box}$.

As may be expected, the semantic consequence relation is defined with respect to the selected models:

Definition 4: $\Gamma^{\Box} \models_{\mathsf{COM}} A$ *iff all* COM*-models of* $\Gamma^{\Box}$ *verify* A.

The following theorem shows the intuitive adequacy of the above definitions.[5] We refer to [7] for its proof:

[4] In [7], the actual semantics for COM is presented as well as a simplified version; we immediately give the simplified one.

[5] Remember that, for inconsistent sets of premises, COM leads to triviality.

Theorem 1: *Where A is a wff of $\mathcal{L}$, $\Gamma^{\Box} \models_{\mathsf{COM}} \Diamond A$ iff $\Gamma \not\models_{\mathsf{CL}} \neg A$ or $\Gamma \models_{\mathsf{CL}} \bot$.*

Let us now turn to the proof theory. As was mentioned above, the proof theory of COM is dynamic: formulas that, at some stage of the proof, are considered to be derived, may at a later stage be 'withdrawn'. Technically, this is realized by attaching, to each line in the proof, a 'condition' (a possibly empty set of abnormalities).

Thus, lines in a COM-proof have the following structure:

i A $j_1, \ldots, j_n$ RULE Δ

The first four elements are as usual: i is the line number, A is the formula that is derived, $j_1, \ldots, j_n$ $(n \geq 0)$ stand for the line numbers of the formulas from which A is derived, and the fourth element is the justification (the rule by means of which A is derived). The fifth element, Δ, is the condition. Intuitively, this set contains the abnormalities that should not be derivable in order for A to be derivable.

The function of the condition is most easily illustrated by means of an example. Consider $\Gamma = \{(\forall x)(Qx \supset \neg Sx),\ (\forall x)(Px \supset Rx),\ Ra \wedge Sa\}$ and suppose that we want to check whether $(\forall x)(Rx \supset Qx)$, $(\forall x)(Px \supset Qx)$ and $(\forall x)(Sx \supset Px)$ are compatible with Γ. In COM, this comes down to checking whether $\Diamond(\forall x)(Rx \supset Qx)$, $\Diamond(\forall x)(Px \supset Qx)$ and $\Diamond(\forall x)(Sx \supset Px)$ are COM-derivable from $\Gamma^{\Box} = \{\Box(\forall x)(Qx \supset \neg Sx),\ \Box(\forall x)(Px \supset Rx),\ \Box(Ra \wedge Sa)\}$. This may be done like this.

First, we enter the premises:

1	$\Box(\forall x)(Qx \supset \neg Sx)$	–	PREM	$\emptyset$
2	$\Box(\forall x)(Px \supset Rx)$	–	PREM	$\emptyset$
3	$\Box(Ra \wedge Sa)$	–	PREM	$\emptyset$

Note that the premises are entered on the empty condition. This is as it should be: the derivability of the premises is not dependent on the normal behaviour of any formula. If a formula A is derived on a line that has the empty set as its fifth element, then A will be said to be derived *unconditionally*.

Next, we add the following lines:

4	$\Diamond(\forall x)(Rx \supset Qx)$	–	RC	$\{\neg\Diamond(\forall x)(Rx \supset Qx)\}$
5	$\Diamond(\forall x)(Px \supset Qx)$	–	RC	$\{\neg\Diamond(\forall x)(Px \supset Qx)\}$
6	$\Diamond(\forall x)(Sx \supset Px)$	–	RC	$\{\neg\Diamond(\forall x)(Sx \supset Px)\}$

The rule RC is a *conditional* rule: it allows one to add $\Diamond A$ to the proof (for any formula A of $\mathcal{L}$) on the condition $\{\neg\Diamond A\}$. This corresponds to the assumption that $\Diamond A$ is COM-derivable from the premises *unless* $\neg\Diamond A$ is S5-derivable from them.

Some readers may object that the condition of line 4 is not satisfied—$\neg\Diamond(\forall x)(Rx \supset Qx)$ *is* S5-derivable from the premises—and hence, that it should not be possible to add this line to the proof. However, as is usual for adaptive logics, it is allowed in COM-proofs that inferences are made on the basis of one's best insights in the premises (that is, on the basis of what is explicitly written down in the proof). So, as long as $\neg\Diamond(\forall x)(Rx \supset Qx)$ has not been derived in the proof, the formula of line 4 will be considered to be derived.

