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THE AXIOM OF MCKINSEY-SOBOCIŃSKI K1 IN THE FRAMEWORK OF DISCUSSIVE LOGICS

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Abstract
In this paper we use Jaśkowski's method of defining a propositional logic with the help of the M-fragment of a given modal logic to express classical logic. We use as weak tools as possible to do this. A strengthening of some results by Scott and Lemmon concerning the McKinsey-Sobociński axiom is presented. This paper is a part of the investigation of building the adaptive logic on the basis of the logic $\mathsf{D_2}$.

Introduction

In [11] a comparison of adaptive and discussive approaches to paraconsistency was presented. As most of inconsistency adaptive logics use classical logic as a so called upper limit logic[1], the question of expressing classical logic with the help of M-fragment[2] of a certain modal logic arises. It appears (see Lemma 3) that in the discussive framework Duns Scotus' law is equivalent to the famous McKinsey-Sobociński axiom[3]:

$$\Box\Diamond A \to \Diamond\Box A. \tag{K1}$$

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[1] For explanation see for example [3].

[2] The *M-fragment* of a given modal logic P is the set $\{\Diamond A : A$ is any formula and $\Diamond A \in \mathsf{P}\}$.

[3] In the literature it is also denoted by (M).

In 1966, Lemmon made a conjecture that each consistent normal logic P is complete with respect to some class of P-frames (the paper was finally published as [10]). In [13] the conjecture was disproved. But Lemmon had already himself sensed the limitation of his conjecture. The McKinsey-Sobociński axiom was given by him as a possible counterexample.

We have the following Lemmon and Scott results concerning the McKinsey-Sobociński axiom ([10], pp. 74–76)[4]:

Theorem 1: *A formula is valid in all frames satisfying the condition m^{∞}: $\forall_w \exists_{\overline{w}} \Big(wR\overline{w} \wedge \forall_{w_1} \forall_{w_2} (\overline{w}Rw_1 \wedge \overline{w}Rw_2 \rightarrow w_1 = w_2) \Big)$ and the condition of transitivity iff it is a theorem of the logic* K4 *with the additional axiom* $(\mathrm{K1})$ *(notation:* K4M*).*

Theorem 2: *A formula is valid in all frames satisfying the condition m^{∞}, the condition of reflexivity, and the condition of transitivity iff it is a theorem of the logic* S4 *with the additional axiom* $(\mathrm{K1})$ *(notation:* S4M*).*

In the class of transitive frames the validity of the axiom $\mathrm{K1}$ is equivalent to the satisfaction of the condition (‡): $\forall_w \exists_{\overline{w}} \Big(wR\overline{w} \wedge \forall_{w'} (\overline{w}Rw' \rightarrow \overline{w} = w') \Big)$[5]. In [12] it was proved that M-counterpart[6] of McKinsey' logic $\mathsf{S4.1} = \mathsf{S4}[\mathrm{K1}]$ [7] is the trivial logic. Of course $\mathsf{S5}[\mathrm{K1}] = \mathsf{Triv}$.

In the paper [14] it was shown that the class of frames for which the completeness result for KM [8] might hold is not definable by the first order condition. The same result was stated in [7] since the likely class of frames for which KM would be complete is not closed on ultraproducts. Finally, in [5] the completeness theorem for KM with respect to some class of finite frames was proved by the normal form method.

Let us add, that logic KM is neither canonical [8], nor compact [15].

In the paper we will use standard notions and results from the field of the modal logic. A short summary can be found in [11]. We will use the notation introduced there.

[4] See also [9] pp. 131–134.

[5] See for example [4], p. 82.

[6] The *M-counterpart* of a given modal logic P is the set $\{A : \Diamond A \in \mathsf{P}\}$.

[7] It is by the definition the smallest normal logic containing S4 and axiom K1.

[8] It is by the definition the smallest normal logic containing axiom K1.

