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FUZZIFICATION OF GROENENDIJK-STOKHOF PROPOSITIONAL EROTETIC LOGIC

LIBOR BĚHOUNEK*

1. *Introduction*

Fuzzy logic is a branch of many-valued logics aimed at capturing comparative degrees of truth and reasoning under vagueness. For a long time, fuzzy sets and fuzzy logic were rather an engineering tool than a well-developed mathematical theory. The advances in metamathematics of fuzzy logic achieved during past few years (esp. [Háj98]), however, set the theory on a firm ground and made it possible to develop fuzzy generalizations of various branches of classical mathematics in the axiomatic way.

One of the fields in which many-valued logics can fruitfully be applied is the logic of questions. The importance of a many-valued approach to questions follows, i.a., from the fact that many questionnaires employ scaled answers rather than simple yes-no ones. In many cases, the scale of answers directly corresponds to comparative degrees of truth, which is the domain of fuzzy logic.[1] Furthermore, many questions in natural language ask for information about predicates which are not 'black and white' (i.e., 'crisp', in fuzzy terminology), but show a natural scale of truth.[2]

This paper develops a fuzzy generalization FGS of Groenendijk-Stokhof's system of erotetic logic (as described in [GS90] and [GS97], further referred to as GS). Since Groenendijk-Stokhof's system (also known as the *partition semantics of questions*) is based on intensional semantics of classical logic,

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[1] E.g., the scale 'yes, rather yes, rather no, no'. However, fuzzy logic is not applicable if the set of answers contains options like 'I don't know', since these are not *truth* degrees.

[2] For instance, if John is middle-sized, then the answer to the question 'Is John tall?' should be neither 'yes' nor 'no', but something in-between.

fuzzy intensional semantics is developed first, within the framework of fuzzy class theory [BC04b]. Our attention is restricted to propositional FGS, i.e. fuzzy yes-no questions.[3]

2. *Classical Groenendijk-Stokhof semantics*

In this section we repeat the basic definitions of intensional semantics for classical propositional logic and classical propositional Groenendijk-Stokhof system. For details, see [GS90] and [GS97].

Definition 2.1: (Intensional semantics) *Let W be a non-empty set. By a* valuation in W *we mean a function $\|\cdot\|$ taking formulae to subsets of W, such that* $\|\neg\varphi\| = W - \|\varphi\|$, $\|\varphi \,\&\, \psi\| = \|\varphi\| \cap \|\psi\|$, $\|\varphi \vee \psi\| = \|\varphi\| \cup \|\psi\|$, $\|\varphi \to \psi\| = (W - \|\varphi\|) \cup \|\psi\|$. *The pair $\mathcal{W} = \langle W, \|\cdot\| \rangle$ is called a* logical space, *the elements of W* indices *or* possible worlds, *the subsets of W* propositions.

The proposition $\|\varphi\|$ is called the intension of φ *(in $\mathcal{W}$). The* extension of φ in $w \in W$ *is the truth value of the statement that $w \in \|\varphi\|$; it will be denoted by $\|\varphi\|_w$.*[4]

A formula φ holds *in a logical space $\mathcal{W} = \langle W, \|\cdot\| \rangle$ (written $\mathcal{W} \models \varphi$) iff $\|\varphi\| = W$. A formula φ is a* tautology *(written $\models \varphi$) iff it holds in any logical space. A formula φ* entails *a formula ψ* in $\langle W, \|\cdot\| \rangle$ *iff $\|\varphi\| \subseteq \|\psi\|$. A formula φ* entails *a formula ψ (written $\varphi \models \psi$) iff φ entails ψ in any logical space.*

Intensional semantics is adequate w.r.t. classical propositional calculus; i.e., a formula is provable in classical propositional calculus iff it is a tautology of intensional semantics. GS extends this semantics to interrogative formulae $?\varphi$ (read *whether* φ), where φ is any propositional formula.

Definition 2.2: (Semantics of interrogative formulae) *Let $\mathcal{W} = \langle W, \|\cdot\| \rangle$ be a logical space. The* extension $\|?\varphi\|_w$ *of $?\varphi$ in $w \in W$ is the proposition $\{w' \in W \mid \|\varphi\|_{w'} = \|\varphi\|_w\}$.*

The intension $\|?\varphi\|$ *of $?\varphi$ in $\mathcal{W}$ is the equivalence relation $\{\langle w, w' \rangle \in W^2 \mid \|\varphi\|_w = \|\varphi\|_{w'}\}$. The partition of W induced by this equivalence relation will be denoted by $W/\|?\varphi\|$.*

[3] While classical propositional GS is trivial, its fuzzified version is less so.

[4] Thus if $w \in \|\varphi\|$, we say that the extension of φ in w is 1 (the truth value 'true'); if $w \notin \|\varphi\|$, we say that it is 0 (the truth value 'false'). The intension of φ can be identified with the function that assigns to each possible world $w \in W$ the extension of φ in w.

Definition 2.3: (Answerhood and entailment of interrogatives)
Let $\langle W, \|\cdot\|\rangle$ be a logical space.

We say that ψ is a direct answer *to $?\varphi$ in $\mathcal{W}$ iff $\|\psi\| \in W/\|?\varphi\|$. We say that ψ is an* answer *to $?\varphi$ in $\mathcal{W}$ (written $\psi \models^{\mathcal{W}} ?\varphi$) iff ψ entails a direct answer to $?\varphi$ in $\mathcal{W}$.*

We say that $?\psi$ entails *$?\varphi$ in $\mathcal{W}$ (written $?\psi \models^{\mathcal{W}} ?\varphi$) iff every answer to $?\psi$ is an answer to $?\varphi$ in $\mathcal{W}$. We say that $?\psi$ and $?\varphi$ are* equivalent *in $\mathcal{W}$ (written $?\psi \equiv^{\mathcal{W}} ?\varphi$) iff $?\psi$ entails $?\varphi$ in $\mathcal{W}$ and vice versa.*

We say that these relations hold generally iff they hold in any logical space.

It is easy to prove that $?\psi \models^{\langle W, \|\cdot\|\rangle} ?\varphi$ iff the partition $W/\|?\psi\|$ refines the partition $W/\|?\varphi\|$, and that equivalence of interrogatives corresponds to the identity of partitions.

3. *T-norm based fuzzy logic*

In this section, the main ideas of t-norm based fuzzy logic are outlined and basic definitions are given. For details see [Háj98].

T-norm based fuzzy logic is founded upon a few natural assumptions regarding the semantics of fuzzy conjunction: truth-functionality, associativity, commutativity, monotonicity, continuity, and classical values on $\{0, 1\}$. Such binary functions on $[0, 1]$ had already been studied in probability theory under the name *continuous triangular norms* (or *continuous t-norms*). Given a continuous t-norm $*$, the semantics of other propositional connectives can be defined in a natural way (e.g., the semantics of implication is the maximal function such that the internalization of modus ponens is valid). Generalizing Tarski's definitions in the obvious way, for each $[0, 1]$-valuation v of propositional variables and any formula φ we get a unique semantic value $\|\varphi\|_v \in [0, 1]$. A formula is a *tautology* w.r.t. a continuous t-norm $*$ iff it gets the value 1 under each valuation v. The set of all tautologies w.r.t. a continuous t-norm $*$ is called the *logic of* $*$ and denoted by $\mathrm{PC}(*)$.