Suppose now that we continue the proof as follows:

7	$\neg\Diamond Qa$	$1, 3$	RU	$\emptyset$
8	$\neg\Diamond(\forall x)(Rx \supset Qx)$	$3, 7$	RU	$\emptyset$

The rule RU is a generic rule that allows one to add B to the proof whenever B is S5-derivable from $A_1, \ldots, A_n$ and $A_1, \ldots, A_n$ occur in the proof. Note that RU is an *unconditional* rule: it does not lead to the introduction of new conditions. So, if B is derived from $A_1, \ldots, A_n$, the condition of the line at which B occurs is simply the union of the conditions of the lines at which $A_1, \ldots, A_n$ occur.

At stage 8 of the proof, it has been established that the condition of line 4 is not satisfied. Hence, at this stage of the proof, the formula of line 4 should no longer be considered as derived in the proof. This will be expressed by 'marking' line 4:

1	$\Box(\forall x)(Qx \supset \neg Sx)$	–	PREM	$\emptyset$	
2	$\Box(\forall x)(Px \supset Rx)$	–	PREM	$\emptyset$	
3	$\Box(Ra \wedge Sa)$	–	PREM	$\emptyset$	
4	$\Diamond(\forall x)(Rx \supset Qx)$	–	RC	$\{\neg\Diamond(\forall x)(Rx \supset Qx)\}$	$\boxed{8}$
5	$\Diamond(\forall x)(Px \supset Qx)$	–	RC	$\{\neg\Diamond(\forall x)(Px \supset Qx)\}$	
6	$\Diamond(\forall x)(Sx \supset Px)$	–	RC	$\{\neg\Diamond(\forall x)(Sx \supset Px)\}$	
7	$\neg\Diamond Qa$	$1, 3$	RU	$\emptyset$	
8	$\neg\Diamond(\forall x)(Rx \supset Qx)$	$3, 7$	RU	$\emptyset$	

A formula is considered to be derived *at a stage* s of a COM-proof from $\Gamma^{\Box}$ iff it occurs on line i that, at that stage of the proof, is not marked. So, the formulas on lines 4 to 6 are all considered to be derived up to stage 7 of the proof, but the one on line 4 is no longer considered to be derived at stage 8. Note that, no matter how the proof is extended, it will not be possible to mark lines 5 and 6. This is why we say that $\Diamond(\forall x)(Px \supset Qx)$ and $\Diamond(\forall x)(Sx \supset Px)$, unlike $\Diamond(\forall x)(Rx \supset Qx)$, are *finally derivable* from the premises (see below for the definition).

We now give the precise formulation of the proof theory for COM. Adding lines to a proof from $\Gamma^{\Box}$ is governed by the rules PREM, RU and RC:

PREM If $A \in \Gamma^{\Box}$, then one may add to the proof a line consisting of
(i) the appropriate line number,

(ii) A,
(iii) "$-$",
(iv) "PREM", and
(v) $\emptyset$.

RU If $A_1, \ldots, A_n \vdash_{\mathsf{S5}} B$, and $A_1, \ldots, A_n$ ($n \geq 0$) occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add to the proof a line consisting of:
(i) the appropriate line number,
(ii) B,
(iii) the line numbers of the A_i or "$-$" if $n = 0$,
(iv) "RU", and
(v) $\Delta_1 \cup \ldots \cup \Delta_n$.

RC At any point of the proof, one may, for any formula A of $\mathcal{L}$, add to the proof a line consisting of:
(i) the appropriate line number,
(ii) $\Diamond A$,
(iii) "$-$",
(iv) "RC", and
(v) $\{\neg\Diamond A\}$.

The marking of lines is governed by the following definition:

Definition 5: *Marking for* COM*: Line i is marked at stage s of a* COM-*proof from $\Gamma^{\Box}$ iff, where Δ is the condition of line i, some $A \in \Delta$ has been* unconditionally *derived at stage s.*

In view of the marking definition, two notions of derivability may be defined: derivability at a stage (see above) and final derivability. The latter is defined as follows:

Definition 6: *A formula A is* finally derived *on line i of a* COM-*proof from $\Gamma^{\Box}$ iff (i) A is the second element of line i, (ii) line i is not marked in the proof, and (iii) line i will not be marked in any extension of the proof.*

As may be expected, the consequence relation is defined with respect to final derivability:

Definition 7: $\Gamma^{\Box} \vdash_{\mathsf{COM}} A$ *(A is finally derivable from $\Gamma^{\Box}$) iff A is finally derived on a line in a* COM-*proof from $\Gamma^{\Box}$.*

As is proven in [7], the semantics of COM is, in a restricted way, sound and complete with respect to its proof theory:

Theorem 2: *Where A is a wff of $\mathcal{L}$, $\Gamma^{\Box} \vdash_{\mathsf{COM}} \Diamond A$ iff $\Gamma^{\Box} \models_{\mathsf{COM}} \Diamond A$.*

3. *Direct Proofs for* COMPAT

In [7], the logic COMPAT is indirectly defined as follows:

Definition 8: $\Gamma \vdash_{\mathsf{COMPAT}} A$ *iff* $\Gamma^{\Box} \vdash_{\mathsf{COM}} \Diamond A$.

