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FOUR KINDS OF SUBMINIMAL NEGATION WITHIN THE CONTEXT OF THE BASIC POSITIVE LOGIC B+

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Abstract
Four subminimal negation completions of the basic positive relevance logic are defined, isolating weak negative principles of contraposition, double negation and reductio by means of weak constructive falsity constants.

1. *Introduction*

The divergence between semantical and proof-theoretical definitions of (sentential) negation can be exploited to obtain positive logics for various kinds of negations, that is, formal systems lacking explicit negation but able to codify patterns of deductive inference involving negation. Results concerning strong negation in positive logics with a constructible falsity constant abound in the literature (see [6] for a general review).

However, a natural point of departure for a *general* study of positive logics for varieties of negation is in fact minimal negation as historically defined by Johansson ([2]) by means of a weak constructive falsity constant. Minimal negation is a "positive" negation in the sense that all its theorems are instances of positive intuitionistic logic (I+). Generally speaking, a logic with minimal negation is a logic including weak double negation, weak contraposition and weak reductio [i.e., $A \to \neg\neg A$, $(A \to B) \to (\neg B \to \neg A)$ and $(A \to \neg A) \to \neg A$]. We call "subminimal" any negation strictly included in minimal negation (see [1]).

Now, positive propositional logics significantly weaker than I+ can still define minimal negation. In [5], we have shown how to introduce minimal negation in such a weak system as the basic positive logic B+, which is the basic relevance logic and only includes as theorems those required for a logic to be complete with respect to the ternary relational semantics (see, for example, [4]). This paper improves that previous result in [5] inasmuch as four fine-grained varieties of negation are defined in the context of B+ at a subminimal level.

Notice that an additional source of interest in defining negations in weak positive logics is the axiomatical and semantical isolation of negative principles within positive logics. In this paper, given B+, we define the logic B+,F (B+ with a falsity constant F added to the sentential language), offering weak contraposition as a rule, to wit:

(i) If A → B, then ¬B → ¬A

Next, we define the logic Bdn (B+,F with double negation) by means of the rule

(ii) If A → ¬B, then B → ¬A

or, equivalently, the axiom

(iii) A → ¬¬A

The logics Bdnr1 and Bdnr2 are two extensions of Bdn with two versions of the reductio rule, to wit:

(iv) If A → B and A → ¬B, then ¬A

for Bdnr1, and

(v) If A → B, then (A → ¬B) → ¬A

for Bdnr2.

It will be shown that (i), (ii) [or (iii)], (iv) and (v) are isolable given B+ with their corresponding different logics B+F, Bdn, Bdnr1, Bdnr2 containing different kinds of subminimal negation.

2. *The logic* B^+

B^+ is axiomatized with

A1. $A \to A$
A2. $(A \wedge B) \to A \qquad (A \wedge B) \to B$
A3. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
A4. $A \to (A \vee B) \qquad B \to (A \vee B)$
A5. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
A6. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$

The rules of derivation being

Modus ponens: $\vdash A, \vdash A \to B \Rightarrow \vdash B$
Adjunction: $\vdash A, \vdash B \Rightarrow \vdash A \wedge B$
Suffixing: $\vdash A \to B \Rightarrow \vdash (B \to C) \to (A \to C)$

Prefixing: $\vdash B \to C \Rightarrow \vdash (A \to B) \to (A \to C)$

The following formulas are B+ theorems useful in the proof of the completeness theorem:

T1. $(A \wedge B) \to (B \wedge A)$
T2. $((A \vee B) \wedge (C \wedge D)) \to ((A \wedge C) \vee (B \wedge D))$
T3. $((A \to C) \vee (B \to D)) \to ((A \wedge B) \to (C \vee D))$
T4. $((A \to C) \wedge (B \to D)) \to ((A \wedge B) \to (C \wedge D))$
T5. $((A \to C) \wedge (B \to D)) \to ((A \vee B) \to (C \vee D))$

3. *Semantics for* B^+

A B^+ model is a quadruple <O,K,R,$\models$> where K is a set, O a non-empty subset of K and R a ternary relation on K subject to the following definitions and postulates for all a,b,c,d $\in$ K with quantifiers ranging over K:

d1 a $\leq$ b $=_{def}$ $\exists$x(x $\in$ O and Rxab)
d2 R^2abcd $=_{def}$ $\exists$x(Rabx and Rxcd)
P1 a $\leq$ a
P2 a $\leq$ b and Rbcd $\Rightarrow$ Racd

Finally, $\models$ is a valuation relation from K to the sentences of B^+ satisfying the following conditions for all wffs p,A,B and a,b,c $\in$ K:

(i) a $\models$ p and a $\leq$ b $\Rightarrow$ b $\models$ p
(ii) a $\models$ A $\vee$ B iff a $\models$ A or a $\models$ B
(iii) a $\models$ A $\wedge$ B iff a $\models$ A and a $\models$ B
(iv) a $\models$ A $\to$ B iff for all b, c $\in$ K, Rabc and b $\models$ A $\Rightarrow$ c $\models$ B

A is valid in B^+ [$\models^+_B$A] iff a $\models$ A for all a $\in$ O in all models. The general strategy shown in [3] or [5] proves:

Theorem 3.1: (Semantic consistency of B+) *If* $\vdash_{B+}$A, *then* $\models_{B+}$A.