It turns out that some formulae are tautologies w.r.t. any continuous t-norm; we call them t-tautologies. It can be proved that the set of all t-tautologies is finitely axiomatizable. This gives rise to Basic Fuzzy Logic BL:

Definition 3.1: (BL) *Propositional logic* BL *is determined by the following axiom schemata and the deduction rule of modus ponens (the primitive* *connectives are* $\to$, $\&$, *and* $\bot$*).*

$$
\begin{array}{rl}
\text{(BL1)} & (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \\
\text{(BL2)} & (\varphi \mathbin{\&} \psi) \to \varphi \\
\text{(BL3)} & (\varphi \mathbin{\&} \psi) \to (\psi \mathbin{\&} \varphi) \\
\text{(BL4)} & (\varphi \mathbin{\&} (\varphi \to \psi)) \to (\psi \mathbin{\&} (\psi \to \varphi)) \\
\text{(BL5a)} & (\varphi \to (\psi \to \chi)) \to ((\varphi \mathbin{\&} \psi) \to \chi) \\
\text{(BL5b)} & ((\varphi \mathbin{\&} \psi) \to \chi) \to (\varphi \to (\psi \to \chi)) \\
\text{(BL6)} & ((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi) \\
\text{(BL7)} & \bot \to \varphi
\end{array}
$$

Further connectives are defined as follows:

$$
\begin{array}{rcl}
\varphi \wedge \psi & \equiv_{\text{df}} & \varphi \mathbin{\&} (\varphi \to \psi) \\
\varphi \vee \psi & \equiv_{\text{df}} & ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) \\
\varphi \leftrightarrow \psi & \equiv_{\text{df}} & (\varphi \to \psi) \mathbin{\&} (\psi \to \varphi) \\
\neg\varphi & \equiv_{\text{df}} & \varphi \to \bot \\
\top & \equiv_{\text{df}} & \neg\bot
\end{array}
$$

There are three salient continuous t-norms:[5] the minimum, also known as the Gödel t-norm $x * y = \min(x, y)$, the product $x * y = x \cdot y$, and the Łukasiewicz t-norm $x * y = \max(0, x + y - 1)$. The sets of all tautologies w.r.t. these t-norms are called Gödel, product, and Łukasiewicz fuzzy logic, denoted G, Π and Ł, respectively.[6] They are axiomatizable by the following respective schematic extensions of BL:

$$
\begin{array}{ll}
\text{(G)} & \varphi \to (\varphi \mathbin{\&} \varphi) \\
\text{(Ł)} & \neg\neg\varphi \to \varphi \\
(\Pi) & (\neg(\varphi \mathbin{\&} \varphi) \to \neg\varphi) \\
& \qquad \mathbin{\&} (\neg\neg\varphi \to (((\psi \mathbin{\&} \varphi) \to (\chi \mathbin{\&} \varphi)) \to (\psi \to \chi)))
\end{array}
$$

[5] Not only are they most often used in applications, but it is proved that any continuous t-norm is a special kind of ordinal sum of these three t-norms (Mostert-Shield's characterization theorem).

[6] Ł and G coincide respectively with Łukasiewicz and Gödel infinite-valued logics. G extends intuitionistic logic with Dummett's prelinearity axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$.

The $[0,1]$-semantics of $\wedge$, $\vee$, $\bot$ and $\top$ in any logic $\mathrm{PC}(*)$ is that of minimum, maximum, 0, and 1, respectively. Furthermore, in any $\mathrm{PC}(*)$, $\|\varphi \to \psi\| = \max\{z \mid z * \|\varphi\| \leq \|\psi\|\}$; in particular, $\|\varphi \to \psi\|_v = 1$ iff $\|\varphi\|_v \leq \|\psi\|_v$. Consequently $\|\varphi \leftrightarrow \psi\|_v = 1$ iff $\|\varphi\|_v = \|\psi\|_v$, and $\|\neg\varphi\|_v = 1$ iff $\|\varphi\|_v = 0$.

Except for G, all $\mathrm{PC}(*)$ lack contraction (i.e., $\varphi \mathbin{\&} \varphi$ is generally stronger than φ), which justifies the presence of min-conjunction $\wedge$. If we add the law of excluded middle (i.e., the schema $\varphi \vee \neg\varphi$) to BL, we get classical logic.

A further unary propositional connective Δ (Baaz's delta) with the $[0,1]$-semantics $\|\Delta\varphi\|_v = 1$ iff $\|\varphi\|_v = 1$, otherwise $\|\Delta\varphi\|_v = 0$, is often introduced. The resulting logics $\mathrm{BL}\Delta$, $\mathrm{G}\Delta$, $\mathrm{Ł}\Delta$, and $\Pi\Delta$ are axiomatized by the axioms of the respective fuzzy logic plus the following axioms for Δ:

$(\Delta 1)$	$\Delta\varphi \vee \neg\Delta\varphi$
$(\Delta 2)$	$\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$
$(\Delta 3)$	$\Delta\varphi \to \varphi$
$(\Delta 4)$	$\Delta\varphi \to \Delta\Delta\varphi$
$(\Delta 5)$	$\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$

The deduction rules for logics with Δ are modus ponens and Δ-necessitation (from φ infer $\Delta\varphi$).

In order to develop fuzzy mathematics, fuzzy predicate calculus is necessary. The syntax of first-order fuzzy logic is classical (except for the differences in propositional connectives, i.e. the presence of two conjunctions and possibly Δ). The quantifiers $\forall$ and $\exists$ are governed by the following axiom schemata (which assume that the term t is substitutable for x in φ and that x is not free in χ):

$(\forall 1)$	$(\forall x)\varphi(x) \to \varphi(t)$
$(\exists 1)$	$\varphi(t) \to (\exists x)\varphi(x)$
$(\forall 2)$	$(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$
$(\exists 2)$	$(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$
$(\forall 3)$	$(\forall x)(\chi \vee \varphi) \to (\chi \vee (\forall x)\varphi)$

The deduction rules are those of propositional logic plus generalization (from φ infer $(\forall x)\varphi$). Equality can be regarded as a logical symbol governed by the axioms of reflexivity $x = x$ and universal intersubstitutivity $x = y \to \Delta(\varphi(x) \leftrightarrow \varphi(y))$.

The *standard semantics* for fuzzy predicate calculi is a straightforward generalization of Tarski's semantics to $[0,1]$. The interpretation of predicates and functors of arity n in a model with the universe M are functions from

M^n to $[0, 1]$ (for predicates) or to M (for functors); equality is interpreted as the identity on M. The semantics of $\forall$ and $\exists$ is that of infimum and supremum, respectively. The first-order logics G and GΔ are complete w.r.t. the standard $[0, 1]$-semantics; first-order BL, Ł and Π (with or without Δ), however, are not.[7]

4. *Fuzzy class theory*

Within fuzzy predicate calculus, axiomatic theory of fuzzy sets can be developed. For most purposes, however, one does not need a full-fledged set theory over fuzzy logic, since it is usually not necessary to consider the membership of *sets* in sets. The theory of membership of (atomic) *individuals* in fuzzy sets—i.e., fuzzy *class* theory—is much simpler; it has been elaborated in [BC04b] over a richer fuzzy logic ŁΠ, which contains all the connectives of GΔ, ŁΔ, and $\Pi\Delta$. An easy inspection of proofs in [BC04b] shows that the theorems of [BC04b] that do not mix connectives of different logics remain valid in its fragments GΔ, ŁΔ, and $\Pi\Delta$. The adaptation of fuzzy class theory FCT developed in [BC04b] for an extension $\mathcal{F}$ of $\mathrm{BL}\Delta$ will be denoted by $\mathcal{F}\mathrm{CT}$.

The language of $\mathcal{F}\mathrm{CT}$ has two sorts of variables: object variables $x, y, \ldots$ and class variables $X, Y, \ldots$ (there are no universal variables). The only primitive predicate is the membership predicate $\in$ between objects and classes. $\mathcal{F}\mathrm{CT}$ enjoys full class comprehension, i.e., for any formula $\varphi(x)$ there is a function symbol[8] $\{x \mid \varphi(x)\}$ and the comprehension axiom $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$. The classes are understood extensionally, therefore $\mathcal{F}\mathrm{CT}$ adopts the axiom of extensionality $(\forall x)\Delta(x \in X \leftrightarrow x \in Y) \rightarrow X = Y$.

The intended models consist of a universe U, which is the range of object variables, and the set $U^{[0,1]}$ of all functions from U to $[0, 1]$, which is the range of class variables. The truth value of the formula $x \in X$ in a model M under an evaluation e of class and object variables is defined as the value of the function $e(X)$ on $e(x)$. The semantic value of the comprehension term $\{x \mid \varphi(x)\}$ in M under e is the function $f\colon U \rightarrow [0, 1]$ such that for any $a \in U$, $f(a)$ is the truth value of $\varphi(x)$ in M under the evaluation $e_{x:a}$, where $e_{x:a}$ coincides with e except that $e_{x:a}(x) = a$. It is easy to prove all

[7] They are complete w.r.t. special classes of distributive residuated lattices. Since our main motivation is the interval $[0, 1]$, we shall not discuss this general semantics (it can be found in [Háj98]). The results of [BC04b] reduce the relevant part of fuzzy class theory (see Section 4) to fuzzy propositional calculus, for which the completeness w.r.t. $[0, 1]$ holds.