In view of the proof theory for COM, a direct proof theory for COMPAT can easily be obtained. It suffices to extend CL with the following conditional rule:

RC' At any point of the proof, one may, for any formula A of $\mathcal{L}$, add to the proof a line consisting of:
- (i) the appropriate line number,
- (ii) A,
- (iii) "$-$",
- (iv) "RC'", and
- (v) $\{\neg A\}$.

and to formulate an appropriate marking definition.

The idea behind RC' is simple and nicely captures the intuition behind compatibility: we assume that A is compatible with the premises unless and until this assumption is proven to be false (that is, unless and until $\neg A$ is CL-derivable). In view of the meta-proofs, we shall, however, rely on a conditional rule that is slightly more general (and from which RC' may be derived). The idea will be that, whenever a disjunction $B \vee \bigvee(\Delta)$ is CL-derivable in the proof, B may be derived on the condition Δ.

Here are the generic rules for COMPAT-proofs:

PREM If $A \in \Gamma$, then one may add to the proof a line consisting of
- (i) the appropriate line number,
- (ii) A,
- (iii) "$-$",
- (iv) "PREM", and
- (v) $\emptyset$.

RU If $A_1, \ldots, A_n \vdash_{\mathsf{CL}} B$, and $A_1, \ldots, A_n$ ($n \geq 0$) occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add to the proof a line consisting of:
- (i) the appropriate line number,
- (ii) B,
- (iii) the line numbers of the A_i or "$-$" if $n = 0$,
- (iv) "RU", and
- (v) $\Delta_1 \cup \ldots \cup \Delta_n$.

RC If $A_1, \ldots, A_n \vdash_{\mathsf{CL}} B \vee \bigvee(\Delta)$, and $A_1, \ldots, A_n$ ($n \geq 0$) occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then one may add to the proof a line consisting of:
(i) the appropriate line number,
(ii) B,
(iii) the line numbers of the A_i or "$-$" if $n = 0$,
(iv) "RC", and
(v) $\Delta \cup \Delta_1 \cup \ldots \cup \Delta_n$.

The following rules are obviously derivable and lead to proofs that are more interesting from a heuristic point of view:

RD1 If A is derived in the proof on the condition $\{C_1, \ldots, C_n\}$ and $\neg A$ is derived on the condition $\emptyset$, then one may add to the proof a line consisting of
(i) the appropriate line number,
(ii) $\bigvee\{C_1, \ldots, C_n\}$,
(iii) the line numbers of A and $\neg A$,
(iv) "RD1", and
(v) $\emptyset$.

RD2 If A is derived in the proof on the condition $\{C_1, \ldots, C_n\}$ and $\neg A$ is derived on the condition $\{D_1, \ldots, D_m\}$, then one may add to the proof a line consisting of
(i) the appropriate line number,
(ii) $\bigvee(\{C_1, \ldots, C_n\} \cup \{D_1, \ldots, D_m\})$,
(iii) the line numbers of A and $\neg A$,
(iv) "RD2", and
(v) $\emptyset$.

The marking definition for COMPAT is somewhat different from that for COM and will be illustrated in the example below:

Definition 9: *Marking for* COMPAT*: Line i is marked at a stage of a proof from Γ iff, where Δ is the condition of line i, $\bigvee(\Delta)$ is unconditionally derived at that stage of the proof.*

As is the case for COM, a formula A is said to be derived at a stage s of a COMPAT-proof from Γ iff A is the second element of a non-marked line at stage s. Also the definitions of final derivability and of the consequence relation are analogous to those for COM:

Definition 10: *A formula A is* finally derived *on line i of a* COMPAT*-proof from Γ iff (i) A is the second element of line i, (ii) line i is not marked in the proof, and (iii) line i will not be marked in any extension of the proof.*

Definition 11: $\Gamma \vdash_{\mathsf{COMPAT}} A$ (*A is finally derivable from Γ*) *iff A is finally derived on a line in a* COMPAT*-proof from Γ.*

Here is the direct proof for the example from the previous section:

1	$(\forall x)(Qx \supset \neg Sx)$	–	PREM	$\emptyset$
2	$(\forall x)(Px \supset Rx)$	–	PREM	$\emptyset$
3	$Ra \wedge Sa$	–	PREM	$\emptyset$
4	$(\forall x)(Rx \supset Qx)$	–	RC	$\{\neg(\forall x)(Rx \supset Qx)\}$ $\boxed{8}$
5	$(\forall x)(Px \supset Qx)$	–	RC	$\{\neg(\forall x)(Px \supset Qx)\}$
6	$(\forall x)(Sx \supset Px)$	–	RC	$\{\neg(\forall x)(Sx \supset Px)\}$
7	$\neg Qa$	$1, 3$	RU	$\emptyset$
8	$\neg(\forall x)(Rx \supset Qx)$	$3, 7$	RU	$\emptyset$

Unlike what was the case for COM, COMPAT allows one to 'conjoin' different compatibility hypotheses. For instance, from the formulas on lines 5 and 6, one may derive:

9	$(\forall x)(Sx \supset Qx)$	$5, 6$	RU	$\{\neg(\forall x)(Sx \supset Px), \neg(\forall x)(Px \supset Qx)\}$

Thus, the proof theory for COMPAT is, in a sense, richer than that for COM. (In COM, it is not possible to derive $\Diamond(\forall x)(Sx \supset Qx)$ from $\Diamond(\forall x)(Px \supset Qx)$ and $\Diamond(\forall x)(Sx \supset Px)$.) That it is not too rich is warranted by the marking definition. Suppose, for instance, that we continue the proof as follows:

10	Pa	$3, 6$	RU	$\{\neg(\forall x)(Sx \supset Px)\}$
11	$(\forall x)(Px \supset \neg Sx)$	$1, 5$	RU	$\{\neg(\forall x)(Px \supset Qx)\}$
12	$\neg Pa$	$3, 11$	RU	$\{\neg(\forall x)(Px \supset Qx)\}$
13	$Pa \wedge \neg Pa$	$10, 12$	RU	$\{\neg(\forall x)(Sx \supset Px), \neg(\forall x)(Px \supset Qx)\}$

Although both Pa and $\neg Pa$ are compatible with the premises, $Pa \wedge \neg Pa$ is obviously not, and hence, should not be finally derivable from them. However, as soon as the following line is added to the proof, lines 9 and 13 are marked:

14	$\neg(\forall x)(Sx \supset Px) \vee \neg(\forall x)(Px \supset Qx)$	10,12	RD2	$\emptyset$

Note that, although both $(\forall x)(Sx \supset Px)$ and $(\forall x)(Px \supset Qx)$ are compatible with the premises (the formulas on lines 5 and 6 are finally derived), they are not *jointly* compatible with them. This is what the formula on line 14 teaches: in view of the premises, either $(\forall x)(Sx \supset Px)$ or $(\forall x)(Px \supset Qx)$ must be false.

In the rest of this section, we prove that the direct proof theory for COMPAT is equivalent to the indirect one from Definition 8. For the sake of generality, $A \vee \bigvee(\emptyset)$ will denote A.[6]

[6] The logic COMPAT is not the first adaptive logic for which a direct proof theory is designed. In [2], for instance, it is shown that the so-called Rescher-Manor consequence

Lemma 1: *If, in a* COMPAT*-proof from* Γ*,* A *occurs as the second element of line* i *and* Δ *as its fifth element, then* $\Gamma \vdash_{\mathsf{CL}} A \vee \bigvee(\Delta)$.

Proof. The proof proceeds by induction on the number of the line at which A occurs. The lemma obviously holds if $i = 1$, for then, in view of the generic rules, $A \in \Gamma$ or $\vdash_{\mathsf{CL}} A \vee \bigvee(\Delta)$. Suppose that the lemma holds for all lines that precede i.

Case 1: The third element of line i is "$-$". Analogous to the case where $i = 1$.

Case 2: The third element of line i is not "$-$". Suppose that the third element of i is $j_1, \ldots, j_n$ ($n \geq 1$) and that $B_1, \ldots, B_n$ are the second elements of lines $j_1, \ldots, j_n$. Both RU and RC warrant that $B_1, \ldots, B_n \vdash_{\mathsf{CL}} A \vee \bigvee(\Delta)$, and hence, that $\vdash_{\mathsf{CL}} ((B_1 \wedge \ldots \wedge B_n) \supset A) \vee \bigvee(\Delta)$. As the fifth elements of lines $j_1, \ldots, j_n$ are subsets of Δ, the supposition warrants that $\Gamma \vdash_{\mathsf{CL}} B_i \vee \bigvee(\Delta)$ for every B_i, and hence, that $\Gamma \vdash_{\mathsf{CL}} (B_1 \wedge \ldots \wedge B_n) \vee \bigvee(\Delta)$. But then, $\Gamma \vdash_{\mathsf{CL}} A \vee \bigvee(\Delta)$. □