4. *Completeness for* B^+

Let us first record some basic definitions. A theory is a set of formulas of B^+ closed under adjunction and provable entailment. A theory $\underline{a}$ is prime if whenever A $\vee$ B $\in$ a, then A $\in$ a or B $\in$ a. We shall call any theory $\underline{a}$ regular if all theorems of B^+ belong to $\underline{a}$.

Now we define the B^+ canonical model. Let K^T be the set of all theories and R^T be defined on K^T as follows: for all formulas A,B and a,b,c $\in K^T$, R^Tabc just in case if $A \to B \in a$ and $A \in b$, then $B \in c$. Further, let K^C be the set of all prime theories, O^C the set of all regular prime theories and R^C the restriction of R^T to K^C. Finally, let $\vDash^C$ be defined as follows: for any wff A and $a \in K^C$, $a \vDash^C A$ iff $A \in a$. Then, the B^+ canonical model is the quadruple $<O^C,K^C;R^C,\vDash^C>$. In what follows in this section we sketch a proof of the completeness theorem. We list a series of lemmas proof of which can be found either in [5] or (though not necessarily stated in the way we have formulated them) in [3].

Lemma 4.1: *Let* A *be a wff,* $a \in K^T$ *and* $A \notin a$. *Then,* $A \notin x$ *for some* $x \in K^C$ *such that* $a \subseteq x$.

Lemma 4.2: *Let* R^Tabc, a,b $\in K^T$, $c \in K^C$. *Then,* R^Txbc *for some* $x \in K^C$ *such that* $a \subseteq x$.

Lemma 4.3: *Let* R^Tabc, a,b $\in K^T$, $c \in K^C$. *Then,* R^Taxc *for some* $x \in K^C$ *such that* $b \subseteq x$.

Lemma 4.4: *If* $\nvdash_{B+}$A, *there is some* $x \in O^C$ *such that* $A \notin x$.

Lemma 4.5: *Let* a,b $\in K^T$. *The set* $x = \{B : \exists A(A \to B \in a \text{ and } A \in b)\}$ *is a theory such that* R^Tabx.

Lemma 4.6: $a \leq^C b$ *iff* $a \subseteq b$.

Lemma 4.7: *Postulates* P1, P2 *hold in the* B^+ *canonical model.*

Lemma 4.8: *The canonical* $\vDash^C$ *is a valuation relation satisfying conditions (i)–(iv) of §3.*

Lemma 4.9: *The* B^+ *canonical model is in fact a* B^+ *model.*

Now, from Lemmas 4.4 and 4.9 we have:

Theorem 4.1: (Completeness of B^+) *If* $\vDash_{B+}$A, *then* $\vdash_{B+}$A.

5. *The logic* B^+,F

The logic B^+,F is defined by adding to the sentential language of B^+ the propositional falsity constant F together with the definition: $\neg A =_{def} A \to F$. Note that, for example, the following are provable in B^+,F:

T6. $\vdash (A \to B) \Rightarrow \vdash (\neg B \to \neg A)$
T7. $\vdash \neg B \Rightarrow \vdash (A \to B) \to \neg A$
T8. $\neg(A \vee B) \to (\neg A \wedge \neg B)$
T9. $(\neg A \vee \neg B) \to \neg(A \wedge B)$

6. *Semantics for* B^+,F

A B^+,F model is a quintuple $<O,K,S,R,\vDash>$ where $<O,K,R,\vDash>$ is a B^+ model and S is a subset of K such that $S \cap O \neq \emptyset$. The clause:

(v) $a \vDash F$ iff $a \notin S$

is satisfied in all models.

A is valid in B^+,F [$\vDash_{B,F} A$] iff $a \vDash A$ for all $a \in O$ in all models. We note that F is not valid: let $a \in S \cap O$. Then, $a \nvDash F$. But $a \in O$. So, $\nvDash_{B+,F} F$.

Theorem 6.1: (Semantic consistency of B^+,F) *If* $\vdash_{B+,F} A$, *then* $\vDash_{B+,F} A$.
Proof. Theorem 4.1.