[8] For function symbols in fuzzy logics see [Háj00].

comprehension axioms as well as extensionality in such models; for details see [BC04b].

We repeat here several definitions and theorems of [BC04b] that will be needed later on.

Definition 4.1: (Fuzzy class operations and relations)

$\emptyset$	$=_{\rm df}$	$\{x \mid \bot\}$	*empty class*
V	$=_{\rm df}$	$\{x \mid \top\}$	*universal class*
$-X$	$=_{\rm df}$	$\{x \mid \neg(x \in X)\}$	*complement*
$X \cup Y$	$=_{\rm df}$	$\{x \mid x \in X \vee x \in Y\}$	*union*
$X \cap Y$	$=_{\rm df}$	$\{x \mid x \in X \;\&\; x \in Y\}$	*strong intersection*
$X \subseteq Y$	$\equiv_{\rm df}$	$(\forall x)(x \in X \to x \in Y)$	*inclusion*

Convention 4.2: *In what follows, let the notation $\varphi(p_1, \ldots, p_n)$ imply that the formula φ contains no propositional variables other than $p_1, \ldots, p_n$. The formula $\varphi \;\&\; \ldots \;\&\; \varphi$ (n times) is abbreviated by φ^n. Furthermore, we abbreviate $(\forall x)(x \in X \to \varphi)$ as $(\forall x \in X)\varphi$, $(\exists x)(x \in X \;\&\; \varphi)$ as $(\exists x \in X)\varphi$, and $\{x \mid x \in X \;\&\; \varphi\}$ as $\{x \in X \mid \varphi\}$. If $\varphi(p_1, \ldots, p_n)$ is a propositional formula and $\psi_1, \ldots, \psi_n$ are any formulae, then $\varphi(\psi_1, \ldots, \psi_n)$ denotes the formula φ in which all occurrences of p_i are replaced by ψ_i (for all $i \leq n$).*

Definition 4.3: (n-ary class operation) *Let φ be a propositional formula. We define the n-ary class operation induced by φ as*

$$\mathrm{Op}_\varphi(X_1, \ldots, X_n) =_{\rm df} \{x \mid \varphi(x \in X_1, \ldots, x \in X_n)\}.$$

The following lemmata are corollaries of more general theorems of [BC04b]; their direct proofs are given in Appendix A.

Lemma 4.4: *Let $\varphi(p_1, \ldots, p_n)$ and $\psi(p_1, \ldots, p_n)$ be propositional formulae. Then $\mathcal{F} \vdash \varphi \to \psi$ iff $\mathcal{F}\mathrm{CT} \vdash \mathrm{Op}_\varphi(X_1, \ldots, X_n) \subseteq \mathrm{Op}_\psi(X_1, \ldots, X_n)$.*

Lemma 4.5: $\mathcal{F}\mathrm{CT} \vdash (X \subseteq Y \;\&\; Y \subseteq Z) \to X \subseteq Z$.

5. *Fuzzy intensional semantics*

We want to generalize classical intensional semantics to fuzzy intensional semantics, i.e., to allow propositions to be fuzzy sets. Since we have a formal theory of fuzzy sets, viz. the theory of fuzzy classes $\mathcal{F}\mathrm{CT}$, we want to define the semantical notions in this theory (thus we shall be able to prove results on entailment within its framework). First we shall give an intuitive motivation for our definitions.

Let us work in $\mathcal{F}\mathrm{CT}$. Given a (possibly fuzzy) class W (to be informally interpreted as a logical space), certain class operations of $\mathcal{F}\mathrm{CT}$ (union, intersection, etc.) on (possibly fuzzy) subclasses of W correspond directly to propositional connectives (disjunction, conjunction, etc., respectively). Subclasses $A \subseteq W$ can therefore aptly be called propositions and taken for the range of intensions of propositional formulae of fuzzy logic $\mathcal{F}$. The extension of a proposition A in $w \in W$ is expressed by the formula '$w \in A$'.[9]

It is natural to say that the proposition A entails B iff for all $w \in W$, the extension of A in w implies that of B in w.[10] This condition can be expressed as $(\forall w \in W)(w \in A \to w \in B)$, i.e., according to the definitions of $\mathcal{F}\mathrm{CT}$, $W \cap A \subseteq B$. Similarly we can say that a proposition A holds in W iff it holds in all indices $w \in W$, formally $(\forall w \in W)(w \in A)$, i.e. $W \subseteq A$.

In these considerations, propositions $A \subseteq W$ represent intensions of propositional formulae of a fuzzy logic $\mathcal{F}$. The assignment $\|\cdot\|$ of propositions $A \subseteq W$ to formulae obeying the rules of correspondence between propositional connectives and class operations (e.g., $\|\varphi \vee \psi\| = \|\varphi\| \cup \|\psi\|$) can therefore be construed as an intensional semantics for propositional formulae in the logical space $\langle W, \|\cdot\|\rangle$.

We of course intend tautologicity to be defined as validity in all logical spaces, i.e., for all couples $\langle W, \|\cdot\|\rangle$. However, the assignment $\|\cdot\|$ is not an object of our theory;[11] thus we cannot quantify over it, and another formal solution is required.

It can be observed that in classical intensional semantics, the function $\|\cdot\|$ is in fact a translation of propositional formulae to the language of a theory of subsets of some basic set. Similarly, we can define fuzzy intensional semantics by giving a *translation* of propositional formulae to the language of a theory of *fuzzy* subsets of some basic set (favourably, a part of fuzzy class theory $\mathcal{F}\mathrm{CT}$).[12] Interpreting propositional variables as class *variables,*

[9] These definitions look the same as in the classical case, but notice that now the extensions can have truth values between 0 and 1 and the propositions can be fuzzy classes.

[10] This definition (of *local* entailment) allows the inference from A to B in w if A entails B (by detachment). Note that the entailment itself is a fuzzy notion.

[11] It could become an object of the theory after some strenghtening of $\mathcal{F}\mathrm{CT}$, which would allow us to encode propositional formulae and classes of classes, but we shall not pursue this line here.

[12] We are thus giving an *interpretation* (a direct syntactic model) of fuzzy propositional calculus in $\mathcal{F}\mathrm{CT}$. By means of this interpretation, any model of $\mathcal{F}\mathrm{CT}$ together with a valuation of free variables yields a fuzzy intensional model for the original propositional formulae.

propositional connectives as the corresponding class operations, and choosing a class *variable* W, we get the generality we need. The translation is adequate in the sense that a propositional formula is provable in fuzzy logic $\mathcal{F}$ iff the general validity of its translation is provable in $\mathcal{F}\mathrm{CT}$ (and so holds in every model of $\mathcal{F}\mathrm{CT}$).

Let us elaborate this idea formally:[13]

Definition 5.1: (Fuzzy intensional semantics) *The translation* $\|\cdot\|$ *of the formulae of propositional fuzzy logic* $\mathcal{F}$ *to* $\mathcal{F}\mathrm{CT}$ *is defined as follows:*

The translation $\|p_i\|$ *of an atomic formula* p_i *is a class variable* A_i*. The translation of a complex formula* $\varphi(p_1, \dots, p_n)$ *is*

$$\|\varphi(p_1, \dots, p_n)\| =_{\mathrm{df}} \mathrm{Op}_\varphi(\|p_1\|, \dots, \|p_n\|).$$[14]

Theorem 5.2: (Adequacy of fuzzy intensional semantics)

$$\mathcal{F} \vdash \varphi \quad \text{iff} \quad \mathcal{F}\mathrm{CT} \vdash W \subseteq \|\varphi\|$$

The proof is given in Appendix A. Similarly it is shown that $\mathcal{F} \vdash \varphi \to \psi$ iff $\mathcal{F}\mathrm{CT} \vdash W \cap \|\varphi\| \subseteq \|\psi\|$. This correspondence justifies writing $\models \varphi$ instead of $W \subseteq \|\varphi\|$, and $\varphi \models \psi$ instead of $W \cap \|\varphi\| \subseteq \|\psi\|$.[15] The notation can conveniently be generalized to any class terms of $\mathcal{F}\mathrm{CT}$, defining $(\models A) \equiv_{\mathrm{df}} (W \subseteq A)$ and $(A \models B) \equiv_{\mathrm{df}} (W \cap A \subseteq B)$.[16] We further define logical equivalence of propositions as their mutual entailment: $(A \equiv B) \equiv_{\mathrm{df}} (A \models B)\ \&\ (B \models A)$.

