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SIEVING OUT RELEVANT AND EFFICIENT QUESTIONS

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Abstract
Wiśniewksi's erotetic logic provides us with two slightly different semantic explications of the intuitive concept of "a question arises from a set of declarative premises". Unfortunately, Wiśniewski's erotetic concepts suffer from the drawback that they allow for the raising of irrelevant and inefficient questions. The aim of this paper is to show that raising such questions can be avoided by changing the underlying logic. Several closely related logical approaches which enable us to eliminate irrelevant and inefficient questions, are presented.

1. *Introduction*

One of the most fascinating problems studied in erotetic logic is the way in which questions *arise* from a set of statements or a set of statements together with another question. A major contribution in this respect is Wiśniewski's logic of questions (see [10]). In this paper we only focus on the raising of a question from a set of declarative sentences. Wiśniewski offers two slightly different *semantic* explications of this concept: *evocation* of a question by a set of declarative premises, and *generation* of a question by a set of declarative premises. Although Wiśniewski's general characterization of erotetic concepts can be applied to any logic of questions that satisfies some minimal conditions (see [10, pp. 226–230]), the standard of deduction in [10] is always Classical Logic (henceforth CL).

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In this paper we are looking for an explication of the raising of *relevant* and *efficient* questions from a set of declarative premises Γ. We show that Wiśniewski's concept of *evocation* permits the raising of questions that are *irrelevant* to the set of premises on hand. The stronger[1] concept of *generation* prevents the raising of some irrelevant questions. However, (i) some generated questions are still irrelevant to Γ, and (ii) some questions that are intuitively relevant to Γ are not generated by Γ. Hence, both the concept of evocation and the concept of generation are unsuitable – unless one changes the underlying logic from CL to some non-standard logic – for explicating the raising of relevant and efficient questions from a set of premises. Evocation is certainly too weak, whereas generation is too weak in some respects and too strong in others.

In this paper we restrict ourselves to the propositional level (PC refers to propositional Classical Logic). This has the advantage that we can leave out the rather complex logical machinery needed for Wiśniewski's general definitions. We show that the problems of relevance and efficiency are mainly caused by the standard properties of the disjunction. We present several logics in which the disjunction has nonstandard properties, and we show that those logics are better suited for the derivation of relevant and efficient questions than PC. It should be clear that the problems of relevance and efficiency also occur at the predicative level, and even more so (because of the additional source of irrelevance due to existentially quantified formulas).

When approaching the outlined problems, most logicians will look for conditions that restrict the definition of question evocation. Inspired by the Ghent tradition, we tackled the problem differently, viz. in terms of logics. More precisely, we shall present the logics RAD, QP, $\mathsf{QP}^{\star}$ and QP^{p}, that are derived from PC – if the basic logic is different, so would be the derived logics. These logics will determine which questions are evoked from a set of premises in cases where PC is the standard of deduction.

In the next section we present Wiśniewski's concepts of evocation and generation of questions (for the underlying logic PC). In section 3 we point to the problem of the raising of irrelevant and inefficient questions, both for the concept of evocation and generation. We also argue that, if one is convinced that pragmatic aspects (concerning the posing and answering of questions) should better be neglected when developing or evaluating a logic of questions, then one should also give up any attempt to develop a sensible and somewhat sophisticated logic of questions. For in that case, a very basic 'logic' of questions will do. In section 4 we present the non-monotonic logic RAD, in which the disjunction behaves in a nonstandard way. Relying on

[1] We say that generation is a stronger concept than evocation because the set of questions generated by Γ is a subset of the set of questions evoked by Γ.

RAD, we develop the logic QP in Section 5. In the next section we define the concept of questions QP-raised by Γ, and present a proof theory for the logic of questions QP^{q}. In section 7 we offer two different approaches for the derivation of questions that are both maximally relevant and maximally efficient.

2. *Wiśniewski's concepts of evocation and generation*

As we already mentioned, Wiśniewski's concepts of question evocation and generation can be applied to any logic of questions that satisfies some minimal requirements.[2] In order to define evocation and generation in a way as general as possible, Wiśniewki uses *partitions* and *multiple-conclusion entailment*, two concepts borrowed from [6]. For our purposes, pointing out the problem of the raising of *irrelevant* and *inefficient* questions, it suffices to focus on a logic of questions that is based on PC. It will become clear, though, that any logic of questions built on a declarative logic in which the disjunction behaves classically[3], suffers from the problem of irrelevance and inefficiency.

We first make some terminological and notational remarks. Let $\mathcal{L}$ be the language of PC, containing the logical constants $\neg$, $\supset$, $\wedge$, $\vee$ and $\equiv$. We use the letters $p, q, r, s, t, u, p_1, \ldots$ for propositional variables. The well-formed formulas (henceforth, wffs) of $\mathcal{L}$ are defined as usual. The language $\mathcal{L}^q$ of PC^{q} is obtained by extending $\mathcal{L}$ with the symbols ?, {, } and ",". By declarative well-formed formulas (henceforth, d-wffs) of $\mathcal{L}^q$ we mean the well-formed formulas of $\mathcal{L}$. We always use $A, B, C, \ldots$ as metavariables for d-wffs, and $\Gamma, \Gamma', \ldots$ for sets of d-wffs. A *question* of $\mathcal{L}^q$ is an expression of the form $?\{A_1, \ldots, A_n\}$, where $n \geq 2$ and $A_1, \ldots, A_n$ are syntactically different d-wffs. Questions are the erotetic wffs (e-wffs) of $\mathcal{L}^q$. A question $?\{A_1, \ldots, A_n\}$ can be read as "Is it the case that A_1, or …, or is it the case that A_n?".[4] We use $Q, Q_1, \ldots$ as metavariables for questions. The set of

[2] An obvious requirement is that its language $\mathcal{L}$ consists of a declarative part and an erotetic part (that allows for questions). The only further requirements are that the declarative part of $\mathcal{L}$ is provided with a proper semantics, and that its erotetic part assigns to each question an at least two-element set of direct answers.

[3] This is manifested, for instance, in Addition being a valid inference rule.

[4] In some cases a different reading seems to be more appropriate: for instance, $?\{p, \neg p\}$ can be read as "Is it the case that p?".

wffs of $\mathsf{PC^q}$, $\mathcal{W}^q$, is the union of the set of d-wffs and the set of e-wffs of $\mathcal{L}^q$.

If $?\{A_1, \ldots, A_n\}$ is a question, then each of the d-wffs $A_1, \ldots, A_n$ is a *direct answer* to this question. The set of direct answers to a question Q is referred to as dQ. Note that dQ is a finite set, and contains at least two elements. Any d-wff that is implied by each member of dQ is called a *presupposition* of the question Q. Where $\{A_1, \ldots, A_n\}$ is a non-empty finite set of formulas, $\bigvee\{A_1, \ldots, A_n\}$ abbreviates $A_1 \vee \ldots \vee A_n$; $\bigwedge\{A_1, \ldots, A_n\}$ abbreviates $A_1 \wedge \ldots \wedge A_n$. For every question $Q = ?\{A_1, \ldots, A_n\}$, we choose $A_1 \vee \ldots \vee A_n$ to be *the* presupposition of Q, and we refer to it by $Pres(Q)$.

Wiśniewski's concept of *evocation* – for the underlying logic PC[5] – can be defined as follows:

Definition 1: *A question* $Q = ?\{A_1, \ldots, A_n\}$ *is* evoked *by a set of d-wffs* Γ, $\mathsf{E}(\Gamma, Q)$, *iff*
(i) $\Gamma \vDash_{\mathsf{PC}} A_1 \vee \ldots \vee A_n$, *and*
(ii) for each $A_i \in dQ$: $\Gamma \nvDash_{\mathsf{PC}} A_i$.

The first clause requires that Q is *sound* relative to Γ: intuitively, this requirement is fulfilled iff Q is truly answerable in case all members of Γ are true. The second clause requires that Q is *informative* relative to Γ: intuitively, this requirement is fulfilled iff Q cannot be answered on the basis of Γ.

Except for border cases, the set of questions evoked by a set of d-wffs Γ is infinite. Unfortunately, the major part of these evoked questions are completely uninteresting, comparable with the repetitive and pointless application of Addition when deriving consequences from a set of d-wffs (the comparison is no accident). Of course, some of the questions evoked by Γ are highly relevant to Γ, but they risk to get lost in the crowd. We give a simple example, which we will continue to use throughout the paper, to illustrate our point. Let $\Gamma = \{p \vee q, r\}$. The (infinite) set of questions evoked by Γ contains the following questions:

(i) $?\{p, q\}, ?\{p, \neg p\}, ?\{q, \neg q\}, ?\{\neg p, \neg q, p \wedge q\}, ?\{p \wedge q, \neg(p \wedge q)\}$
(ii) $?\{p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q\}, ?\{p, q, \neg r\}, ?\{p, q, t \wedge \neg t\}, \ldots$
(iii) $?\{p, q, s\}, ?\{p, q \vee s\}, ?\{p \vee s, \neg(\neg q \wedge \neg s), t\}, \ldots$
(iv) $?\{p \wedge q, p \wedge \neg q, \neg p \wedge q\}, ?\{p \wedge r, q \wedge r\}, ?\{p \wedge r, \neg p \wedge r\}, ?\{p \wedge r, \neg p \wedge r, q \wedge r, \neg q \wedge r\}, ?\{r \wedge t, r \wedge \neg t\}, ?\{r \wedge t, r \wedge \neg t, p \wedge s, q \wedge s, \neg s\}, \ldots$

[5] Note that PC is supposed to be characterized here in a way that allows for continuous disjunctions and conjunctions. From now on, PC refers to such characterization.