1. *Duns Scotus' law in the discussive framework*

Let us start with:

Lemma 3: *On the basis of the logic* K *the McKinsey-Sobociński formula*

$$\Box\Diamond A \to \Diamond\Box A \tag{K1}$$

is equivalent the discussive version of Duns Scotus' law

$$\Diamond\Big(\Diamond A \to (\Diamond \sim A \to B)\Big) \tag{JDS}$$

Proof. We prove the axiom K1 on the basis of K[JDS] [9]:

1. $\Box\Diamond A \to \Big(\Box\Diamond \sim A \to \Diamond\bot\Big)$ — the law t5 of distributivity of „$\Diamond$" with respect to „$\to$" and JDS: $B/\bot$
2. $\Diamond\bot \leftrightarrow \bot$ — the theorem of K
3. $\Box\Diamond A \wedge \Box\Diamond \sim A \to \bot$ — the law of importation, extensionality, 1. and 2.
4. $\sim(\Box\Diamond A \wedge \Box\Diamond \sim A)$ — 3. via classical logic
5. $\Box\Diamond A \to \Diamond\Box A$ — 4. and the law of negation of implication and inter-definability of „$\Diamond$" and „$\Box$"

On the other hand we prove JDS in KM:

1. $\Box \sim B \to \Big(\Box\Diamond A \to \Diamond\Box A\Big)$ — K1 via the classical logic
2. $\Box\Diamond A \to \Big(\Box \sim B \to \Diamond\Box A\Big)$ — 2. and interchange of premises
3. $\Big(\Box \sim B \to \Diamond\Box A\Big) \to \Big(\sim\Diamond\Box A \to \sim\Box \sim B\Big)$ — the law of contraposition
4. $\Big(\Box \sim B \to \Diamond\Box A\Big) \to \Big(\Box\Diamond \sim A \to \Diamond B\Big)$ — 3. and the inter-definability between $\Box$ and $\Diamond$
5. $\Box\Diamond A \to (\Box\Diamond \sim A \to \Diamond B)$ — the law of syllogism, 2. and 4.
6. $\Diamond\Big(\Diamond A \to (\Diamond \sim A \to B)\Big)$ — the law of syllogism and t5

[9] We use auxiliary facts, definitions and notations as presented in [11].

Corollary 4: *The logic* $\mathsf{KD^{*}T^{*}}[\mathrm{K1}]$ [10] *is the minimal normal logic containing axioms* $\Diamond(Ax1)^{d}$ — $\Diamond(Ax13)^{d}$ [11] *and Duns Scotus' law with the discussive interpretation of „→", and closed under the rule:* $\Diamond A, \Diamond(\Diamond A \to B) \vdash \Diamond B$.

Proof. Follows directly from the theorem 7 from [11] and Lemma 3.

The theorem states that the logic $\mathsf{KD^{*}T^{*}}$ extended with JDS is the minimal normal logic determining discussive classical logic.

2. *Semantics of the logic* $\mathsf{S5_M}[\mathrm{K1}]$

Now we give conditions for frames which establish the completeness result for the logic under consideration. We'll use theorem 12 of [11] and the following observations semantically characterizing logic $\mathsf{KD^{*}T^{*}}[\mathrm{K1}]$.

Lemma 5: *The set of all frames satisfying the McKinsey condition:*

($\ddagger$) $\qquad \forall_{w}\exists_{\overline{w}}\Big(wR\overline{w} \wedge \forall_{w'}(\overline{w}Rw' \to \overline{w} = w')\Big)$

is contained in the intersection of the set of all frames satisfying the condition ($\circledast$) $\forall_{w}\exists_{w'}\Big(wRw' \wedge \forall_{w''}(w'Rw'' \to wRw'')\Big)$ *and the set of all frames satisfying the condition* ($\circledcirc$) $\forall_{w}\exists_{\overline{w}}\Big(wR\overline{w} \wedge \forall_{w'}\forall_{w''}(\overline{w}Rw' \wedge w'Rw'' \to wRw'')\Big)$.

Proof. Let us consider the frame $\langle W, R\rangle$ fulfilling the condition ($\ddagger$). Let us take any $w \in W$. Let $\overline{w}$, be a world such that $wR\overline{w}$, the existence of which is stated in the condition ($\ddagger$). We prove that $\forall_{w''}(\overline{w}Rw'' \to wRw'')$. Let w'' be any world such that $\overline{w}Rw''$. By ($\ddagger$): $\overline{w} = w''$, since $wR\overline{w}$, so also wRw''. Because w is any world, we have shown that given frame satisfies the condition ($\circledast$).