[13] Since W can be construed as only a part of a larger logical space W' (whose subclass W is the class of those worlds to which we currently restrict our attention), we shall not further require that propositions be subclasses of W. The relativization of quantifiers in the definitions guarantees that only the worlds in W are taken into account when evaluating entailment of propositions.

[14] It can be observed that the definition works naturally, i.e., $\|\varphi \vee \psi\| = \|\varphi\| \cup \|\psi\|$, etc.

[15] Formulae of $\mathcal{F}$ are translated by $\|\cdot\|$ to class terms of $\mathcal{F}\mathrm{CT}$ (thus their semantical values in models of $\mathcal{F}\mathrm{CT}$ are fuzzy propositions). The semantical notions of tautologicity and entailment are expressed as certain formulae of $\mathcal{F}\mathrm{CT}$. They can combine to complex semantical statements like $(\varphi \models \chi)\ \&\ (\psi \models \chi) \to (\varphi\ \&\ \psi \models \chi)$, which again are formulae of $\mathcal{F}\mathrm{CT}$ (so in models they may have truth values between 0 and 1). If they are provable in $\mathcal{F}\mathrm{CT}$, we take them for valid semantical laws (they are 1-true in all models.)

[16] Thus we can also write $A \models \|\varphi\|$, or shortly $A \models \varphi$, etc.

Theorem 5.3: (Properties of fuzzy entailment) *It is provable in* $\mathcal{F}\mathrm{CT}$ *that* $\Delta(W \subseteq W \cap W)$ *implies*[17]

$$[(A \models B)\ \&\ (B \models C)] \ \rightarrow\ (A \models C) \tag{1}$$

$$[(A \equiv B)\ \&\ (B \equiv C)] \ \rightarrow\ (A \equiv C) \tag{2}$$

$$[(A \equiv A')\ \&\ (B \equiv B')] \ \rightarrow\ [(A \models B) \leftrightarrow (A' \models B')] \tag{3}$$

$$(\varphi \models \psi) \ \rightarrow\ (\neg\psi \models \neg\varphi) \tag{4}$$

Proof. See Appendix A. □

It should be stressed that the semantic notions defined here are graded, and can have truth values between 0 and 1. Thus even though the theorems on fuzzy answerhood derived here have syntactically the same form as their classical counterparts, in $\mathcal{F}\mathrm{CT}$ they express more general statements, namely that the truth value of the consequent is not less than that of the antecedent. Thus, e.g., the formula (4) should be interpreted as '$\neg\psi$ entails $\neg\varphi$ *at least in the degree* in which φ entails ψ', rather than a crisp statement that '$\neg\varphi$ entails $\neg\psi$ if φ entails ψ'. The same is true about the notions of answerhood and entailment of questions defined in the next Section.

6. *Fuzzy semantics for questions*

Having defined intensional semantics in $\mathcal{F}\mathrm{CT}$ for propositional formulae, we want to extend this semantics to interrogative formulae $?\varphi$. There are two (classically equivalent) options as to how to understand the question $?\varphi$:

(a) What is the truth value of φ?
(b) Is it the case that φ?

We shall discuss both cases separately. We first interpret $?\varphi$ as the question about the truth value of φ.

Let us fix some *crisp* logical space W.[18] Then ψ answers such a question (which fact we shall symbolize $\psi \models_{\mathrm{t}} ?\varphi$) iff the truth value of ψ determines the truth value of φ. This amounts to the condition that for any indices $w, w' \in W$, if $\|\psi\|_w = \|\psi\|_{w'}$, then $\|\varphi\|_w = \|\varphi\|_{w'}$. Since the identity of truth values is expressed by the equivalence connective defuzzified by Δ

[17] This condition is automatically satisfied in G, or if W is crisp. For each of the statements it can be somewhat weakened: e.g., the condition $W \subseteq W \cap W \cap W$ is sufficient for (1).

[18] Fuzzy W is also meaningful, but the definitions would need much more careful discussion.

(see Section 3), and the extension of φ in w is expressed by $w \in \|\varphi\|$ (see Section 5), the defining condition for $\psi \models_{\mathrm{t}} ?\varphi$ in $\mathcal{F}\mathrm{CT}$ reads

$$(\forall w, w' \in W)[\Delta(w \in \|\psi\| \leftrightarrow w' \in \|\psi\|) \\ \rightarrow \Delta(w \in \|\varphi\| \leftrightarrow w' \in \|\varphi\|)]. \quad (5)$$

Again we can extend the notation and write $A \models_{\mathrm{t}} ?\varphi$, $\psi \models_{\mathrm{t}} ?B$, and $A \models_{\mathrm{t}} ?B$ for arbitrary class terms A and B, not restricting our definition to propositions definable by propositional formulae.

If we define the truth-equivalence relation R_X induced by (a proposition) X as[19]

$$R_X =_{\mathrm{df}} \{\langle u, v\rangle \mid \Delta(u \in X \leftrightarrow v \in X)\}$$

then the answerhood condition can be rewritten as

$$A \models_{\mathrm{t}} ?B \ \equiv_{\mathrm{df}} \ W^2 \cap R_A \subseteq R_B.$$

Following GS, we can identify the intension of $?\varphi$ and the relation $R_{\|\varphi\|}$. The proposition $\{w' \in W \mid \langle w, w'\rangle \in R_{\|\varphi\|}\}$ can be understood as the direct true answer to $?\varphi$ in w, i.e., the extension of $?\varphi$ in w. Truth-value based entailment and equivalence of $?B$ and $?C$ can be defined standardly as

$$?B \models_{\mathrm{t}} ?C \quad \equiv_{\mathrm{df}} \quad (\forall A)[(A \models_{\mathrm{t}} ?B) \rightarrow (A \models_{\mathrm{t}} ?C)] \quad (6)$$
$$?B \equiv_{\mathrm{t}} ?C \quad \equiv_{\mathrm{df}} \quad (?B \models_{\mathrm{t}} ?C) \ \&\ (?C \models_{\mathrm{t}} ?B) \quad (7)$$

It can be observed that these notions of answerhood and entailment are crisp. In fact, they correspond to answerhood and entailment for questions $?\alpha(\|\varphi\|_w = \alpha)$ of classical predicative GS in the intended models of $\mathcal{F}\mathrm{CT}$.[20] As such, they bring little new to the topic; there is, however, a natural fuzzification of these semi-classical notions, obtained by omitting one or both of the Δ's in (5):

[19] We need to extend the language of $\mathcal{F}\mathrm{CT}$ by tuples of objects $\langle x_1, \dots, x_n\rangle$ here. This can be done by adding functors for forming tuples and accessing their components, and axiom schemata saying that tuples equal iff their respective components equal. We then define $W^2 =_{\mathrm{df}} \{\langle u, v\rangle \mid u \in W \ \&\ v \in W\}$. For details see [BC04b]

[20] See Section 4 and [GS97].

Definition 6.1: (Fuzzy truth-value based answerhood)

$$\begin{aligned} A \models_{\mathrm{ft}} ?B \quad &\equiv_{\mathrm{df}} \quad (\forall w, w' \in W) \\ &\quad [\Delta(w \in A \leftrightarrow w' \in A) \to (w \in B \leftrightarrow w' \in B)] \\ A \models_{\mathrm{fft}} ?B \quad &\equiv_{\mathrm{df}} \quad (\forall w, w' \in W) \\ &\quad [(w \in A \leftrightarrow w' \in A) \to (w \in B \leftrightarrow w' \in B)] \end{aligned}$$

The corresponding notions of entailment and equivalence of questions are defined as in (6) and (7), respectively.

It can be observed that the third option, viz. discarding only the first Δ in (5), would lead to a counter-intuitive notion of answerhood, since it would admit cases when φ itself does not answer $?\varphi$ (this follows from the fact that $\chi \to \Delta\chi$ is not a theorem of BLΔ).