(v) $?\{s, \neg s\}, ?\{t, \neg t\}, ?\{s\wedge t, s\wedge\neg t, \neg s\wedge t, \neg s\wedge\neg t\}, ?\{p, q\wedge s, q\wedge\neg s\}, \ldots$
(vi) $?\{(p\wedge q)\wedge r, \neg p\wedge\neg q, t, \neg p\wedge s, \neg q\wedge r, \neg s\}, ?\{p\wedge r, q, s\}, ?\{s\wedge r, \neg s\wedge r\}, \ldots$

We think everyone would agree that all questions in (i) are intuitively *relevant* to Γ. Moreover, they all are *efficient* (see below). We label all other questions as *irrelevant*, *inefficient*, or both. We briefly and somewhat intuitively[6], try to explain the reasons for this labelling.

The questions in (ii) all have at least one direct answer that is self-contradictory or that is incompatible with the d-wffs in Γ: for every question Q in (ii), there is an $A_i \in dQ$ such that $\Gamma \cup \{A_i\} \vDash_{\mathsf{PC}} A \wedge \neg A$. Clearly, each such question is irrelevant: for each such irrelevant question Q, there is a question Q' which is also evoked by Γ, $dQ' \subsetneq dQ$ and dQ' does not contain any direct answer that is self-contradictory or incompatible with Γ.

The questions in (iii) all contain at least one direct answer that is irrelevant to Γ: the direct answer contains a propositional letter that is not mentioned in Γ, and the only reason why that propositional letter pops up in the evoked question is because of the properties of the disjunction (and hence also implication) in PC (think of Addition and Irrelevance). So, it could have been any propositional letter, and indeed, all questions in (iii) in which the new letter is replaced by another one, are also evoked by Γ.

These irrelevant questions in (ii) and (iii) are not evoked when the underlying logic is changed from PC to RAD, as we will do in section 4.

Some of the questions in (iv) could be considered as being relevant to Γ, but all these questions are *inefficient* with respect to Γ: they all have at least one direct answer that contains a *superfluous* part. To define the "superfluous part" of a direct answer, we first define when a formula is in *negation normal form*:

Definition 2: *A formula A is in* negation normal form $N(A)$ *iff*

(1) *A does not contain the connectives $\supset$ or $\equiv$, and*
(2) *every occurrence of $\neg$ in A is immediately followed by a propositional letter.*

Every formula A can be brought into negation normal form by replacing equivalences $A \equiv B$ and implications $C \supset D$ by resp. $(A \supset B) \wedge (B \supset A)$

[6] The non-standard logics we present in the next sections offer an explication of these intuitions.

and $\neg C \vee D$, using De Morgan's laws to push negations inside, and eliminating double negations. It is obvious that any PC-wff A is PC-equivalent with its negation normal form $N(A)$.

Definition 3: *A question Q evoked by Γ has a direct answer A containing a* superfluous *part iff*

(i) *the negation normal form of A, N(A), contains a (sub)formula of the form $B_1 \wedge \ldots \wedge B_n (n \geq 2)$, and*

(ii) *there is a B_i $(1 \leq i \leq n)$ such that, if A' is the result of replacing in $N(A)$ (an occurrence of) the (sub)formula $B_1 \wedge \ldots \wedge B_n$ by $\bigwedge(\{B_1, \ldots, B_n\} \setminus \{B_i\})$, it holds that $\Gamma \cup \{A'\} \vdash_{\mathsf{PC}} A$.*

A filter that sieves efficient questions from those containing superfluous parts is imposed by changing the underlying logic in the definition of evocation from PC to QP, as we will do in section 5. As QP is built on the logic RAD, all questions in (i) are QP-evoked, and none of the questions in (ii)-(iv) are QP-evoked by Γ.

It is obvious that all the questions in (v) are irrelevant to Γ, for the same reasons as the questions in (iii). Some of these questions could be eliminated by the concept of generation (see below), but to eliminate them all, we shall take a completely different road in section 7.

The questions in (vi) are mixed cases, being both irrelevant and inefficient with respect to Γ.

An alternative explication of the concept of the arising of a question from a set of declarative sentences, is given by Wiśniewski's concept of generation. For the underlying logic PC, the concept of generation can be defined as follows:

Definition 4: *A question Q is* generated *by a set of declarative premises Γ, $\mathsf{G}(\Gamma, Q)$, iff*

(i) $\mathsf{E}(\Gamma, Q)$, *and*

(ii) $\emptyset \nvDash_{\mathsf{PC}} Pres(Q)$.

Hence, not all questions evoked by Γ are also generated by Γ, and in this sense the concept of generation is a stronger one than that of evocation. This can be stated more clearly by using Wiśniewksi's concept of a safe question:

Definition 5: *A question Q is* safe *iff* $\emptyset \vDash_{\mathsf{PC}} Pres(Q)$. *Otherwise, Q is a* risky *question.*

Hence, a question Q is generated by Γ iff (i) Q is evoked by Γ and (ii) Q is a risky question.

By imposing an extra condition, it seems plausible that the concept of generation provides a better explication of the raising of relevant and efficient questions, but actually, it does not. A lot of questions we considered to be irrelevant or inefficient are also generated by Γ: for instance, the second and third question in (ii) and all questions in (iii) and (iv) are generated by $\Gamma = \{p \vee q, r\}$. Moreover, although neither the first question in (ii) nor any question in (v) is generated by Γ (which is a good thing), the questions $?\{p, \neg p\}$, $?\{q, \neg q\}$ and $?\{p \wedge q, \neg(p \wedge q)\}$ are also not generated by Γ and they are both relevant and efficient.

So the concept of generation is not suited for explicating the raising of relevant and efficient questions by a set of premises Γ, because it is too weak in some respects and too strong in others.

3. *Pragmatic aspects of raising questions*

Pursuing answers to questions that are irrelevant to one's set of premises at hand may be a pleasant way to kill time, but taking into account that most of us do not dispose of unlimited resources and time, it is not a very rational way to operate.

Once a question is raised by Γ, one can decide to actually pose the question (addressing the question to someone, possibly oneself, or something). In most cases one then expects the addressed party to come up with some (preferably correct) answer to the question (which of course is not always straightforward). An answer to a question can be reached by several means: by observation, by consulting some knowledge source (an encyclopedia, a textbook, an expert, a database, etc.), by performing tests or experiments, or by reasoning. More often than not, a combination of these actions has to be performed to come up with an answer to the question. Of course, all these actions take time, effort, and hence – in most cases – money. Apart from 'real' costs, every action comes also – in economic parlance – with an opportunity cost: the cost of the most valuable forgone alternative. For instance, instead of wasting resources on trying to come up with a correct answer to a question irrelevant to one's knowledge set, one could have used these resources to obtain an answer to one or more highly relevant questions (the information or problem one is really interested in).

In light of the above considerations it seems rational to only pose questions (and pursue answers to them) that are relevant to the set of premises at hand. Moreover, it is important to pose (maximally) efficient questions: the person or source that is trying to answer the question posed, should not waste its resources on coming up with a more informative answer, when a less informative answer – together with the initial set of premises – would have been sufficient to derive a direct answer to the question. Note that the informativeness requirement in the definition of evocation and generation ($\Gamma \nvDash A_i$, for any $A_i \in dQ$), is a (minimal) interpretation of the requirement that a question raised by Γ should be efficient with respect to Γ. Roughly, it comes down to the following maxim: you should not ask a question of which you already know the answer. We will strengthen this maxim in the following way: you should not ask a question of which you already know some part(s) of the answer.

The reader may wonder whether these pragmatic aspects should be taken into consideration when one is trying to build a decent logic of questions. However, if pragmatic aspects are ignored, one hardly needs any logic of questions of some sophistication at all, because one can just as well investigate by brute force. This method can be described – in a somewhat caricatural way – as follows: if one doesn't know everything there is to know, one should ask the question "What are all the true non-tautological propositions statable in at most n words from vocabulary V?", where n is a fixed, very large number and V for instance the set of all English words tokens of which have occurred in print at least once.[7] A direct answer to this question is finite. We make our point somewhat more accurately. Let us introduce the following definition:

Definition 6: *A set of premises* Γ PC*-decides (the truth value of) a formula* A *iff* $\Gamma \vDash_{\mathsf{PC}} A$ *or* $\Gamma \vDash_{\mathsf{PC}} \neg A$.

Suppose Γ is a non-empty, finite set of declarative formulas. Then Γ contains at most n ($n \geq 1$) propositional variables $A_1, \ldots, A_n \in \mathcal{S}$. If Γ decides all of them, i.e. for each A_i ($1 \leq i \leq n$) either $\Gamma \vDash_{\mathsf{PC}} A_i$ or $\Gamma \vDash_{\mathsf{PC}} \neg A_i$, then there is no need for asking a (relevant) question. But if for at least some A_i, $\Gamma \nvDash_{\mathsf{PC}} A_i$ and $\Gamma \nvDash_{\mathsf{PC}} \neg A_i$, then one could ask – remember the brute force – the following question $?\{A_1 \wedge \ldots \wedge A_n, A_1 \wedge \ldots \wedge \neg A_n, \ldots, \neg A_1 \wedge \ldots \wedge \neg A_n\}$. But, if pragmatic aspects don't matter anyway, one could just as well ask a much wilder question. Let $B_1, \ldots, B_m$ be a very long (but finite) list of propositional variables not occurring in Γ. Then the following question is evoked by Γ: $?\{A_1 \wedge \ldots \wedge A_n \wedge B_1 \wedge \ldots \wedge B_m, \ldots, \neg A_1 \wedge \ldots \wedge \neg A_n \wedge \neg B_1 \wedge \ldots \wedge \neg B_m\}$. We think nobody would seriously consider asking such a question, unless one meets an angel, of course.