Now we show that for the chosen above world $\overline{w}$ the following is satisfied $\forall_{w'}\forall_{w''}(\overline{w}Rw' \wedge w'Rw'' \to wRw'')$. Let us consider any worlds w' and w'', such that $\overline{w}Rw'$ and $w'Rw''$. We have to prove that wRw''. Once more by ($\ddagger$) we have: $\overline{w} = w'$; since $w'Rw''$, therefore also $\overline{w}Rw''$. Using ($\ddagger$) once

[10] $\mathsf{KD^{*}T^{*}}$ is the minimal normal logic containing axioms: (D^{*}) $\Box\Diamond A \to \Diamond A$ and (T^{*}) $\Box\Diamond\Diamond A \to \Diamond A$. It was proved by Dziobiak that $\mathsf{KD^{*}T^{*}}$ is equal to Perzanowski's system $\mathsf{S5_M}$.

[11] For a propositional variable A, $A^{\mathrm{d}} = A$, and for any formulas B, C: $(B \vee C)^{\mathrm{d}} = B^{\mathrm{d}} \vee C^{\mathrm{d}}$, $(B \wedge C)^{\mathrm{d}} = B^{\mathrm{d}} \wedge \Diamond C^{\mathrm{d}}$, $(B \to C)^{\mathrm{d}} = \Diamond B^{\mathrm{d}} \to C^{\mathrm{d}}$, $(\sim B)^{\mathrm{d}} = \sim(B^{\mathrm{d}})$, and $(B \leftrightarrow C)^{\mathrm{d}} = (\Diamond B^{\mathrm{d}} \to C^{\mathrm{d}}) \wedge \Diamond(\Diamond C^{\mathrm{d}} \to B^{\mathrm{d}})$, while $Ax1$ — $Ax13$ are axioms of the propositional part of logic CLuN i.e. the full positive classical logic plus Clavius' law. For details see [2] and [11].

again, we get $\overline{w} = w''$, but $wR\overline{w}$, i.e. wRw'', which shows that the condition $(\circledast)$ is fulfilled.

The proof of the next theorem is based on the analogous proof the completeness theorem 1 for the logic $\mathsf{K4M}$. We strengthen the theorem 1 using, as we'll see, the weaker logic. The result shows the importance of the M-fragment of a given logic.

Theorem 6: *A formula is valid in all frames fulfilling the condition*

$$(\ddagger) : \forall_w \exists_{\overline{w}} \Big(wR\overline{w} \wedge \forall_{w'} (\overline{w}Rw' \to \overline{w} = w') \Big)$$

iff it is provable in the logic $\mathsf{S5_M}$ *with the additional axiom* (K1) *(notation:* $\mathsf{S5_M}[\mathrm{K1}]$*).*

Proof. $(\Leftarrow)$ We show that the axioms D^*, T^* and K1 are valid in all frames fulfilling the condition $(\ddagger)$. By the theorems stating the completeness results for logics $\mathsf{D}^\star$ and $\mathsf{T}^\star$ (see [11]), axioms D^* and T^* are valid in all frames satisfying conditions $(\circledast)$ and $(\circledast)$ respectively. However, by lemma 5 we know that each frame satisfying the condition $(\ddagger)$ also satisfies the conjunction of conditions $(\circledast)$ and $(\circledast)$. So it is enough to show that the axiom K1 is valid in each frame satisfying the condition $(\ddagger)$. Let us assume otherwise, i.e. that there is a Kripke frame for which the condition is fulfilled while the formula K1 is not valid. So there is a world w and a valuation v, such that $w \not\models_v$ K1, therefore $w \models_v \Box\Diamond p$ and $w \not\models_v \Diamond\Box p$. By the definition of truth in a model we have $\forall_{\overline{w}}(wR\overline{w} \Rightarrow \overline{w} \not\models_v \Box p)$, in particular, for a world w' such that wRw', existence of which is mentioned in $(\ddagger)$, we have $w' \not\models_v \Box p$. Using the definition of truth in a model for „$\Box$", we have that there is w'', that $w'Rw''$ and $w'' \not\models_v p$. By the condition $(\ddagger)$ we see that $w' = w''$, i.e. $w' \not\models_v p$ (notation $\bullet$). Because $w \models_v \Box\Diamond p$, so for any world which is accessible from the world w, in particular for w' holds $w' \models_v \Diamond p$. By the definition of truth there is a world $\breve{w}$, for which $w'R\breve{w}$ and $\breve{w} \models_v p$. But by $(\ddagger)$ $\breve{w} = w$, so $w' \models_v p$, which contradicts $(\bullet)$.