All $A \models_{\mathrm{t}} ?B$, $A \models_{\mathrm{ft}} ?B$, and $A \models_{\mathrm{fft}} ?B$ are 1-true in a model if the partition of W by the truth-levels of A refines the partition by the truth-levels of B.[21] Unlike crisp $\models_{\mathrm{t}}$, which otherwise is absolutely false, its graded variants $\models_{\mathrm{ft}}$ and $\models_{\mathrm{fft}}$ partially tolerate the flaws in the match of truth-levels. The truth value of $A \models_{\mathrm{ft}} ?B$ is high iff the truth value of B does not change *too much* within the truth-levels of A.[22] In other words, a proposition *more-or-less* answers $?B$ if its truth value *more-or-less* determines the truth value of B. Different t-norm logics provide different measures of tolerance for imperfection in satisfying the answerhood condition.

The answerhood notion $\models_{\mathrm{fft}}$ strengthens the condition and requires further that the closeness of the truth values of the answer imply the closeness of those being asked for. In Ł, $A \models_{\mathrm{fft}} ?B$ is 1-true iff for any $w, w' \in W$, the difference of the truth values of A in w and w' does not exceed the difference of the truth values of B in w and w'. For Π and G, replace the word 'difference' in the previous sentence respectively by 'ratio' and 'smaller' (where 'the smaller of the truth values' means 1 if they are equal).

Obviously $\models_{\mathrm{ft}}$ is the weakest of the three notions:

[21] We slightly abuse the language here for brevity's sake. It would be more accurate to speak about the partition of the *evaluation of* W and the truth-levels of the *evaluations* of A and B in the model. We use a similar license in the following paragraphs.

[22] The exact meaning of 'does not change too much' is given by the semantics of the equivalence connective, which in t-norm logics expresses the closeness of truth values. In particular, in Ł and Π it respectively expresses the difference and ratio of truth values, while in G it yields the smaller of the truth values (unless they are equal, in which case it is 1-true).

Theorem 6.2: $\mathcal{F}\mathrm{CT}$ *proves*

$$(A \models_{\mathrm{fft}} ?B) \quad \rightarrow \quad (A \models_{\mathrm{ft}} ?B) \tag{8}$$
$$(A \models_{\mathrm{t}} ?B) \quad \rightarrow \quad (A \models_{\mathrm{ft}} ?B) \tag{9}$$

For the proof see Appendix A; counter-examples to the remaining implications are easy to find.[23]

Theorem 6.3: *Let* $\circ$ *be* ft *or* fft. *Then* $\mathcal{F}\mathrm{CT}$ *proves*

$$(A \equiv B) \quad \rightarrow \quad (A \models_{\circ} ?B) \tag{10}$$
$$(A \equiv B) \quad \rightarrow \quad (?A \equiv_{\circ} ?B) \tag{11}$$
$$(A \equiv B) \quad \rightarrow \quad [(A \models_{\mathrm{fft}} ?C) \leftrightarrow (B \models_{\mathrm{fft}} ?C)] \tag{12}$$
$$(?A \models_{\circ} ?B) \quad \rightarrow \quad [(?B \models_{\circ} ?C) \rightarrow (?A \models_{\circ} ?C)] \tag{13}$$
$$(?A \equiv_{\circ} ?B) \quad \rightarrow \quad [(?B \equiv_{\circ} ?C) \rightarrow (?A \equiv_{\circ} ?C)] \tag{14}$$

For the proofs see Appendix A. Two-element counter-examples show that (10) and (11) are not valid for t in place of $\circ$, and that fft in (12) cannot be replaced by t or ft.

Because of their motivation, the notions of answerhood defined above are sensitive with respect to operations that can change the (exact or approximate) match of truth values. Therefore, answerhood is not preserved by usual logical operations (except for equivalence).[24] An example of preservation properties that can be proved is the following theorem:

Theorem 6.4: *Let* $\diamond$ *be a (primitive or defined) connective congruent w.r.t.* $\leftrightarrow$, *i.e., such that* $\mathcal{F} \vdash [(\varphi \leftrightarrow \psi) \mathbin{\&} (\varphi' \leftrightarrow \psi')] \rightarrow [(\varphi \diamond \varphi') \leftrightarrow (\psi \diamond \psi')]$. *Then* $\mathcal{F}\mathrm{CT}$ *proves*

$$[(\varphi \models_{\mathrm{ft}} ?\psi) \mathbin{\&} (\varphi \models_{\mathrm{ft}} ?\psi')] \rightarrow [\varphi \models_{\mathrm{ft}} ?(\psi \diamond \psi')]$$

For the proof, see Appendix A. In particular, the statement holds for $\&$, $\wedge$, $\vee$, or $\leftrightarrow$ substituted for $\diamond$, and can easily be generalized to any arity of $\diamond$. A

[23] E.g., to disprove $(A \models_{\mathrm{ft}} ?B) \rightarrow (A \models_{\mathrm{fft}} ?B)$, use a two-element intended model with the universe $\{a, b\}$ and assign the function $\{\langle a, 0.5\rangle, \langle b, 0.6\rangle\}$ to A, $\{\langle a, 0.5\rangle, \langle b, 0.7\rangle\}$ to B, and $\{\langle a, 1\rangle, \langle b, 1\rangle\}$ to W. Then the truth value of $A \models_{\mathrm{ft}} ?B$ in Ł is 1, while $A \models_{\mathrm{fft}} ?B$ evaluates only to 0.9.

[24] Again, the two-element counter-examples to $[(\varphi \models_{\circ} ?\chi) \mathbin{\&} (\psi \models_{\circ} ?\chi)] \rightarrow (\varphi \diamond \psi \models_{\circ} ?\chi)$ for $\diamond$ replaced by $\&$, $\wedge$, or $\vee$ are easy to find.

further discussion of the truth-value based notion of answerhood, entailment, and equivalence of questions is given in Section 7.

Let us now investigate the other interpretation of $?\varphi$. We again work in $\mathcal{F}\mathrm{CT}$ and now allow W to be fuzzy. The yes-no question 'Is it the case that φ?' is answered by a proposition A iff A either entails φ (then it is an affirmative answer) or entails $\neg\varphi$ (a negative answer):

Definition 6.5: *A proposition A is an* affirmative *answer to $?\varphi$ iff $A \models \varphi$. It is a* negative *answer to $?\varphi$ iff $A \models \neg\varphi$. It is a* yes-no *answer (in symbols $A \models ?\varphi$) iff it is an affirmative answer or a negative answer:*

$$A \models ?\varphi \equiv_{\mathrm{df}} (A \models \varphi) \vee (A \models \neg\varphi)$$

Since entailment of fuzzy propositions is generally a fuzzy notion, so is yes-no answerhood: answers can be, not only fully affirmative or negative, but also *partially* affirmative or *partially* negative (or neither).

Theorem 6.6: $\mathcal{F}\mathrm{CT}$ *proves that* $\Delta(W \subseteq W \cap W)$ *implies*[25]

$$(A \models B) \to [(B \models ?\varphi) \to (A \models ?\varphi)] \tag{15}$$
$$(A \equiv B) \to [(B \models ?\varphi) \leftrightarrow (A \models ?\varphi)] \tag{16}$$
$$(\varphi \equiv \psi) \to [(A \models ?\varphi) \to (A \models ?\psi)] \tag{17}$$

Proof. See Appendix A. □

Theorem 6.7: $\mathcal{F}\mathrm{CT}$ *proves that if* $\Delta(W \subseteq W \cap W)$*, then affirmative and negative answers exclude each other, i.e.,*

$$[(\psi^+ \models \varphi) \mathbin{\&} (\psi^- \models \neg\varphi)] \to (\models \neg(\psi^+ \mathbin{\&} \psi^-))$$

Proof. See Appendix A. □

It can be noticed that the consequent in Theorem 6.7 cannot be strengthened to $\models \neg(\psi^+ \wedge \psi^-)$. In Ł, e.g., an answer ψ can be *both* partially affirmative and partially negative (only $\psi \mathbin{\&} \neg\psi$ must be false).[26]

[25] See footnote 17 on page 176.

[26] An example from natural language for such a situation is, e.g., an answer to the question 'Is he old?' giving some middle age, which both partially affirms and partially denies seniority. (Yet, since *old* and *not old* are mutually exclusive, the truth degrees of affirmation and denial must be low enough for their strong conjunction to be false.)