[7] This question is one of the candidates for being the best question one can pose (to an angel to whom you are allowed to pose exactly one question that will be truthfully answered). See [7] for a survey of the (pragmatic) complications with respect to finding the best question to ask, and some amusing reflections on this point.

4. *The logic* RAD

In [8] the logic RAD is developed to solve the paradoxes of material implication. The logic RAD is a propositional logic based on PC. A good way to get an intuitive grasp on it, is to consider it as PC on which a special filter is imposed, screening for “superfluous” disjuncts within a formula. Only those PC-consequences of Γ that do not contain superfluous disjuncts are RAD-consequences of Γ. The meta-theoretic properties of RAD are extensively studied in [8].[8] The language and wffs of RAD are those of PC.

4.1. *Semantics*

The semantic characterization of RAD is rather unusual, because it is given in terms of *sets* of classical propositional truth-value assignments (PC-models for short). Let M range over the PC-models, and S over the sets of PC-models. Let $M \vDash A$ mean that the PC-model M verifies A, and let $S \vDash_{\mathsf{PC}} A$ mean that all PC-models in S verify A. We also need the following definitions:

Definition 7: $S_{\mathsf{PC}}(\Gamma) = \{M \mid M \vDash \Gamma\}$.

Definition 8: *An n-tuple* $(S_1, \ldots, S_n)$ *is an* n-partition *of* S *iff* $S_1 \cup \ldots \cup S_n = S$ *and* $S_i \cap S_j = \emptyset$ *for* $1 \leq i, j \leq n$ *and* $i \neq j$.

The logic RAD is semantically characterized by the following clauses:

(1) For A a propositional letter, $S \vDash_{\mathsf{RAD}} A$ iff for all $M \in S$, $M \vDash A$.
(2) For A a propositional letter, $S \vDash_{\mathsf{RAD}} \neg A$ iff for all $M \in S$, $M \vDash \neg A$.
(3) $S \vDash_{\mathsf{RAD}} A_1 \wedge \ldots \wedge A_n$ iff $S \vDash_{\mathsf{RAD}} A_1$ and … and $S \vDash_{\mathsf{RAD}} A_n$.
(4) $S \vDash_{\mathsf{RAD}} A_1 \vee \ldots \vee A_n$ iff
  (i) $S \vDash_{\mathsf{PC}} A_1 \vee \ldots \vee A_n$, and
  (ii) for all n-partitions $(S_1, \ldots, S_n)$ of S for which $S_1 \vDash_{\mathsf{PC}} A_1$ and … and $S_n \vDash_{\mathsf{PC}} A_n$, it holds that:
    (a) $S_1 \neq \emptyset$ and … and $S_n \neq \emptyset$, and
    (b) $S_1 \vDash_{\mathsf{RAD}} A_1$ and … and $S_n \vDash_{\mathsf{RAD}} A_n$.
(5) $S \vDash_{\mathsf{RAD}} \neg(A_1 \wedge \ldots \wedge A_n)$ iff $S \vDash_{\mathsf{RAD}} \neg A_1 \vee \ldots \vee \neg A_n$.
(6) $S \vDash_{\mathsf{RAD}} \neg(A_1 \vee \ldots \vee A_n)$ iff $S \vDash_{\mathsf{RAD}} \neg A_1 \wedge \ldots \wedge \neg A_n$.
(7) $S \vDash_{\mathsf{RAD}} \neg\neg A$ iff $S \vDash_{\mathsf{RAD}} A$.
(8) $S \vDash_{\mathsf{RAD}} A \supset B$ iff $S \vDash_{\mathsf{RAD}} \neg A \vee B$.
(9) $S \vDash_{\mathsf{RAD}} \neg(A \supset B)$ iff $S \vDash_{\mathsf{RAD}} A \wedge \neg B$.

[8] We give a characterization of RAD that is slightly different from the one given in [8] and [9], allowing for the use of *continuous* conjunctions and disjunctions.

(10) $S \models_{\mathsf{RAD}} A \equiv B$ iff $S \models_{\mathsf{RAD}} A \supset B$ and $S \models_{\mathsf{RAD}} B \supset A$
(11) $S \models_{\mathsf{RAD}} \neg(A \equiv B)$ iff $S \models_{\mathsf{RAD}} (A \wedge \neg B) \vee (\neg A \wedge B)$

Definition 9: $\Gamma \models_{\mathsf{RAD}} A$ *iff* $S_{\mathsf{PC}}(\Gamma) \models_{\mathsf{RAD}} A$.

As is shown in [8], a semantic characterization of PC in terms of sets of PC-models can be easily obtained by (i) replacing clause (4) by the following clause: $S \models_{\mathsf{PC}} A_1 \vee \ldots \vee A_n$ iff S has an n-partition $(S_1, \ldots, S_n)$ such that $S_1 \models_{\mathsf{PC}} A_1$, and ..., and $S_n \models_{\mathsf{PC}} A_n$, (ii) replacing in all other clauses the subscript RAD by the subscript PC, and (iii) adding the following definition:

Definition 10: $\Gamma \models^{S}_{\mathsf{PC}} A$ *iff* $S_{\mathsf{PC}}(\Gamma) \models_{\mathsf{PC}} A$.

It can be easily shown (see [8]) that this semantic characterization of PC is equivalent with the standard characterization:

Theorem 1: $S \models_{\mathsf{PC}} A$ *iff* $M \models A$ *for all* $M \in S$.

Corollary 1: $\Gamma \models^{S}_{\mathsf{PC}} A$ *iff* $\Gamma \models_{\mathsf{PC}} A$.

We will rely on Theorem 1 in the next section.

It should be obvious that the crucial and only difference between RAD and PC is located in clause (4). In view of clauses (5), (6), (7), (8), (9), (10) and (11) of RAD, we restrict our attention to formulas in negation normal form. The first part (i) of clause (4) is exactly the clause for PC. Part (ii) (a) prevents that any disjunct of $A_1 \vee \ldots \vee A_n$ is superfluous. Part (ii) (b) is needed for the suitable handling of nested disjunctions (disjunctions occurring within a disjunct). We try to clarify this with two examples.
Suppose $S \models_{\mathsf{RAD}} p$. From (i) it follows that $S \models_{\mathsf{PC}} p$. The formula $p \vee q$ contains the superfluous disjunct q in the context of p being verified by S. For the partition $(S_1 = S, S_2 = \emptyset)$ of S the following hold: $S_1 \models_{\mathsf{PC}} p$, $S_2 \models_{\mathsf{PC}} q$, $S_1 \neq \emptyset$, $S_2 = \emptyset$, $S_1 \models_{\mathsf{RAD}} p$ and $S_2 \models_{\mathsf{RAD}} q$. Here part (ii) (a) prevents $p \vee q$ from being RAD-verified by S.
Suppose $S \models_{\mathsf{RAD}} p \vee (q \wedge r)$. From (i) $S \models_{\mathsf{PC}} p \vee (q \wedge r)$ it follows there is a partition (S_1, S_2) of S such that $S_1 \models_{\mathsf{PC}} p$ and $S_2 \models_{\mathsf{PC}} q \wedge r$. The formula $p \vee (q \wedge (r \vee s))$ is not RAD-verified by S. Indeed, $S_1 \models_{\mathsf{PC}} p$, $S_2 \models_{\mathsf{PC}} q \wedge (r \vee s)$, $S_1 \models_{\mathsf{RAD}} p$ but not $S_2 \models_{\mathsf{RAD}} q \wedge (r \vee s)$. The latter does not hold because $q \wedge (r \vee s)$ contains the superfluous disjunct s in the context of $q \wedge r$ being verified by S_2. Here part (ii) (b) prevents $p \vee (q \wedge (r \vee s))$ from being RAD-verified by S.

4.2. *A measure for disjunctions*

In preparation for the presentation of the proof theory, we define $\rho(A)$, which provides us with a kind of measure for disjunctions.

Definition 11: *For a formula A in negation normal form, A' is a* d-fragment *of A iff A' is the result of replacing in A a disjunction $B_1 \vee \ldots \vee B_n$ by $\bigvee \Phi$, for a Φ such that $\emptyset \subset \Phi \subset \{B_1, \ldots, B_n\}$.*

To give an example, $p \vee (q \wedge r) \vee \neg t$ has the following d-fragments: $p, q \wedge r, \neg t, p \vee (q \wedge r), p \vee \neg t$, and $(q \wedge r) \vee \neg t$.

The following theorem shows the relation between RAD and PC, and was proven in [8]:

Theorem 2: *For any formula A: $S \vDash_{\mathsf{RAD}} A$ iff*

(1) $S \vDash_{\mathsf{PC}} A$,
(2) *the negation normal form $N(A)$ of A has no d-fragment A' for which $S \vDash_{\mathsf{PC}} A'$.*

In view of this theorem, the logic RAD can be considered to be the logic PC on which a relevance filter is imposed. This will become clearer in the (dynamic) proof theory below.