$(\Rightarrow)$. Firstly we show that for any formulas $A_1, \ldots, A_n$ the following $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Diamond\Big((A_1 \to \Box A_1) \wedge \cdots \wedge (A_n \to \Box A_n)\Big)$ holds. To get this result we infer in $\mathsf{S5_M}[\mathrm{K1}]$ a theorem:

$$\Box(\Box\Diamond A \wedge \Box\Diamond B) \to \Diamond(A \wedge B). \qquad (\star)$$

1. $\Box\Diamond A \wedge \Box\Diamond B \to \Diamond\Box A \wedge \Box\Diamond B$
   addition of a new right conjunct $(\Box\Diamond B)$ to arguments of implication K1
2. $\Box(\neg A \vee \neg B) \to (\Box\neg A \vee \Diamond\neg B)$ — the substitution into the theorem

$\Box(A \vee B) \to (\Box A \vee \Diamond B)$ of logic K: $A/\neg A, B/\neg B$

3. $\neg(\Box\neg A \vee \Diamond\neg B) \to \neg\Box(\neg A \vee \neg B)$ the contraposition of 2.
4. $\Diamond A \wedge \Box B \to \Diamond(A \wedge B)$ 3., de Morgan's law and the inter-definability between $\Box$ and $\Diamond$
5. $\Diamond(A \wedge B) \to \Diamond(B \wedge A)$ commutativity of $\wedge$ and monotonicity rule
6. $\Diamond A \wedge \Box B \to \Diamond(B \wedge A)$ the law of syllogism, 4., and 5.
7. $\Diamond\Box A \wedge \Box\Diamond B \to \Diamond(\Diamond B \wedge \Box A)$ a substitution into 6.: $A/\Box A$ and $B/\Diamond B$
8. $\Diamond B \wedge \Box A \to \Diamond(A \wedge B)$ a substitution into 6.:A/B, B/A
9. $\Diamond(\Diamond B \wedge \Box A) \to \Diamond\Diamond(A \wedge B)$ 8. and monotonicity rule
10. $\Box\Diamond A \wedge \Box\Diamond B \to \Diamond\Diamond(A \wedge B)$ the law of syllogism, 1., 7. and 9.
11. $\Box(\Box\Diamond A \wedge \Box\Diamond B) \to \Box\Diamond\Diamond(A \wedge B)$ 10. and monotonicity rule
12. $\Box\Diamond\Diamond(A \wedge B) \to \Diamond(A \wedge B)$ the axiom T^*: $A/(A \wedge B)$
13. $\Box(\Box\Diamond A \wedge \Box\Diamond B) \to \Diamond(A \wedge B)$ and the law of syllogism, 11. and 12.

Now we are ready to prove that in $\mathsf{S5_M}[\mathrm{K1}]$ the following formula is a theorem:

$$\Diamond\Big((A_1 \to \Box A_1) \wedge \cdots \wedge (A_n \to \Box A_n)\Big). \qquad (\star\star)$$

Proof by the induction on n. For $n = 1$ the required theorem *via* the law of distribution of „$\Diamond$” with respect to „$\to$” is equivalent to the axiom D^*.

We also consider the case $n = 2$, since the induction step will go through similarly. By substitution into the schema $(\star)$ $A/(A_1 \to \Box A_1)$ and $B/(A_2 \to \Box A_2)$ we have: $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Box\Big(\Box\Diamond(A_1 \to \Box A_1) \wedge \Box\Diamond(A_2 \to \Box A_2)\Big) \to \Diamond\Big((A_1 \to \Box A_1) \wedge (A_2 \to \Box A_2)\Big)$. Using the distributivity law, the axiom D^* is equivalent to $\Diamond(A_1 \to \Box A_1)$, by Gödel's rule we get $\Box\Diamond(A_1 \to \Box A_1)$, and by the law of adjunction and once more by RG we have: $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Box\Big(\Box\Diamond(A_1 \to \Box A_1) \wedge \Box\Diamond(A_2 \to \Box A_2)\Big)$, which is the antecedent of our substitution into the theorem $(\star)$, so by MP also the consequent $\Diamond\Big((A_1 \to \Box A_1) \wedge (A_2 \to \Box A_2)\Big)$ is a theorem.

INDUCTIVE STEP. By the induction hypothesis we have that for any formulae $A_1 \ldots A_{n-1}$: $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Box\Diamond\Big[\Big(A_1 \to \Box A_1\Big) \wedge \cdots \wedge \Big(A_{n-1} \to \Box A_{n-1}\Big)\Big]$. We will show that the required theorem holds also for n.