Following GS, we can define yes-no entailment and equivalence of questions in the standard way:

Definition 6.8: (Yes-no entailment and equivalence of questions)

$$
\begin{array}{lll}
?\varphi \models ?\psi & \equiv_{\mathrm{df}} & (\forall A)[(A \models ?\varphi) \to (A \models ?\psi)] \\
?\varphi \equiv ?\psi & \equiv_{\mathrm{df}} & (?\varphi \models ?\psi) \mathbin{\&} (?\psi \models ?\varphi)
\end{array}
$$

Theorem 6.9: *It is provable in* $\mathcal{F}\mathrm{CT}$ *that* $\Delta(W \subseteq W \cap W)$ *implies*

$$(?\varphi \models ?\psi) \quad \to \quad [(?\psi \models ?\chi) \to (?\varphi \models ?\chi)] \tag{18}$$
$$(?\varphi \equiv ?\psi) \quad \to \quad [(?\psi \equiv ?\chi) \to (?\varphi \equiv ?\chi)] \tag{19}$$
$$(\varphi \equiv \varphi') \quad \to \quad [(?\varphi \models ?\psi) \to (?\varphi' \models ?\psi)] \tag{20}$$
$$(\psi \equiv \psi') \quad \to \quad [(?\varphi \models ?\psi) \to (?\varphi \models ?\psi')] \tag{21}$$
$$(\varphi \equiv \psi) \quad \to \quad (?\varphi \models ?\psi) \tag{22}$$
$$(?\varphi \models ?\psi) \quad \to \quad (?\varphi \models ?\neg\psi) \tag{23}$$

Proof. See Appendix A. □

Since obviously $\varphi \models \varphi$, from (22) and (23) it follows that $?\varphi \models ?\varphi$ and $?\varphi \models ?\neg\varphi$. The converse, $?\neg\varphi \models ?\varphi$, does not hold generally, since $\neg\neg\varphi \to \varphi$ is not a theorem of BL. There are examples from natural language that this result does not contradict intuition: if negation behaves in some context as the bivalent negation of G or Π (there are such contexts—e.g., *not guilty* can be regarded as bivalent, even though there are degrees of *guilt*), then a negative answer to $?\neg\varphi$ need not be affirmative enough to $?\varphi$. The equivalence of $?\varphi$ and $?\neg\varphi$ does, however, hold in Ł or for crisp φ.

7. *Conclusions*

We have seen that the two interpretations of the question $?\varphi$ in fuzzy logic give rise to two different kinds of fuzzy answerhood notions. Although these notions coincide in classical logic, their properties in fuzzy logic are considerably different. It appears that the number of interesting theorems that can be derived in $\mathcal{F}\mathrm{CT}$ is larger with yes-no answerhood than with truth-value based answerhood. The following observations can shed some light upon this fact.

The definition of yes-no answerhood $\models$ conforms better to the methodology of [BC04a], according to which the truth-value semantics of fuzzy logic is only secondary to the rules of inference that hold for fuzzy propositions. Since there is little sense in asserting that some fuzzy proposition (e.g., 'John loves Mary') is true exactly in the degree (say) 0.7845, fuzzy truth values must only be regarded as a *model* underlying the rules of inference valid for fuzzy propositions (even though these rules may originally have been descried by means of this model). The doctrine of not speaking explicitly about the truth degrees, but rather hiding them in the semantical meta-level of a formal theory, is one of the design principles of $\mathcal{F}\mathrm{CT}$, which has been used here as the framework for fuzzy intensional semantics and erotetic logic.

Although formulated formally in $\mathcal{F}\mathrm{CT}$ (ergo, without an explicit reference to truth values), the definitions of $\models_{\mathrm{t}}$, $\models_{\mathrm{ft}}$, and $\models_{\mathrm{fft}}$ capture in fact the answerhood conditions for the question about the *truth value* of φ, rather than about the fuzzy proposition φ itself.[27] Therefore these notions, though useful when working with particular models, are not particularly well-suited for investigation in $\mathcal{F}\mathrm{CT}$, which only captures general laws valid in all models, rather than particular truth values. Nevertheless, the theorems of Section 6 show that at least some properties of truth-value answerhood are universally valid and can be proved in $\mathcal{F}\mathrm{CT}$.

Fuzzy intensional semantics developed here for the purposes of fuzzy erotetic logic is general enough to serve as the basis for a similar fuzzification of other kinds of modal (epistemic, deontic, etc.) logic. Since our semantic notions of entailment and answerhood are defined as certain formulae of $\mathcal{F}\mathrm{CT}$, they are compatible with the formalism proposed in [BC04a] and [BC04b] as a unified framework for a large part of fuzzy mathematics, and directly applicable in other formal theories within the framework.

Appendix A. *Formal proofs*

In this Appendix, we give the formal proofs of the theorems of the preceding sections. In the proofs we shall freely use the transitivity of implication, (i.e., the axiom (BL1) plus twice modus ponens), (BL3), (BL5a), and (BL5b) without explicit notices. All statements of the form $\mathrm{BL} \vdash \varphi$ or $\mathrm{BL}\Delta \vdash \varphi$ refer to [Háj98] where they are proved.

[27] This can be seen from the fact that propositions stronger than A need not answer $?\varphi$, even if A itself does. This would be counter-intuitive for answerhood of the question about φ, but is quite natural for querying about truth values, since the truth values of stronger propositions may be much different from those of A and the distinctions may become lost.

Lemma A.1: *The following formulae are theorems of* BLΔ*:*

$$(\forall w)(\varphi \to \psi) \to [(\forall w)\varphi \to (\forall w)\psi] \tag{24}$$
$$(\varphi \to \psi) \to [(\chi \to \varphi) \to (\chi \to \psi)] \tag{25}$$
$$(\varphi \to \psi) \to [\nu \to (\varphi \to \psi)] \tag{26}$$
$$[(\varphi \to \psi) \mathbin{\&} (\varphi' \to \psi')] \to [(\varphi \mathbin{\&} \varphi') \to (\psi \mathbin{\&} \psi')] \tag{27}$$
$$[(\varphi \to \psi) \mathbin{\&} (\varphi' \to \psi')] \to [(\varphi \vee \varphi') \to (\psi \vee \psi')] \tag{28}$$
$$[(\varphi \to \psi) \mathbin{\&} (\varphi' \to \psi')] \to [(\varphi \wedge \varphi') \to (\psi \wedge \psi')] \tag{29}$$
$$[\varphi \to (\psi \to \chi)] \to [(\varphi \to \psi) \to (\varphi^2 \to \chi)] \tag{30}$$
$$(\varphi \to \psi) \to (\Delta\varphi \to \psi) \tag{31}$$
$$(\Delta\varphi \to \Delta\psi) \to (\Delta\varphi \to \psi) \tag{32}$$
$$[(\nu \to \nu^3) \mathbin{\&} ((\nu \mathbin{\&} \varphi) \to \psi) \mathbin{\&} ((\nu \mathbin{\&} \psi) \to \chi)] \to ((\nu \mathbin{\&} \varphi) \to \chi) \tag{33}$$
$$[(\nu \to \nu^2) \mathbin{\&} ((\nu \mathbin{\&} \varphi) \to \psi)] \to ((\nu \mathbin{\&} \neg\psi) \to \neg\varphi) \tag{34}$$
$$\Delta(\nu \to \nu^2) \to (\nu \to \nu^3) \tag{35}$$

Proof. (24) and (27) are proved in [Háj98].

(26) is an instance of $\mathrm{BL} \vdash \varphi \to (\psi \to \varphi)$.

(25) follows from (BL1) by (BL5a), (BL3), and (BL5b).

(28) From $(\varphi \to \psi) \to [(\psi \to (\psi \vee \psi')) \to (\varphi \to (\psi \vee \psi'))]$, which is an instance of (BL1), and $\mathrm{BL} \vdash \psi \to (\psi \vee \psi')$ we get $(\varphi \to \psi) \to (\varphi \to (\psi \vee \psi'))$. Similarly, using in addition $\mathrm{BL} \vdash (\psi' \vee \psi) \to (\psi \vee \psi')$, we get $(\varphi' \to \psi') \to (\varphi' \to (\psi \vee \psi'))$. Thus by (27) we get $[(\varphi \to \psi) \mathbin{\&} (\varphi' \to \psi')] \to [(\varphi \to (\psi \vee \psi')) \mathbin{\&} (\varphi' \to (\psi \vee \psi'))]$, whence (28) follows from $\mathrm{BL} \vdash [(\varphi \to \chi) \mathbin{\&} (\varphi' \to \chi)] \to [(\varphi \vee \varphi') \to \chi]$.