Definition 12: $\rho(A) = \delta(N(A))$ *where $\delta(A)$ is constructed as follows:*

(1) $\delta(A) = A$ *for A a propositional letter.*
(2) $\delta(\neg A) = \neg A$ *for A a propositional letter.*
(3) $\delta(A_1 \wedge \ldots \wedge A_n) = \{\delta(A_1), \ldots, \delta(A_n)\}$ *where $\{\ldots\}$ denotes a set.*
(4) $\delta(A_1 \vee \ldots \vee A_n) = [\delta(A_1), \ldots, \delta(A_n)]$ *where $[\ldots]$ denotes a multiset.*

We can define a partial order relation $<_d$ over all $\rho(A)$, where A is an RAD-wff:

Definition 13: $\rho(A) <_d \rho(B)$ *iff*

- $\rho(A) \in \rho(B)$ *and $\rho(B)$ is a multiset, or*
- $\rho(A) = [\alpha_1, \ldots, \alpha_n, \beta_1]$ *and* $\rho(B) = [\alpha_1, \ldots, \alpha_n, \beta_2]$ *and $\beta_1 <_d \beta_2$, or*
- $\rho(A) = \{\alpha_1, \ldots, \alpha_n, \beta_1\}$ *and* $\rho(B) = \{\alpha_1, \ldots, \alpha_n, \beta_2\}$ *and $\beta_1 <_d \beta_2$.*

The partial order relation $<_d$ can be extended to the transitive order relation $<$ in the following way:

Definition 14: *The order relation $<$ is the transitive extension of the partial order relation $<_d$ and is obtained in the following way:*

- *If $\rho(A) <_d \rho(B)$, then $\rho(A) < \rho(B)$.*
- *If $\rho(A) < \rho(B)$ and $\rho(B) < \rho(C)$, then $\rho(A) < \rho(C)$.*

4.3. *Proof Theory of* RAD

As all adaptive[9] logics, RAD has a *dynamic* proof theory: in the course of the development of a proof, there may be, apart from an increase of conclusions, a process of revising, viz. rejecting, previously drawn conclusions (because of an increased insight in the premises).

As is usual for adaptive logics, a line in a dynamic RAD-proof consists of five elements: (i) the line number, (ii) the formula derived on that line, (iii) the numbers of the lines relied upon to derive the second element, (iv) the rule that justifies the derivation of the formula in (ii), and (v) the condition on which the second element is derived.

RAD-proofs are governed by a conditional premise rule, a generic conditional derivation rule, and a marking definition. By the rules, lines can be added to the proof. At each stage of the proof, a line is either marked or unmarked, which is determined by the Marking Definition.

Prem If $A \in \Gamma$, one may write down a line containing the following elements: (i) an appropriate line number, (ii) A, (iii) - , (iv) '*Prem*' and (v) $\rho(A)$.

PC If $A_1, \ldots, A_n \vdash_{\mathsf{PC}} B$ and $A_1, \ldots, A_n$ are the second elements of lines $i_1, \ldots, i_n$ (which may be marked or unmarked[10]), one may add a line with the following elements: (i) the appropriate line number, (ii) B, (iii) $i_1, \ldots, i_n$ (iv) '*PC*' and (v) $\rho(B)$.

The marking of lines in a proof is governed by the Marking Definition:[11]

Definition 15: *A line i with A as second element and ρ_1 as fifth element, is* marked *at stage s of the proof iff there is another line j in the proof, marked*

[9] For a general characterization of adaptive logics, see [1].

[10] Of course, one could introduce – e.g. for heuristical purposes – the requirement that each line should be unmarked. This would not alter the logic, though, as the marking definition blocks all irrelevant derivations anyway.

[11] This is the proof-theoretical equivalent of the semantic requirement (see the second clause of theorem 2) that the negation normal form $N(A)$ of a wff A has no d-fragment A' for which $S \vDash_{\mathsf{PC}} A'$.

or unmarked at that stage, with B as second element and ρ_2 as fifth element and $\rho_2 < \rho_1$.

At every stage of a proof, the marking definition determines which lines are marked and unmarked at that stage. The marking of line i in view of line j is indicated by putting the mark $\surd_j$ at the end of line i. Note that the dynamics in an RAD-proof is much more limited than the dynamics occurring in most other adaptive logics: once a line in an RAD-proof is marked, it remains marked in any extension of the proof, and one can conclude that the formula occurring on the marked line is not RAD-derivable from the premises.

In view of the marking definition, two forms of derivability can be defined – derivability at a stage and final derivability:

Definition 16: *A formula is* derived *at stage s of a proof from Γ iff it is the second element of an unmarked line at stage s of the proof.*

Definition 17: *A formula A is* finally derived *in an* RAD*-proof from Γ iff A occurs as the second element of an unmarked line at stage s of a proof from Γ, and this line is not marked in any further extension of the proof.*

The proof-theoretical consequence relation of RAD is defined with respect to final derivability:

Definition 18: $\Gamma \vdash_{\mathsf{RAD}} A$ *iff A is finally derived in an* RAD*-proof from Γ.*

We now give an example of an RAD-proof from a simple set of premises $\Gamma = \{p, p \supset (q \vee r)\}$:

1	p	-	*Prem*	p	
2	$p \supset (q \vee r)$	-	*Prem*	$[\neg p, q, r]$	$\surd_3$
3	$q \vee r$	1,2	*PC*	$[q, r]$	
4	$q \vee r \vee t$	3	*PC*	$[q, r, t]$	$\surd_3$

At stage 3 of the proof, line 2 is marked in view of line 3 because $[q, r] < [\neg p, q, r]$. This illustrates that RAD does not satisfy Reflexivity: it is possible that a premise is not an RAD-consequence of the set of premises (this is only because the premise in question is considered to be irrelevant or containing superfluous disjuncts). At stage 4 of the proof, line 4 is immediately marked in view of line 3, as $[q, r] < [q, r, t]$. As remarked above, the set of RAD-consequences of Γ is always a proper subset of the set of PC-consequences of Γ.[12]

4.4. RAD-*derivation of questions*

We can now define the concept of evocation for the underlying logic RAD:

Definition 19: *A question Q is RAD-evoked by a set of declarative sentences Γ, $\mathsf{E}_{\mathsf{RAD}}(\Gamma, Q)$, iff $\Gamma \vDash_{\mathsf{RAD}} \bigvee dQ$.*

This definition contains only one explicit requirement, i.e. the soundness requirement; the requirement that Q should be informative with respect to Γ is now taken care of by the logic itself. For if $Q = ?\{A_1, \ldots, A_n\}$ and, say, $\Gamma \vdash_{\mathsf{RAD}} A_{n-1}$, then Q is not RAD-evoked by Γ because $\Gamma \nvDash_{\mathsf{RAD}} A_1 \vee \ldots \vee A_n$.

It can easily be proven that the set of questions RAD-evoked by Γ is a proper subset of the set of questions PC-evoked by Γ, for any Γ except for border cases. Let us return to our example wherein $\Gamma = \{p \vee q, r\}$. The questions in (i) are all RAD-evoked by Γ. None of the questions in (ii) or (iii) is RAD-evoked by Γ. Unfortunately, the questions in (iv) and (v) (and some of the questions in (vi)) are RAD-evoked by Γ. Therefore, we will impose a stronger filter in the next section.

We can also define the concept of RAD-generation:

Definition 20: *A question Q is RAD-generated by a set of d-wffs Γ, $\mathsf{G}_{\mathsf{RAD}}$ (Γ, Q), iff*
(i) $\Gamma \vDash_{\mathsf{RAD}} \bigvee dQ$, and
(ii) $\emptyset \nvDash_{\mathsf{RAD}} \bigvee dQ$.

None of the questions in (ii) or (iii) is RAD-generated by $\Gamma = \{p \vee q, r\}$. All questions in (iv), and some in (vi) and (v) are RAD-generated by Γ. But of course, the questions in (i) that are not PC-generated by Γ, are also not RAD-generated by Γ: $?\{p, \neg p\}, ?\{q, \neg q\}, ?\{\neg p, \neg q, p \wedge q\}, ?\{p \wedge q, \neg(p \wedge q)\}$. As was the case for PC, we think that the concept of generation is not

[12] Take any RAD-proof from a set of wffs Γ. If one ignores or erases the mark (if any) at the end of each line in the proof (and thus considers all lines as belonging to the proof), and if one ignores or erases the fifth element of each line, one ends up with a PC-proof from Γ. Marked lines in an RAD-proof from Γ just contain formulas that are PC-consequences of Γ, but not RAD-consequences from Γ.

suited for explicating the raising of relevant and efficient questions.

We now show how easy it is to upgrade the logic RAD to the logic of questions $\mathsf{RAD}^{\mathsf{q}}$. The dynamic proof theory of $\mathsf{RAD}^{\mathsf{q}}$ provides a proof-theoretic counterpart of the abstract semantic definition of RAD-evocation.

The language and wffs of $\mathsf{RAD}^{\mathsf{q}}$ are those of PC^{q} (as defined in section 2). $\mathsf{RAD}^{\mathsf{q}}$-proofs are governed by the Prem-rule and PC-rule of RAD, and the following rule:[13]

Q If $A_1 \vee \ldots \vee A_n$ (where $n \geq 2$) occurs as the second element of an unmarked line i which has ρ as fifth element, one may add a line containing the following elements: (i) an appropriate line number, (ii) $?\{A_1, \ldots, A_n\}$, (iii) i (iv) ‘*Q*’ and (v) ρ.

The marking definition is that of RAD, slightly modified to allow for the marking of lines with a question as second element:

Definition 21: *A line i with $X \in \mathcal{W}^q$ as second element and ρ_1 as fifth element, is* marked *at stage s of the proof iff there is another line j in the proof, marked or unmarked at that stage, with B as second element and ρ_2 as fifth element and $\rho_2 < \rho_1$.*

The marking of line i in view of line j is indicated by putting the mark $\surd_j$ at the end of line i.