1. $\Diamond\Big[\Big(A_1 \to \Box A_1\Big) \wedge \cdots \wedge \Big(A_{n-1} \to \Box A_{n-1}\Big)\Big]$ the induction hypothesis
2. $\Box\Diamond\Big[\Big(A_1 \to \Box A_1\Big) \wedge \cdots \wedge \Big(A_{n-1} \to \Box A_{n-1}\Big)\Big]$ RG and 1.
3. $\Diamond(A_n \to \Box A_n)$ the distributivity of the functor „$\Diamond$”

with respect to „$\to$" and D^*

4. $\Box\Diamond(A_n \to \Box A_n)$ RG and 3.

5. $\Box\Diamond\Big[\Big(A_1 \to \Box A_1\Big) \wedge \cdots \wedge \Big(A_{n-1} \to \Box A_{n-1}\Big)\Big] \wedge \Box\Diamond\Big[A_n \to \Box A_n\Big]$
2., 4. and the law of adjunction

6. $\Box\Big\{\Box\Diamond\Big[\Big(A_1 \to \Box A_1\Big) \wedge \cdots \wedge \Big(A_{n-1} \to \Box A_{n-1}\Big)\Big] \wedge \Box\Diamond\Big[A_n \to \Box A_n\Big]\Big\}$ RG and 5.

7. $\Box\Big\{\Box\Diamond\Big[\Big(A_1 \to \Box A_1\Big) \wedge \cdots \wedge \Big(A_{n-1} \to \Box A_{n-1}\Big)\Big] \wedge\Box\Diamond\Big[A_n \to \Box A_n\Big]\Big\} \to$
$\to \Diamond\Big\{\Big[\Big(A_1 \to \Box A_1\Big) \wedge \cdots \wedge \Big(A_{n-1} \to \Box A_{n-1}\Big)\Big] \wedge \Big[A_n \to \Box A_n\Big]\Big\}$
a substitution into ($\star$): $A/\Big[\Big(A_1 \to \Box A_1\Big) \wedge \cdots \wedge \Big(A_{n-1} \to \Box A_{n-1}\Big)\Big]$
and $B/\Big[A_n \to \Box A_n\Big]$.

8. $\Diamond\Big\{\Big[\Big(A_1 \to \Box A_1\Big) \wedge \cdots \wedge \Big(A_{n-1} \to \Box A_{n-1}\Big)\Big] \wedge \Big[A_n \to \Box A_n\Big]\Big\}$
MP 7. and 6.

Now we consider the canonical model of the logic $\mathsf{S5_M}[\mathrm{K1}]$. We show the canonical frame satisfies the condition ($\ddagger$).

To this end we prove that for any possible world w the set of formulas $\mathcal{W}_w := \{A; \Box A \in w\} \cup \{A \to \Box A; A \in For\}$ is consistent with respect to the logic $\mathsf{S5_M}[\mathrm{K1}]$. Assume for contradiction that there are formulas $\Box A_1, \ldots, \Box A_n \in w$ and some other formulas $A'_1, \ldots, A'_m$ that $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \neg\Big(A_1 \wedge \cdots \wedge A_n \wedge (A'_1 \to \Box A'_1) \wedge \cdots \wedge (A'_m \to \Box A'_m)\Big)$. By the negation of implication we have: $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Big(A_1 \wedge \cdots \wedge A_n\Big) \to \neg\Big((A'_1 \to \Box A'_1) \wedge \cdots \wedge (A'_m \to \Box A'_m)\Big)\Big)$, and by the law of contraposition: $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Big((A'_1 \to \Box A'_1) \wedge \cdots \wedge (A'_m \to \Box A'_m)\Big) \to \neg\Big(A_1 \wedge \cdots \wedge A_n\Big)$. Later by RG and the law of distribution of „$\Box$" with respect to „$\to$" on „$\Diamond$"-s, and the inter-definability of „$\Diamond$" and „$\Box$" we get $\vdash_{\mathsf{S5_M}[\mathrm{K1}]} \Diamond\Big((A'_1 \to \Box A'_1) \wedge \cdots \wedge (A'_m \to \Box A'_m)\Big) \to \neg\Box\Big(A_1 \wedge \cdots \wedge A_n\Big)$. However in the presence of ($\star\star$) using MP we would have $\neg\Box\Big(A_1 \wedge \cdots \wedge A_n\Big) \in w$; next for any $1 \le i \le n$: $\Box A_i \in w$, so by the laws of adjunction and regularity, using MP we have $\Box\Big(A_1 \wedge \cdots \wedge A_n\Big) \in w$. So our assumption results in a contradiction, since the set w is maximally consistent.