(29) is proved similarly as (28), only using $\wedge$ instead of $\vee$ and antecedents instead of consequents of implications.

(30) follows from the instance $[(\varphi \to (\psi \to \chi)) \mathbin{\&} (\varphi \to \psi)] \to [(\varphi \mathbin{\&} \varphi) \to (\psi \mathbin{\&} (\psi \to \chi))]$ of (27) and $\mathrm{BL} \vdash [\psi \mathbin{\&} (\psi \to \chi)] \to \chi$.

(31) follows from (Δ3) and the instance $(\Delta\varphi \to \varphi) \to [(\varphi \to \psi) \to (\Delta\varphi \to \psi)]$ of (BL1).

(32) follows from (Δ3) and the instance $(\Delta\psi \to \psi) \to [(\Delta\varphi \to \Delta\psi) \to (\Delta\varphi \to \psi)]$ of (25).

(33) Take the instance of (BL1)
$((\nu \mathbin{\&} \varphi) \to \psi) \to [(\psi \to \chi) \to ((\nu \mathbin{\&} \varphi) \to \chi)]$; thence by (26) we get
$((\nu \mathbin{\&} \varphi) \to \psi) \to [\nu \to [(\psi \to \chi) \to ((\nu \mathbin{\&} \varphi) \to \chi)]]$; applying (30) we get
$((\nu \mathbin{\&} \varphi) \to \psi) \to [[\nu \to (\psi \to \chi)] \to [((\nu^3 \mathbin{\&} \varphi) \to \chi)]]$,
whence (33) readily follows.

(34) From $\mathrm{BL} \vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$ by (26) we get $\nu \to [(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)]$; then by (30) we have $[(\nu \mathbin{\&} \varphi) \to \psi] \to [(\nu^2 \mathbin{\&} \neg\psi) \to \neg\varphi]$, whence (34).

(35) The instance $[\Delta(\nu \to \nu^2) \mathbin{\&} \nu] \to \nu^2$ of (Δ3) used twice in (27) and $\mathrm{BL}\Delta \vdash (\Delta\varphi \mathbin{\&} \Delta\varphi) \leftrightarrow \Delta\varphi$ yield $[\Delta(\nu \to \nu^2) \mathbin{\&} \nu^2] \to \nu^4$, i.e., $\Delta(\nu \to \nu^2) \to (\nu^2 \to \nu^4)$. Since $\Delta(\nu \to \nu^2) \to (\nu \to \nu^2)$ by (Δ3), we get also $\Delta(\nu \to \nu^2) \to (\nu \to \nu^4)$, whence by (BL2) we obtain (35). □

Proof of Lemma 4.4. The substitution of $x \in X_i$ for p_i (for all $i \leq n$) everywhere in the proof of $\varphi \to \psi$ in $\mathcal{F}$ transforms it into the proof of

$$\varphi(x \in X_1, \ldots, x \in X_n) \to \psi(x \in X_1, \ldots, x \in X_n)$$

in first-order $\mathcal{F}$. Generalization on x then yields

$$(\forall x)(\varphi(x \in X_1, \ldots, x \in X_n) \to \psi(x \in X_1, \ldots, x \in X_n))$$

which is exactly $\mathrm{Op}_\varphi(X_1, \ldots, X_n) \subseteq \mathrm{Op}_\psi(X_1, \ldots, X_n)$ by the definitions and axioms of $\mathcal{F}\mathrm{CT}$.

Conversely, let e be an evaluation that refutes $\varphi \to \psi$ (we use the Completeness Theorem for propositional fuzzy logics here, see [Háj98]). We construct a model M of $\mathcal{F}\mathrm{CT}$ that refutes $\mathrm{Op}_\varphi(X_1, \ldots, X_n) \subseteq \mathrm{Op}_\psi(X_1, \ldots, X_n)$ as follows. Let the universe of M contain a single element a, and let the class variables X_i be represented by the functions that assign $e(p_i)$ to a. It is trivial to check that M models $\mathcal{F}\mathrm{CT}$ and refutes $\mathrm{Op}_\varphi(X_1, \ldots, X_n) \subseteq \mathrm{Op}_\psi(X_1, \ldots, X_n)$. By the Soundness Theorem of the first-order logic $\mathcal{F}$ (see [Háj98]) the proof is done. □

Proof of Lemma 4.5: From the instance $(x \in X \to x \in Y) \to [(x \in Y \to x \in Z) \to (x \in X \to x \in Z)]$ of (BL1), generalization on x and distribution of the quantifier by (24) yields the required formula $[(\forall x)(x \in X \to x \in Y) \mathbin{\&} (\forall x)(x \in Y \to x \in Z)] \to (\forall x)(x \in X \to x \in Z)$. □

Proof of Theorem 5.2. In BL it is provable that φ is equivalent to $\top \to \varphi$. Since further $\mathrm{Op}_\top = \mathrm{V}$, we get from Lemma 4.4:

$$\begin{aligned} &\mathcal{F} \vdash \varphi(p_1, \ldots, p_n) \quad \text{iff} \\ &\quad \text{iff} \quad \mathcal{F} \vdash \top \to \varphi(p_1, \ldots, p_n) \\ &\quad \text{iff} \quad \mathcal{F}\mathrm{CT} \vdash \mathrm{Op}_\top(\|p_1\|, \ldots, \|p_n\|) \subseteq \mathrm{Op}_\varphi(\|p_1\|, \ldots, \|p_n\|) \\ &\quad \text{iff} \quad \mathcal{F}\mathrm{CT} \vdash \mathrm{V} \subseteq \|\varphi(p_1, \ldots, p_n)\| \\ &\quad \text{iff} \quad \mathcal{F}\mathrm{CT} \vdash W \subseteq \|\varphi(p_1, \ldots, p_n)\| \end{aligned}$$

The last equivalence follows in one direction from the monotonicity of $\subseteq$ (Lemma 4.5); the other direction is obtained by generalization on W and specification to V. $\square$

Proof of Theorem 5.3: (1) follows from (33) and a general theorem of [BC04b], but it is not difficult to derive it from (33) directly. Substitute $w \in W$, $w \in A$, $w \in B$, and $w \in C$ into (33) for ν, φ, ψ, and χ, respectively. Then generalizing on w and distributing the quantifier by (24) (using (BL5)), we get (1) expanded according to the definitions of $\mathcal{F}\mathrm{CT}$. (Use (35) to get the stated precondition of the theorem.)

(2) follows from (27) and the instances of (1) $[(A \models B)\ \&\ (B \models C)] \rightarrow (A \models C)$ and $[(C \models B)\ \&\ (B \models A)] \rightarrow (C \models A)$.

(3) It follows from (1) that
$[(A' \models A)\ \&\ (B \models B')] \rightarrow [(A \models B) \rightarrow (A' \models B')]$ and
$[(A \models A')\ \&\ (B' \models B)] \rightarrow [(A' \models B') \rightarrow (A \models B)]$. Now use (27).

(4) follows from (34) in the same way as (1) from (33). (Note that the converse of (4) does not hold.) $\square$

Proof of Theorem 6.2: (8) is proved by generalization on $w \in W$ and $w' \in W$ of the instance $[(w \in A \leftrightarrow w' \in A) \rightarrow (w \in B \leftrightarrow w' \in B)] \rightarrow [\Delta(w \in A \leftrightarrow w' \in A) \rightarrow (w \in B \leftrightarrow w' \in B)]$ of (31), and distribution of both quantifiers over the principal implication by (24). (The rule of bounded generalization follows from (26); the analogue of (24) for quantifiers relativized to a *crisp* domain follows easily from (30).)

(9) is proved in the same way as (8) from the instance $[\Delta(w \in A \leftrightarrow w' \in A) \rightarrow \Delta(w \in B \leftrightarrow w' \in B)] \rightarrow [\Delta(w \in A \leftrightarrow w' \in A) \rightarrow (w \in B \leftrightarrow w' \in B)]$ of (32). $\square$

Proof of Theorem 6.3. In the proof, the restriction of all quantifiers to W is omitted for simplicity's sake. It is an easy, but tedious exercise to verify that the proof works in the same way with all quantifiers restricted to crisp W. We shall use Xww' as shorthand for $w \in X \leftrightarrow w' \in X$.