In this way we obtain a dynamic proof theory for RAD-evocation of questions. It is obvious that $\mathsf{E}_{\mathsf{RAD}}(\Gamma, Q)$ iff $\Gamma \vdash_{\mathsf{RAD}^{\mathsf{q}}} Q$. In the next sections we will present some more logics of questions.

5. *The Logic* QP

In this section we present the logic QP, which will allow us to define when a question is (in)efficient with respect to Γ. First we present the semantics and a theorem that connects the semantic consequence relation of QP with the semantic consequence relation of RAD. Next we give the dynamic proof theory of QP, and discuss the properties of material implication in QP.

[13] Note that the condition that $A_1 \vee \ldots \vee A_n$ occurs as the second element of an unmarked line in an RAD-proof, automatically implies that $A_1, \ldots, A_n$ are syntactically different wffs.

5.1. *Semantics*

To obtain the logic QP we first introduce a kind of modal operator $\Diamond$. To keep things as simple as possible, we choose $\Diamond$ to be an operator belonging to the meta-language only. The operator is defined for internal use, and should be seen as a mere abbreviation for a property holding between a set of PC-models, a logic (in this case RAD), and a wff:[14]

Definition 22: $S \vDash_{\mathsf{RAD}} \Diamond A$ *iff there is a non-empty* $S' \subseteq S$ *such that* $S' \vDash_{\mathsf{RAD}} A$.

Intuitively, $\Diamond$ captures some notion of possibility: $\Diamond A$ can be read as "It is possible that A", where "possible" should be interpreted in the sense of RAD-verification.

We now list the semantic clauses for QP:

(1) For A a propositional letter, $S \vDash_{\mathsf{QP}} A$ iff for all $M \in S$, $M \vDash A$.
(2) For A a propositional letter, $S \vDash_{\mathsf{QP}} \neg A$ iff for all $M \in S$, $M \vDash \neg A$.
(3) $S \vDash_{\mathsf{QP}} A_1 \wedge \ldots \wedge A_n$ iff $S \vDash_{\mathsf{QP}} A_1$ and ... and $S \vDash_{\mathsf{QP}} A_n$.
(4) $S \vDash_{\mathsf{QP}} A_1 \vee \ldots \vee A_n$ iff
  (I) $S \vDash_{\mathsf{RAD}} A_1 \vee \ldots \vee A_n$, and
  (II) 1. $S \vDash_{\mathsf{RAD}} \Diamond\neg A_1$, and
  ...
  n. $S \vDash_{\mathsf{RAD}} \Diamond\neg A_n$.
(5) $S \vDash_{\mathsf{QP}} \neg(A_1 \wedge \ldots \wedge A_n)$ iff $S \vDash_{\mathsf{QP}} \neg A_1 \vee \ldots \vee \neg A_n$.
(6) $S \vDash_{\mathsf{QP}} \neg(A_1 \vee \ldots \vee A_n)$ iff $S \vDash_{\mathsf{QP}} \neg A_1 \wedge \ldots \wedge \neg A_n$.
(7) $S \vDash_{\mathsf{QP}} \neg\neg A$ iff $S \vDash_{\mathsf{QP}} A$.
(8) $S \vDash_{\mathsf{QP}} A \supset B$ iff $S \vDash_{\mathsf{QP}} \neg A \vee B$.
(9) $S \vDash_{\mathsf{QP}} \neg(A \supset B)$ iff $S \vDash_{\mathsf{QP}} A \wedge \neg B$.
(10) $S \vDash_{\mathsf{QP}} A \equiv B$ iff $S \vDash_{\mathsf{QP}} A \supset B$ and $S \vDash_{\mathsf{QP}} B \supset A$
(11) $S \vDash_{\mathsf{QP}} \neg(A \equiv B)$ iff $S \vDash_{\mathsf{QP}} (A \wedge \neg B) \vee (\neg A \wedge B)$

Relying on definition 7 ($S_{\mathsf{PC}}(\Gamma) = \{M \mid M \vDash \Gamma\}$), we define the semantic QP-consequence relation as follows:

Definition 23: $\Gamma \vDash_{\mathsf{QP}} A$ *iff* $S_{\mathsf{PC}}(\Gamma) \vDash_{\mathsf{QP}} A$.

[14] Of course, the operator can also be defined in the object language (and it has some nice connections with the non-standard properties of material implication in RAD en QP), but we cannot go into that in this paper.

The only difference between QP and RAD is located in clause (4). Again, we can restrict our attention to wffs in negation normal form. Part (II) prevents that any disjunct $A_i (1 \leq i \leq n)$ from the disjunction $A_1 \vee \ldots \vee A_n$ contains a superfluous conjunct. We illustrate this with a simple example, in which we rely on Theorem 1, the semantics of RAD, Theorem 2 and the semantics of QP.

Let $\Gamma = \{p \vee q\}$. With respect to p and q, there are three sorts of models in $S_{\mathsf{PC}}(\Gamma)$: $S^1 = \{M \mid M \vDash p \wedge \neg q\}$, $S^2 = \{M \mid M \vDash \neg p \wedge q\}$ and $S^3 = \{M \mid M \vDash p \wedge q\}$. Note that $S^1 \neq \emptyset$, $S^2 \neq \emptyset$ and $S^3 \neq \emptyset$. It is clear that $S_{\mathsf{PC}}(\Gamma) \vDash_{\mathsf{PC}} p \vee q$. Any 2-partition (S_1, S_2) of $S_{\mathsf{PC}}(\Gamma)$ for which $S_1 \vDash_{\mathsf{PC}} p$ and $S_2 \vDash_{\mathsf{PC}} q$, will be of the form $(S^1 \cup X, S^2 \cup Y)$ such that $X \cup Y = S^3$ and $X \cap Y = \emptyset$. It follows that $S_1 \neq \emptyset$, $S_2 \neq \emptyset$, $S_1 \vDash_{\mathsf{RAD}} p$, $S_2 \vDash_{\mathsf{RAD}} q$ and hence $S \vDash_{\mathsf{RAD}} p \vee q$.[15] In view of $S^2 \vDash_{\mathsf{RAD}} \neg p$ and $S^1 \vDash_{\mathsf{RAD}} \neg q$, it follows that $S_{\mathsf{PC}}(\Gamma) \vDash_{\mathsf{RAD}} \Diamond\neg p$ and $S_{\mathsf{PC}}(\Gamma) \vDash_{\mathsf{RAD}} \Diamond\neg q$. Hence it holds that $\Gamma \vDash_{\mathsf{QP}} p \vee q$.

We also have that $\Gamma \vDash_{\mathsf{RAD}} (p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge q)$ (for analogous reasons). Let us have a closer look at this RAD-consequence. The first disjunct contains the superfluous conjunct p (as $\Gamma \cup \{\neg q\} \vDash_{\mathsf{PC}} p$), and the second disjunct contains the superfluous conjunct q (as $\Gamma \cup \{\neg p\} \vDash_{\mathsf{PC}} q$). So $(p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge q)$ should not be a semantic QP-consequence of Γ. It is precisely the second condition of clause (4) that does the job: it is not the case that $S_{\mathsf{PC}}(\Gamma) \vDash_{\mathsf{RAD}} \Diamond\neg(p \wedge \neg q)$ and $S_{\mathsf{PC}}(\Gamma) \vDash_{\mathsf{RAD}} \Diamond\neg(\neg p \wedge q)$ and $S_{\mathsf{PC}}(\Gamma) \vDash_{\mathsf{RAD}} \Diamond\neg(p \wedge q)$. For suppose $S_{\mathsf{PC}}(\Gamma) \vDash_{\mathsf{RAD}} \Diamond\neg(p \wedge \neg q)$. Then there is an $S' \subseteq S_{\mathsf{PC}}(\Gamma)$, such that $S' \vDash_{\mathsf{RAD}} \neg p \vee q$ and hence $S' \vDash_{\mathsf{PC}} \neg p \vee q$. As all M in S^1 falsify $\neg p \vee q$, it follows that $S' \subseteq (S^2 \cup S^3)$. But as all M in $S^2 \cup S^3$ verify q, $S' \vDash_{\mathsf{PC}} q$ and hence $S' \nvDash_{\mathsf{RAD}} \neg p \vee q$, a contradiction. Hence, $\Gamma \nvDash_{\mathsf{QP}} (p \wedge \neg q) \vee (\neg p \wedge q) \vee (p \wedge q)$.

Note that the logic QP allows for a rather subtle use of disjunctive formulas (and hence questions, see below). We give some simple examples:

- Let $\Gamma = \{p\}$. Then $\Gamma \vDash_{\mathsf{QP}} (q \wedge r) \vee (q \wedge \neg r) \vee (\neg q \wedge r) \vee (\neg q \wedge \neg r)$, but $\Gamma \nvDash_{\mathsf{QP}} ((q \wedge r) \vee (q \wedge \neg r)) \vee ((\neg q \wedge r) \vee (\neg q \wedge \neg r))$ and $\Gamma \nvDash_{\mathsf{QP}} ((q \wedge r) \vee (\neg q \wedge r)) \vee ((q \wedge \neg r) \vee (\neg q \wedge \neg r))$.
- Let $\Gamma = \{p \vee (q \wedge r) \vee s, (\neg p \wedge q) \supset r\}$. Then $\Gamma \vDash_{\mathsf{QP}} p \vee (q \wedge r) \vee s$, but $\Gamma \nvDash_{\mathsf{QP}} (p \vee (q \wedge r)) \vee s$ (because there is no $S' \subseteq S_{\mathsf{PC}}(\Gamma)$ such that $S' \vDash_{\mathsf{RAD}} \neg p \wedge (\neg q \vee \neg r)$). However, both $\Gamma \vDash_{\mathsf{QP}} (p \vee q) \vee s$ and $\Gamma \vDash_{\mathsf{QP}} p \vee q \vee s$ hold.