Thus by Lindenbaum's lemma, for each world w there is a world $\overline{w}$ that $\mathcal{W}_w \subseteq \overline{w}$. We show that $wR\overline{w}$. Let A be any formula that $\Box A \in w$. By the definition of the set $\mathcal{W}_w$, we have $A \in \mathcal{W}_w$, i.e. $A \in \overline{w}$. So by the definition of the accessibility relation in the canonical frame we get: $wR\overline{w}$.

Now we prove that for any possible world w and for indicated above $\overline{w}$ the following holds: $\forall_{w'}(\overline{w}Rw' \rightarrow \overline{w} = w')$. Let us assume that there is w', such that $\overline{w}Rw'$ and $\overline{w} \neq w'$, i.e. that there is a formula A that either ($A \in w'$ and $A \notin \overline{w}$) or ($A \in \overline{w}$ and $A \notin w'$). In the first case by the maximality of the set $\overline{w}$ we have $\neg A \in \overline{w}$, via the definition of $\mathcal{W}_w$ we observe that $\neg A \rightarrow \Box\neg A \in \mathcal{W}_w$, thus $\neg A \rightarrow \Box\neg A \in \overline{w}$, and so by MP, since every maximally consistent set is closed under MP, also $\Box\neg A \in \overline{w}$. However, since $\overline{w}Rw'$ we have: $\neg A \in w'$, which gives us a contradiction, because w' is also consistent. In the second case by the definition of $\mathcal{W}_w$ we have $A \rightarrow \Box A \in \overline{w}$, from where via MP we obtain $\Box A \in \overline{w}$, but again, by the definition of the accessibility relation, we get $A \in w'$ which is a contradiction. We have shown, that the canonical frame of the logic $\mathsf{S5}_\mathsf{M}$[K1] fulfills the condition (‡).

Assume that some formula A is valid in all frames satisfying the condition (‡). In the presence of the above observation it is also true in the canonical model of the logic $\mathsf{S5}_\mathsf{M}$[K1]. But any formula true in the canonical model of a given logic is a theorem. Thus a given formula A is a theorem of $\mathsf{S5}_\mathsf{M}$[K1].

A fortiori:

Corollary 7: *The logic* $\mathsf{S5}_\mathsf{M}$[K1] *is canonical.*

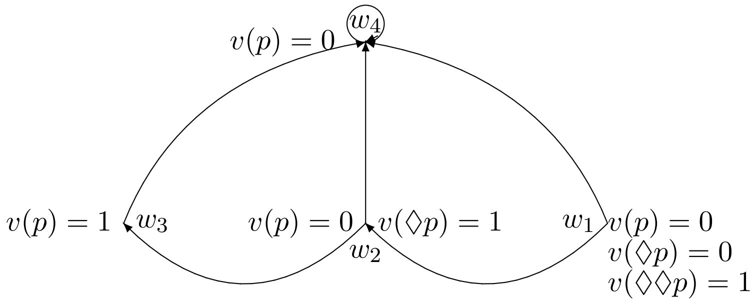
Proof. Follows directly from the previous theorem.

We now state:

Theorem 8: *The logic* $\mathsf{S5}_\mathsf{M}$[K1] *is weaker than* K4M.

Proof. Firstly we show that in K4M D^* and T^* are provable. By substitution into the axiom D (which clearly belongs to both logics) we have $\Box\Diamond A \rightarrow \Diamond\Diamond A$, and by the axiom 4 and transitivity of „→" we get $\Box\Diamond A \rightarrow \Diamond A$ i.e. the axiom D*. To prove T^* it is enough to see that via 4 and the monotonicity rule we obtain $\Box\Diamond\Diamond A \rightarrow \Box\Diamond A$. But in the presence of already proved D^* by the transitivity of implication we have $\Box\Diamond\Diamond A \rightarrow \Diamond A$.

Using the completeness results for both logics it is enough to indicate a frame satisfying the condition (‡) in which the axiom 4 is not valid.



For any world w_i, w_4 is a world the existence of which is postulated in the condition ‡. Indeed we have $\forall_{w'}(w_4 R w' \rightarrow w_4 = w')$. One can see that in w_1 the axiom $\Diamond\Diamond p \rightarrow \Diamond p$ is false.

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