(10) $A \equiv B$ amounts to $A \subseteq B\ \&\ B \subseteq A$ here, whence by specification we get $(A \equiv B) \rightarrow [(w' \in A \rightarrow w' \in B)\ \&\ (w \in B \rightarrow w \in A)]$. The transitivity of implication entails $[(w' \in A \rightarrow w' \in B)\ \&\ (w \in B \rightarrow w \in A)\ \&\ (w \in A \leftrightarrow w' \in A)] \rightarrow (w \in B \rightarrow w' \in B)$; thus we get $[(A \equiv B)\ \&\ (w \in A \leftrightarrow w' \in A)] \rightarrow (w \in B \rightarrow w' \in B)$. Similarly $[(A \equiv B)\ \&\ (w' \in A \leftrightarrow w \in A)] \rightarrow (w' \in B \rightarrow w \in B)$. Since BL $\vdash [(\chi \rightarrow \varphi)\ \&\ (\chi \rightarrow \psi)] \rightarrow [(\chi \rightarrow (\varphi \wedge \psi))]$ and BL $\vdash [(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)] \rightarrow (\varphi \leftrightarrow \psi)$, we get $(A \equiv B) \rightarrow [(w \in A \leftrightarrow w' \in A) \rightarrow (w \in B \leftrightarrow w' \in B)]$. Generalization on w and w' plus the axiom ($\forall 2$) conclude the proof. (10) for $\models_{\mathrm{ft}}$ follows a fortiori (see Theorem 6.2).

(11) In the proof of (10) we have proved $(A \equiv B) \to (Aww' \to Bww')$. By (25) with χ instantiated to Cww' we get $(A \equiv B) \to [(Cww' \to Aww') \to (Cww' \to Bww')]$. Generalization on w, w' plus (∀2) and (24), and generalization on C plus (∀2) conclude the proof. (11) for $\models_{\mathrm{ft}}$ is proved in the same way, only using $\Delta Cww'$ when instantiating χ in (25).

(12) As in the proof of (10) we prove $(A \equiv B) \to (Bww' \to Aww')$. Since further $\mathrm{BL} \vdash [Aww' \mathbin{\&} (Aww' \to Cww')] \to Cww'$, we get $[(A \equiv B) \mathbin{\&} Bww' \mathbin{\&} (Aww' \to Cww')] \to Cww'$, i.e., $(A \equiv B) \to [(Aww' \to Cww') \to (Bww' \to Cww')]$. Generalization on w and w' plus the axiom (∀2) conclude the proof.

(13) From the instance $((D \models ?A) \to (D \models ?B)) \to [((D \models ?B) \to (D \models ?C)) \to ((D \models ?A) \to (D \models ?C))]$ of (BL1), by generalization on D and distribution of the quantifier by (24) we get (13).

(14) follows from (13) by (27). ☐

Proof of Theorem 6.4. Let us denote $\|\varphi\|$, $\|\psi\|$, $\|\psi'\|$, and $\|\psi \diamond \psi'\|$ by A, B, B', and C respectively, and adopt the conventions of the Proof of Theorem 6.3.

The precondition of the present theorem gives $(Bww' \mathbin{\&} B'ww') \to Cww'$, whence $[\Delta Aww' \to (Bww' \mathbin{\&} B'ww')] \to (\Delta Aww' \to Cww')$ by (25). Thence by (27) and $\mathrm{BL}\Delta \vdash \Delta\chi \to (\Delta\chi \mathbin{\&} \Delta\chi)$ we get $[(\Delta Aww' \to Bww') \mathbin{\&} (\Delta Aww' \to B'ww')] \to (\Delta Aww' \to Cww')$. By generalization on w and w' and distribution of the quantifiers by (24) we get the required formula.

That $\&$, $\wedge$, $\vee$, and $\leftrightarrow$ substituted for $\diamond$ satisfy the precondition of the theorem follows from (27), (29), (28), and transitivity of $\leftrightarrow$, respectively. ☐

Proof of Theorem 6.6: (15) From (1) we have
$[(A \models B) \mathbin{\&} (B \models \varphi)] \to (A \models \varphi)$ and $[(A \models B) \mathbin{\&} (B \models \neg\varphi)] \to (A \models \neg\varphi)$.
Thence by (28) it follows that
$[((A \models B) \mathbin{\&} (B \models \varphi)) \vee ((A \models B) \mathbin{\&} (B \models \neg\varphi))] \to ((A \models \varphi) \vee (A \models \neg\varphi))$;
now by $\mathrm{BL} \vdash [(\chi \mathbin{\&} \psi) \vee (\chi \mathbin{\&} \psi')] \leftrightarrow [\chi \mathbin{\&} (\psi \vee \psi')]$ we get
$[((A \models B) \mathbin{\&} ((B \models \varphi) \vee (B \models \neg\varphi)))] \to ((A \models \varphi) \vee (A \models \neg\varphi))$, which is (15).

(16) From (15) we get
$(A \models B) \to [(B \models ?\varphi) \to (A \models ?\varphi)]$ and $(B \models A) \to [(A \models ?\varphi) \to (B \models ?\varphi)]$,
whence by (27) we get (16).

(17) From (1) it follows that
$(\varphi \models \psi) \to [(A \models \varphi) \to (A \models \psi)]$ and, using (4),
$(\psi \models \varphi) \to [(A \models \neg\varphi) \to (A \models \neg\psi)]$. Then

$[(\varphi \equiv \psi)\ \&\ ((A \models \varphi) \vee (A \models \neg\varphi))] \to ((A \models \psi) \vee (A \models \neg\psi))$ as in (15). □

Proof of Theorem 6.7: By (27) we get
$[(\psi^+ \to \varphi)\ \&\ (\psi^- \to \neg\varphi)] \to [(\psi^+\ \&\ \psi^-) \to (\varphi\ \&\ \neg\varphi)]$.
Since BL $\vdash (\chi \to (\varphi\ \&\ \neg\varphi)) \to \neg\chi$, we have
$[(\psi^+ \to \varphi)\ \&\ (\psi^- \to \neg\varphi)] \to \neg(\psi^+\ \&\ \psi^-)$.
Then proceed as in the proof of (34) and (1). □

Proof of Theorem 6.9: (18) and (19) are proved exactly as (13) and (14).

(20) From (17) we have $(\varphi \equiv \varphi') \to [(A \models ?\varphi') \to (A \models ?\varphi)]$. Thus from $((A \models ?\varphi') \to (A \models ?\varphi)) \to$
$$\to [((A \models ?\varphi) \to (A \models ?\psi)) \to ((A \models ?\varphi') \to (A \models ?\psi))],$$
which is an instance of (BL1), we get
$(\varphi \equiv \varphi') \to [((A \models ?\varphi) \to (A \models ?\psi)) \to ((A \models ?\varphi') \to (A \models ?\psi))]$.
Then generalize on A and distribute the quantifier according to ($\forall$2) and (24).

(21) From (17) we have $(\psi \equiv \psi') \to [(A \models ?\psi) \to (A \models ?\psi')]$.
As in the proof of (20) we derive
$(\psi \equiv \psi') \to [((A \models ?\varphi) \to (A \models ?\psi)) \to ((A \models ?\varphi) \to (A \models ?\psi'))]$
and proceed as in the previous case.

(22) From (17) we have $(\varphi \equiv \psi) \to [(A \models ?\varphi) \to (A \models ?\psi)]$.
Then generalize on A and use ($\forall$2).

(23) From BL $\vdash \psi \to \neg\neg\psi$ we can infer $\psi \models \neg\neg\psi$ and by (1) get
$(A \models \psi) \to (A \models \neg\neg\psi)$, whence
$[(A \models ?\varphi) \to (A \models ?\psi)] \to [(A \models ?\varphi) \to (A \models ?\neg\psi)]$.
To finish the proof we generalize on A and apply (24). □

Institute of Computer Science,
Academy of Sciences of the Czech Republic,
Pod Vodárenskou věží 2,
182 07 Praha 8,
Czech Republic
E-mail: behounek@cs.cas.cz

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