In preparation of the dynamic proof theory of QP, we first prove some properties of QP.

[15] It might be easier to see that $\Gamma \vdash_{\mathsf{RAD}} p \vee q$.

Lemma 1: *If* $S \vDash_{\mathsf{RAD}} A$, $S \subseteq S'$ *and* $S' \vDash_{\mathsf{PC}} A$, *then* $S' \vDash_{\mathsf{RAD}} A$.

Proof. Suppose $S \vDash_{\mathsf{RAD}} A$, $S \subseteq S'$, $S' \vDash_{\mathsf{PC}} A$ and $S' \nvDash_{\mathsf{RAD}} A$. In view of Theorem 2, the negation normal form $N(A)$ of A has a d-fragment A' for which $S' \vDash_{\mathsf{PC}} A'$. From Theorem 1 it follows that for all $M \in S'$, $M \vDash A'$. As $S \subseteq S'$, also for all $M \in S$, $M \vDash A'$. In view of Theorem 1 and 2, it follows that $S \vDash_{\mathsf{PC}} A'$ and that $S \nvDash_{\mathsf{RAD}} A$, which contradicts our assumption. □

Theorem 3: *For* $n \geq 2$, $S \vDash_{\mathsf{QP}} A_1 \vee \ldots \vee A_n$ *iff*

(α) $S \vDash_{\mathsf{RAD}} A_1 \vee \ldots \vee A_n$,

(β) 1. $S \vDash_{\mathsf{RAD}} A_1 \vee \neg A_1$,

. . .

n. $S \vDash_{\mathsf{RAD}} A_n \vee \neg A_n$.

Proof. We first prove the left-right direction. From clause (4) of the semantics of QP, $S \vDash_{\mathsf{QP}} A_1 \vee \ldots \vee A_n$ iff

(I) $S \vDash_{\mathsf{RAD}} A_1 \vee \ldots \vee A_n$, and

(II) $S \vDash_{\mathsf{RAD}} \Diamond\neg A_1$, and ..., and $S \vDash_{\mathsf{RAD}} \Diamond\neg A_n$.

From (I) it follows that

(i) $S \vDash_{\mathsf{PC}} A_1 \vee \ldots \vee A_n$, and

(ii) for all n-partitions $(S_1, \ldots, S_n)$ of S for which $S_1 \vDash_{\mathsf{PC}} A_1$, and ... and $S_n \vDash_{\mathsf{PC}} A_n$, it holds that:

  (a) $S_1 \neq \emptyset$ and ... and $S_n \neq \emptyset$,

  (b) $S_1 \vDash_{\mathsf{RAD}} A_1$ and ... and $S_n \vDash_{\mathsf{RAD}} A_n$.

In view of Theorem 1, for any such S_1, $S_1 \subseteq \{M \in S \mid M \vDash A_1\}$. $S_1 \neq \emptyset$ implies that $\{M \in S \mid M \vDash A_1\} \neq \emptyset$. As $S_1 \vDash_{\mathsf{RAD}} A_1$ and $\{M \in S \mid M \vDash A_1\} \vDash_{\mathsf{PC}} A_1$ (Theorem 1), it follows from Lemma 1 that $\{M \in S \mid M \vDash A_1\} \vDash_{\mathsf{RAD}} A_1$.

From (II) it follows that there is a non-empty $S' \subseteq S$ such that $S' \vDash_{\mathsf{RAD}} \neg A_1$. In view of Theorem 2, $S' \vDash_{\mathsf{PC}} \neg A_1$ and in view of Theorem 1, $S' \subseteq \{M \in S \mid M \vDash \neg A_1\}$. $S' \neq \emptyset$ implies that $\{M \in S \mid M \vDash \neg A_1\} \neq \emptyset$. As $\{M \in S \mid M \vDash \neg A_1\} \vDash_{\mathsf{PC}} \neg A_1$ (Theorem 1), it follows from Lemma 1 that $\{M \in S \mid M \vDash \neg A_1\} \vDash_{\mathsf{RAD}} \neg A_1$.

S has only one 2-partition (S_1, S_2) for which it holds that both $S_1 \vDash_{\mathsf{PC}} A_1$ and $S_2 \vDash_{\mathsf{PC}} \neg A_1$, namely $S_1 = \{M \in S \mid M \vDash A_1\}$ and $S_2 = \{M \in S \mid M \vDash \neg A_1\}$. As $\{M \in S \mid M \vDash A_1\} \neq \emptyset$, $\{M \in S \mid M \vDash A_1\} \vDash_{\mathsf{RAD}} A_1$, $\{M \in S \mid M \vDash \neg A_1\} \neq \emptyset$ and $\{M \in S \mid M \vDash \neg A_1\} \vDash_{\mathsf{RAD}} \neg A_1$, $S \vDash_{\mathsf{RAD}} A_1 \vee \neg A_1$.

All other cases are completely analogous.

The other direction follows immediately from clause (4) of the RAD-semantics. □

We will rely on this theorem to construct a measure for disjunctions and conjunctions within disjunctions that will be used in the proof theory of QP.

5.2. *Proof Theory of* QP

We now introduce $\sigma(A)$, which plays a role similar to that of $\rho(A)$ in RAD, and provides us with a measure for (superfluous) disjunctions and (superfluous) conjunctions within disjunctions.

Definition 24: $\sigma(A) = \tau(N(A))$ *where* $\tau(A)$ *is constructed as follows:*

(1) $\tau(A) = \{\rho(A)\} = \{A\}$ *for* A *a propositional letter.*
(2) $\tau(\neg A) = \{\rho(\neg A)\} = \{\neg A\}$ *for* A *a propositional letter.*
(3) $\tau(A_1 \wedge \ldots \wedge A_n) = \tau(A_1) \cup \ldots \cup \tau(A_n)$.
(4) $\tau(A_1 \vee \ldots \vee A_n) = \{\rho(A_1 \vee \ldots \vee A_n), \rho(A_1 \vee \neg A_1), \ldots, \rho(A_n \vee \neg A_n)\}$.

The premise rule Prem and the derivation rule PC of QP only differ from the rules of RAD with respect to the condition. The fifth element of a line in a QP-proof with second element A is the set $\sigma(A)$, constructed as above. The marking definition of QP is as follows:[16]

Definition 25: *A line* i *with* A *as second element and* $\sigma(A)$ *as fifth element, is marked (noted* $\surd_j$*) at stage* s *of the proof iff there is a line* j *in the proof, marked or unmarked at that stage, with* B *as second element and* $\sigma(B)$ *as fifth element such that there is a* $\Phi \in \sigma(A)$ *and a* $\Psi \in \sigma(B)$ *such that* $\Psi < \Phi$.

We use the same example again to illustrate the proof theory of QP. Let $\Gamma = \{p \vee q, r\}$.

1	$p \vee q$	-	*Prem*	$\{[p, q], [p, \neg p], [q, \neg q]\}$	
2	r	-	*Prem*	$\{r\}$	
3	$(p \wedge r) \vee (q \wedge r)$	1,2	*PC*	$\{[\{p, r\}, \{q, r\}], [\{p, r\}, [\neg p, \neg r]],$ $[\{q, r\}, [\neg q, \neg r]]\}$	$\surd_5$
4	$(p \wedge r) \vee q$	1,2	*PC*	$\{[\{p, r\}, q], [\{p, r\}, [\neg p, \neg r]],$	

[16] The relation $<$ was defined in section 4.2.

				$[q, \neg q]\}$	$\surd_5$
5	$(p \wedge r) \vee \neg p$	2	*PC*	$\{[\{p, r\}, \neg p], [\{p, r\}, [\neg p, \neg r]],$	
				$[p, \neg p]\}$	$\surd_5$

Lines 3 and 4 are marked in view of line 5 because $[\{p, r\}, \neg p] < [\{p, r\}, [\neg p, \neg r]]$. For the same reason, line 5 marks itself. This is one of the peculiarities of the proof theory of QP (in its present form): a line in a QP-proof from Γ that causes the marking of another line in the proof because of QP-specific[17] reasons, will also cause the marking of the line itself.
Let DC_1 abbreviate $((s \wedge t) \vee (s \wedge \neg t)) \vee (\neg s \wedge t) \vee (\neg s \wedge \neg t)$, DC_2 abbreviate $(s \wedge t) \vee (s \wedge \neg t) \vee (\neg s \wedge t) \vee (\neg s \wedge \neg t)$, and DC_3 abbreviate $((s \wedge t) \vee (s \wedge \neg t)) \vee (\neg s \wedge (\neg s \vee t))$. We now continue the proof in the following way:

6	DC_1	-	*PC*	$\{[[\{s, t\}, \{s, \neg t\}], \{\neg s, t\}, \{\neg s, \neg t\}], [[\{s, t\}, \{s, \neg t\}],$ $\{[\neg s, \neg t], [\neg s, t]\}], [\{\neg s, t\}, [s, \neg t]], [\{\neg s, \neg t\}, [s, t]]\}$	$\surd_8$
7	DC_2	-	*PC*	$\{[\{s, t\}, \{s, \neg t\}, \{\neg s, t\}, \{\neg s, \neg t\}], [\{s, t\}, [\neg s, \neg t]],$ $[\{s, \neg t\}, [\neg s, t]], [\{\neg s, t\}, [s, \neg t]], [\{\neg s, \neg t\}, [s, t]]$	
8	DC_3	-	*PC*	$\{[[\{s, t\}, \{s, \neg t\}], \{\neg s, [\neg s, t]\}], [[\{s, t\}, \{s, \neg t\}],$ $\{[\neg s, \neg t], [\neg s, t]\}], [\{\neg s, [\neg s, t]\}, [s, \{s, \neg t\}]]\}$	$\surd_8$
9	$p \vee q \vee s$				
		1	*PC*	$\{[p, q, s], [p, \neg p], [q, \neg q], [s, \neg s]\}$	$\surd_1$

Line 6 is marked in view of line 8 because $[[\{s, t\}, \{s, \neg t\}], \{\neg s, [\neg s, t]\}] < [[\{s, t\}, \{s, \neg t\}], \{[\neg s, \neg t], [\neg s, t]\}]$. For the same reason, line 8 marks itself. Note that line 7 is unmarked, and will remain so in any extension of the proof. Hence, $(s \wedge t) \vee (s \wedge \neg t) \vee (\neg s \wedge t) \vee (\neg s \wedge \neg t)$ is a QP-consequence of Γ. The marking of line 9 – because of $[p, q] < [p, q, s]$ – illustrates the fact that every line that would be RAD-marked (in the RAD-analogue of this proof), is also QP-marked. Hence, QP can be seen as RAD on which an additional filter is imposed.

In view of the soundness and completeness of RAD and theorem 3, proving that QP is sound and complete is straightforward.

[17] What we mean, is that if one would consider the proof developed so far to be an RAD-proof from Γ (by replacing all σ-conditions by the appropriate ρ-conditions), the line in question would not cause the marking of the other line.

5.3. *Material Implication in* QP

As $A \supset B$ is defined by $\neg A \vee B$ in RAD and QP, material implication exhibits some remarkable properties in those systems. We concisely point to some properties contributing to the assertability of material implication as indicative conditional, and to the well-known paradoxes of material implication in PC.

5.3.1. *Assertability*

Material implication in RAD is more adequate for the logical representation of assertable indicative conditionals than material implication in PC for at least three reasons. The two main paradoxes of material implication are eliminated, a variant of Mackie's inus conditions is fulfilled and no disjunctions in the consequent are irrelevant.

As $\Gamma \vDash_{\mathsf{QP}} A$ implies that $\Gamma \vDash_{\mathsf{RAD}} A$, the properties shown (in [8]) to hold for material implication in RAD, also hold for material implication in QP:

- the Von Wright–Geach–Smiley criterion:
  $\Gamma \vDash_{\mathsf{QP}} A \supset B$ implies that $\Gamma \nvDash_{\mathsf{PC}} \neg A$ and that $\Gamma \nvDash_{\mathsf{PC}} B$,
- a variant of Mackie's inus conditions in the antecedent:
  $\Gamma \vDash_{\mathsf{QP}} (A \wedge B) \supset C$ implies that $\Gamma \nvDash_{\mathsf{PC}} A \supset C$ and that $\Gamma \nvDash_{\mathsf{PC}} B \supset C$,
- the requirement of relevantly assertable disjunctions in the consequent:
  $\Gamma \vDash_{\mathsf{QP}} A \supset (B \vee C)$ implies that $\Gamma \nvDash_{\mathsf{PC}} A \supset B$ and that $\Gamma \nvDash_{\mathsf{PC}} A \supset C$.

There are some additional properties that hold for material implication in QP, but not in RAD:

- the 'possibility' of the disjuncts in the antecedent:
  $\Gamma \vDash_{\mathsf{QP}} (A \vee B) \supset C$ implies that $\Gamma \vDash_{\mathsf{RAD}} \Diamond A$ and that $\Gamma \vDash_{\mathsf{RAD}} \Diamond B$,
- the 'independence' of the disjuncts in the antecedent:
  $\Gamma \vDash_{\mathsf{QP}} (A \vee B) \supset C$ implies that $\Gamma \nvDash_{\mathsf{PC}} A \supset B$ and that $\Gamma \nvDash_{\mathsf{PC}} B \supset A$,
- the 'non-necessity' of the conjuncts in the consequent:
  $\Gamma \vDash_{\mathsf{QP}} A \supset (B \wedge C)$ implies that $\Gamma \vDash_{\mathsf{RAD}} \Diamond\neg B$ and that $\Gamma \vDash_{\mathsf{RAD}} \Diamond\neg C$,
- the 'independence' of the conjuncts in the consequent:
  $\Gamma \vDash_{\mathsf{QP}} A \supset (B \wedge C)$ implies that $\Gamma \nvDash_{\mathsf{PC}} B \supset C$ and that $\Gamma \nvDash_{\mathsf{PC}} C \supset B$.

Concerning the assertability of the material implication as indicative conditional, the additional properties in QP contribute to a notion of assertability that is too strict for everyday use of indicative conditionals. They require a compactness (in the non-logical sense) of the implication that is not standard for the use of indicative conditionals in natural language. Nevertheless, as compact information is more economic to convey or to store, there might be contexts in which such requirements are useful, for example computational contexts.

5.3.2. *Paradoxes*

In [8] it is shown that in RAD most paradoxes of material implication disappear: for instance, $\neg p \nvdash_{\mathsf{RAD}} p \supset q$, $q \nvdash_{\mathsf{RAD}} p \supset q$, $\nvdash_{\mathsf{RAD}} \neg p \supset (p \supset q)$, and $\nvdash_{\mathsf{RAD}} q \supset (p \supset q)$. The only paradoxes that are not resolved in RAD are those in which a material implication occurs as a negative part of a formula.[18] These remaining paradoxes could perhaps be resolved in an intuitively appealing way by using the quasi-modal operator $\Diamond$ (as defined in section 5.1). The idea is to treat a material implication occurring in a 'negative context' as a 'strict' implication (with respect to $\Diamond$). For instance, suppose that a set of premises Γ contains a premise of the form $\neg(A \supset B)$. Hence, $A \supset B$ occurs in a 'negative context'. Furthermore, suppose that $\neg(A \supset B)$ does not contain any other material implications occurring in 'negative contexts'. Then $\neg(A \supset B)$ should express that there is a non-empty $S' \subseteq S_{\mathsf{PC}}(\Gamma)$ such that $S' \vDash A \wedge \neg B$, which could be abbreviated as $S_{\mathsf{PC}}(\Gamma) \vDash \Diamond(A \wedge \neg B)$. It goes without saying that this proposal has to be further elaborated.

6. QP-*derivable questions and the logic of questions* QP^{q}

We now define when a question Q is QP-derivable from a set of declarative sentences Γ:

Definition 26: *A question Q is* QP*-derivable from a set of declarative premises Γ iff $\Gamma \vDash_{\mathsf{QP}} \bigvee dQ$.*

As is illustrated by the QP-proof from $\Gamma = \{p \vee q, r\}$ given above, we have the following situation: all questions in (i) are QP-derivable, and no question from (ii), (iii) or (iv) is QP-derivable from Γ. Most questions from (v) are QP-derivable from Γ though.[19] Concerning the mixed cases in (vi),

[18] This concerns the following paradoxes: $\neg(p \supset q) \vdash_{\mathsf{RAD}} p \wedge \neg q$ and $(p \supset q) \supset p \vdash_{\mathsf{RAD}} p$.

[19] It should be clear from the proof above that some questions from (v), e.g. $?\{(s \wedge t) \vee (s \wedge \neg t), \neg s \wedge t, \neg s \wedge \neg t\}$, are not QP-derivable from Γ.

none of them is QP-derivable from Γ.

We give the proof theory for the logic of questions $\mathsf{QP^q}$. Its language and wffs are those of $\mathsf{RAD^q}$. $\mathsf{QP^q}$-proofs are governed by the Prem-rule and PC-rule of $\mathsf{RAD^q}$, and the following question-rule:

Q If $Pres(Q)$ occurs as the second element of an unmarked line i which has $\sigma(Pres(Q))$ as fifth element, one may add a line containing the following elements: (i) an appropriate line number, (ii) Q, (iii) i (iv) 'Q' and (v) $\sigma(Pres(Q))$.

$\mathsf{QP^q}$ has the following marking definition:

Definition 27: *A line i with $X \in \mathcal{W}^q$ as second element and σ_1 as fifth element, is* marked *at stage s of the proof iff there is a line j in the proof, marked or unmarked at that stage, with B as second element and σ_2 as fifth element such that there is a $\Phi \in \sigma_1$ and a $\Psi \in \sigma_2$ such that $\Psi < \Phi$.*

The marking of line i in view of line j is indicated by putting the mark $\surd_j$ at the end of line i.

Those questions from (v) that are $\mathsf{QP^q}$-derivable from Γ all have the property of not having any propositional letter in common with Γ. We can easily eliminate those questions by imposing an additional relevance filter, as we will do in the next section.

7. *On introducing new topics*

The last type of questions we want to eliminate are questions that contain propositional letters which do not occur in Γ. But sometimes we might want to ask a question about something of which we do not know anything yet. How can we distinguish such cases? How can we express logically that some things are at issue while others are not, when we do not know anything about them?