Theorem 3: $\Gamma \vdash_{\mathsf{COMPAT}} A$ *iff* $\Gamma \vdash_{\mathsf{CL}} A$ *or there is a non-empty set* Δ *such that* $\Gamma \vdash_{\mathsf{CL}} A \vee \bigvee(\Delta)$ *and* $\Gamma \nvdash_{\mathsf{CL}} \bigvee(\Delta)$.

Proof. The right-left direction immediately follows by inspection of the proof theory: if $\Gamma \vdash_{\mathsf{CL}} A$, then $\Gamma \vdash_{\mathsf{COMPAT}} A$ in view of RU; if, for some non-empty set Δ, $\Gamma \vdash_{\mathsf{CL}} A \vee \bigvee(\Delta)$ and $\Gamma \nvdash_{\mathsf{CL}} \bigvee(\Delta)$, then $\Gamma \vdash_{\mathsf{COMPAT}} A$ in view of RC and the marking definition.

For the left-right direction, suppose that $\Gamma \vdash_{\mathsf{COMPAT}} A$. In that case, A is finally derived at some line j of a COMPAT-proof from Γ. Hence, where Δ is the fifth element of line j, $\Gamma \vdash_{\mathsf{CL}} A \vee \bigvee(\Delta)$ in view of Lemma 1.

It only remains to be shown that, if Δ is non-empty, then $\Gamma \nvdash_{\mathsf{CL}} \bigvee(\Delta)$. Suppose that Δ is non-empty and that $\Gamma \vdash_{\mathsf{CL}} \bigvee(\Delta)$. As CL is compact, there is an extension of the proof in which $\bigvee(\Delta)$ occurs unconditionally. But then, line j is marked in that extension, and will remain marked in any further extension. This contradicts that A is finally derived at line j. □

Theorem 4: $\Gamma \vdash_{\mathsf{COMPAT}} A$ *iff* $\Gamma \nvdash_{\mathsf{CL}} \neg A$ *or* $\Gamma \vdash_{\mathsf{CL}} \bot$.

relations (the Free, Strong, Argued, C-based and Weak consequence relations) can be characterized by means of specific inconsistency-adaptive logics (which incorporate a paraconsistent negation as well as the classical one). The direct proof theories, that proceed entirely in the language of CL, can be found in [8]. An important difference with the present case is that the equivalence proof for COMPAT does not require that a correspondence is established between the indirect proofs and the direct ones.

Proof. For the left-right direction, suppose that $\Gamma \vdash_{\mathsf{COMPAT}} A$. Suppose further that $\Gamma \vdash_{\mathsf{CL}} \neg A$ and that $\Gamma \nvdash_{\mathsf{CL}} \bot$. It follows, in view of Theorem 3, that (i) $\Gamma \vdash_{\mathsf{CL}} A$ or (ii) that, for some non-empty set Δ, $\Gamma \vdash_{\mathsf{CL}} A \vee \bigvee(\Delta)$ and $\Gamma \nvdash_{\mathsf{CL}} \bigvee(\Delta)$. However, both (i) and (ii) are impossible in view of the supposition.

For the right-left direction, suppose first that $\Gamma \nvdash_{\mathsf{CL}} \neg A$. It follows, in view of $\vdash_{\mathsf{CL}} A \vee \neg A$, RC and the marking definition, that A is finally derived at some line in a COMPAT-proof from Γ, and hence, that $\Gamma \vdash_{\mathsf{COMPAT}} A$. The case where $\Gamma \vdash_{\mathsf{CL}} \bot$ is obvious in view of RU. □

Theorem 5: $\Gamma \vdash_{\mathsf{COMPAT}} A$ *iff* $\Gamma^{\Box} \vdash_{\mathsf{COM}} \Diamond A$.

Proof. Immediate in view of Theorem 1, Theorem 2, the Completeness of CL and Theorem 4. □

4. *In Conclusion*

In this paper, we presented a direct proof theory for classical compatibility and showed that it is equivalent to the indirect one from [7]. Important open problems concern the design of a direct proof theory for paraconsistent compatibility (as studied in [9]) and the formulation of proof theories for other forms of ampliative reasoning. Given the central role that compatibility plays in ampliative reasoning, the results from the present paper should constitute a good point of departure for this.

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