Let us have a closer look at tautologies. Is there, logically speaking, any difference between $\Gamma_1 = \{p\}$ and $\Gamma_2 = \{p, q \vee \neg q\}$? In terms of PC, for instance, there seems to be no difference, as $Cn_{\mathsf{PC}}(\Gamma_1) = Cn_{\mathsf{PC}}(\Gamma_2)$, and moreover $q \vee \neg q$ is a tautology. By adopting another, say Gricean, perspective there seems to be a (subtle) difference between the two sets. One could say that, by adopting or making the statement $q \vee \neg q$, one has introduced the fact that one lacks knowledge on the truth value of q – provided that neither q nor $\neg q$ is derivable from one's knowledge – and that one may be interested in filling that gap. We present two ways to approach this issue.

7.1. QP*

One way to exclude this kind of questions, is by interpreting the premises in a Gricean way such that only the 'gaps' indicated by the premises are taken into account. This requires that the premises are in a form that does not allow to indicate 'gaps' which are 'filled' by the premises themselves. Starting from QP, we call this the QP-satisfiable form:

Definition 28: $S \models_{\mathsf{QP}} \Gamma$ *iff* $S \models_{\mathsf{QP}} A$ *for all* $A \in \Gamma$.

Definition 29: *A premise set* Γ *is* QP*-satisfiable iff there is a set of* PC*-models* S*, such that* $S \models_{\mathsf{QP}} \Gamma$.

The consequence relation that interprets the premises in the Gricean way described above comes down to quantifying over the sets of PC-models that QP-verify all of the premises:

Definition 30: *For a* QP*-satisfiable premise set* Γ*,* $\Gamma \models_{\mathsf{QP}^\star} A$ *iff* $S \models_{\mathsf{QP}} A$ *for all sets of* PC*-models* S *for which* $S \models_{\mathsf{QP}} \Gamma$.

Definition 31: *A question* Q *is* QP*-*evoked by a set of declarative sentences* Γ *iff* $\Gamma \models_{\mathsf{QP}^\star} \bigvee dQ$.

A side effect of QP*, is that disjunctions (questions) cannot be combined, for example:

$$\{p \vee q, r \vee s\} \not\models_{\mathsf{QP}^\star} (p \wedge r) \vee (p \wedge s) \vee (q \wedge r) \vee (q \wedge s).$$

The consequence relation QP* can be extended for arbitrary premise sets by defining a procedure that puts arbitrary premise sets in QP-satisfiable form. Note that this can be done in at least two different ways. The premise set $\{p \vee q, p\}$ for example may result in $\{p\}$ or in $\{p, q \vee \neg q\}$, depending on how the irrelevant disjunct q is treated.

7.2. QP$^{\mathsf{p}}$ *and* QP$^{\mathsf{pq}}$

Another way to exclude this kind of questions is by literally not allowing new propositional letters. We first define the set of propositional letters occurring in a wff and in a set of wffs:

Definition 32: *The set of propositional letters occurring in a wff* A*,* $SPL(A)$*, is defined as follows:*

- $SPL(A) = \{A\}$, *for* A *a propositional letter;*
- $SPL(\neg A) = SPL(A)$*;*
- $SPL(A_1 \star \ldots \star A_n) = SPL(A_1) \cup \ldots \cup SPL(A_n)$, *where* $\star$ *is either* $\vee$ *or* $\wedge$*;*
- $SPL(A \star B) = SPL(A) \cup SPL(B)$, *where* $\star$ *is either* $\supset$ *or* $\equiv$.

Definition 33: *The set of propositional letters occurring in* Γ, $SPL(\Gamma)$, *is defined by:* $SPL(\Gamma) = \{A \mid A \in SPL(B)$ *and* $B \in \Gamma\}$.

The proof theory of the logic $\mathsf{QP^p}$ is obtained by extending the rules and marking definition of QP with the following additional marking definition:

Definition 34: (MARK-2) *A line* i *with* A *as second element and* $\sigma(A)$ *as fifth element, is* Γ*-marked (noted* $\surd_\Gamma$*) at stage* s *of the proof iff* $SPL(A) \setminus SPL(\Gamma) \neq \emptyset$.

We can now define when a question Q is $\mathsf{QP^p}$-derivable from Γ:

Definition 35: *A question* Q *is* $\mathsf{QP^p}$*-derivable from a set of declarative premises* Γ *iff* $\Gamma \vdash_{\mathsf{QP^p}} \bigvee dQ$.

The reader can easily verify that all questions in (i) are $\mathsf{QP^p}$-derivable from $\Gamma = \{p \vee q, r\}$, and that none of the questions from (ii)-(vi) is.

The logic of questions $\mathsf{QP^{pq}}$ is obtained by taking the rules Prem, PC and Q, the marking definition from $\mathsf{QP^q}$, and the marking definition from $\mathsf{QP^p}$. Moreover, the following rule is added:

New Let A be either (i) a wff B (compatible with Γ) that was obtained as an answer to a question posed, or (ii) a topic $C \vee \neg C$ that is new with respect to Γ. Then one may write down a line containing the following elements: (i) an appropriate line number, (ii) A, (iii) - , (iv) ‘*New*’ and (v) $\sigma(A)$.

This rule allows for the introduction of an answer to a posed question. As in realistic applications not all answers one obtains will be direct answers to questions posed, we do not impose restrictions on B, except for being compatible with Γ.[20] As $\mathsf{QP^{pq}}$ is a non-monotonic logic, introducing an obtained answer or a new topic will have some effects: some lines may be marked, and some previously marked lines may become unmarked. This will become

[20] Of course, in realistic applications, one will sometimes obtain an answer incompatible with Γ, but handling this requires some mechanism of belief revision or update mechanism, which falls outside the scope of this paper.

clear by the following example:

Let $\Gamma = \{p \vee q\}$. Hence, $SPL(\Gamma) = \{p, q\}$.

1	$p \vee q$	-	*Prem*	$\{[p, q], [p, \neg p], [q, \neg q]\}$	
2	$r \vee \neg r$	-	*CL*	$\{[r, \neg r]\}$	$\surd_{\Gamma}$
3	$?\{p, q\}$	1	*Q*	$\{[p, q], [p, \neg p], [q, \neg q]\}$	
4	$r \supset p$	-	*New*	$\{[\neg r, p], [r, \neg r], [p, \neg p]\}$	

The question $?\{p, q\}$ is derived at stage 3 of the proof (and finally derivable from Γ), and asked to some source. The partial answer $r \supset p$ is received and added to the proof on line 4. This means that the set of premises is extended to $\Gamma' = \Gamma \cup \{r \supset p\}$. As $SPL(\Gamma') = \{p, q, r\}$, line 2 has to be unmarked. We can now proceed as follows:

5	$?\{r, \neg r\}$	2	*Q*	$\{[r, \neg r]\}$	$\surd_{6}$
6	r	-	*New*	$\{r\}$	
7	p	4,6	*PC*	$\{p\}$	
8	$t \vee \neg t$	-	*New*	$\{[t, \neg t]\}$	
9	$?\{t, \neg t\}$	8	*Q*	$\{[t, \neg t]\}$	

The question $?\{r, \neg r\}$ derived on line 5 (and finally derivable from Γ') is posed to some source, and the obtained answer is added to the proof on line 6. In this way, the set of premises is extended to $\Gamma'' = \Gamma' \cup \{r\}$. This causes the marking of line 5 ($?\{r, \neg r\}$ is not finally derivable from Γ''), which makes perfect sense: once a question is fully answered, it vanishes. On line 7, p is derived, in view of which both line 1 and 3 are marked (note that the partial answer $r \supset p$ did not cause the marking of line 3). On line 8, a new topic in introduced, and hence Γ'' is extended to $\Gamma''' = \Gamma'' \cup \{t \vee \neg t\}$. The question $?\{t, \neg t\}$ is (finally) derived from Γ''' on line 9 (it was not derivable from any subset of Γ''').

We only presented the proof theory of $\mathsf{QP^q}$ and $\mathsf{QP^{pq}}$. We will not give the semantics in this paper, but it can be obtained by working with sets of *partial* PC-models instead of working with sets of complete PC-models. In a partial PC-model truth-values are assigned to a subset of $\mathcal{S}$. Only those partial PC-models M for which $\{A \mid M \vDash A$ and $A \in \mathcal{S}\} = SPL(\Gamma)$ will be $\mathsf{QP^q}$-models of Γ. This gives the general outline for the semantics $\mathsf{QP^q}$, but we will not elaborate on this any further.

8. *Conclusions and Open Problems*

We have pointed out that Wiśniewksi's concepts of evocation and generation allow for the raising of questions that are irrelevant and/or inefficient with respect to the evoking set Γ. We have argued that the problems are due to the underlying logic PC (and by extension, any standard logic in which disjunction behaves classically). We have defined several (non-monotonic) logics that can be used as the underlying logic for a logic of questions. Each of them solves some aspects of the problems of irrelevance and/or inefficiency. Moreover, the logics of questions defined from them have a (dynamic) proof theory.

Of course, many open problems remain. First, an alternative characterization of QP – with conditions and a marking definition that are not directly based on RAD – would possibly yield a more intuitive (and certainly more direct) proof theory. This might be realised by introducing the modal operator into the object language. Also, QP^{p} should be provided with a decent semantics, and the relations with the QP^*-approach should be spelled out. Second, all logics presented in this paper are only defined at the propositional level, and should be worked out at the predicative level. Third, Wiśniewski's interesting concept of erotetic implication, i.e. the derivation of a question on the basis of another question and/or a set of d-wffs, should be studied from the perspective of the logics presented in this paper. In [2]and [3], and in [4], two different roads are taken to eliminate the derivation of irrelevant and/or inefficient questions from a main question and a set of d-wffs Γ. The results that will obtain by defining the concept of erotetic implication for the logics presented in this paper should be compared with those approaches.

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