7. *Completeness for* B^+,F

We define the B^+,F canonical model as the quintuple $<O^C,K^C,S^C,R^C,\vDash^C>$ where $<O^C,K^C,R^C,\vDash^C>$ is the B^+ canonical model and S^C is interpreted as the set of all consistent theories. A theory a is consistent iff $F \notin a$. Now we need to prove:

Lemma 7.1: $S^C \cap O^C$ *is not empty.*
Proof. Given that $\nvDash_{B+,F} F$ [see §6], we have, by theorem 6.1, $\nvdash_{B+,F} F$, i.e., $F \notin B^+,F$. As B^+,F is a theory, Lemma 4.1 applies and there is some $x \in K^C$ such that $B^+,F \subseteq x$ and $F \notin x$. Obviously, $x \in O^C$ [since $B^+,F \subseteq x$]. As $F \notin x$, $x \in S^C$.

Lemma 7.2: *Clause (v) holds in the canonical model.*
Proof. Lemma 7.1 and definition of S^C.

Lemma 7.3: *The* B^+,F *canonical model is in fact a* B^+,F *model.*
Proof. Lemmas 4.7, 4.8, 4.9 and 7.2.

Finally, we prove

Theorem 7.1: (Completeness of B^+,F) *If* $\vDash_{B+,F}$A, *then* $\vdash_{B+,F}$A.
Proof. Analogues of Lemmas 4.4 and 7.3.

8. *Adding weak double negation to* B^+: *the logic* Bdn

Bdn is the result of adding the double negation axiom:

A7. $A \to ((A \to F) \to F)$

To B^+,F. We remark that, together with T6–T9, the following are exemplar Bdn theorems:

T10. $A \to \neg\neg A$
T11. $\neg\neg\neg A \to \neg A$
T12. $\vdash A \to \neg B \Rightarrow \vdash B \to \neg A$
T13. $(\neg A \wedge \neg B) \to \neg(A \vee B)$

Note: Bdn can be axiomatized with

A7′. $\vdash A \to (B \to F) \Rightarrow \vdash B \to (A \to F)$

instead of A7, as the reader can verify.

9. *Semantics for* Bdn

A Bdn model is a B^+,F model but with the addition of the postulate:

P3. Rabc and $c \in S \Rightarrow \exists x[x \in S$ and Rbax]

$\vDash_{Bdn}$A iff a $\vDash$ A for all a $\in$ O in all models. Hence,

Theorem 9.1: (Semantic consistency of Bdn) *If* $\vdash_{Bdn}$A, *then* $\vDash_{Bdn}$A.
Proof. It must only be proved that A7 is valid. Use P3.

10. *Adding reductio to* Bdn: *the logic* Bdnr1

We add to Bdn the rule:

A8. $\vdash A \to B$ and $\vdash A \to \neg B \Rightarrow \vdash \neg A$

Noting that, in addition to T6–T13, the following, for example, are provable in Bdnr1

T14. $\neg(A \wedge \neg A)$

T15. $\neg\neg(A \vee \neg A)$
T16. $\vdash A \Rightarrow\ \vdash (B \rightarrow \neg A) \rightarrow \neg B$

Note: Bdnr1 can be alternatively axiomatized with T14 instead of A8. We leave to the reader the proof of this fact.

11. *Semantics for* Bdnr1

Models for Bdnr1 are defined similarly as those for Bdn but with the addition of the postulate:

P4. $a \in S \Rightarrow \exists x(x \in S$ and Raax)

Note: In [1] it is shown that P4 is equivalent to

P4′. Rabc and $c \in S \Rightarrow \exists x$(Rcbx and $x \in S$)

Given intuitionistic propositional logic without contraction but with weak double negation, weak contraposition and reductio. Here this equivalence does not hold: P4 is weaker (see [3]).

Theorem 11.1: (Semantic consistency of Bdnr1) *If* $\vdash_{\text{Bdnr1}} A$*, then* $\vDash_{\text{Bdnr1}} A$.

Proof. Use P4 to show the validity of A8.

12. *Adding a stronger version of reductio to* Bdn: *the logic* Bdnr2

We add to Bdn the rule

A9. $\vdash A \rightarrow B \Rightarrow\ \vdash (A \rightarrow \neg B) \rightarrow \neg A$

Note that, in addition to T6–T16, the following, for example, are provable in Bdnr2:

T17. $\vdash A \rightarrow \neg B \Rightarrow\ \vdash (A \rightarrow B) \rightarrow \neg A$
T18. $(A \rightarrow \neg A) \rightarrow \neg A$
T19. $(A \rightarrow \neg B) \rightarrow \neg(A \wedge B)$
T20. $(A \rightarrow B) \rightarrow \neg(A \wedge \neg B)$

Note: Bdnr2 can be axiomatized with T17, T18, T19 or T20 instead of A8, among other possibilities.

13. *Semantics for* Bdnr2

Models for Bdnr2 are defined similarly as those for Bdn but with the addition of the postulate

P5. Rabc and $c \in S \Rightarrow \exists x \exists y$(Rabx and Rxby and $y \in S$)

which can be used to prove:

Theorem 13.1: (Semantic consistency of Bdnr2) *If* $\vdash_{Bdnr2}$A, $\vDash_{Bdnr2}$A.

14. *Completeness for* Bdn, Bdnr1 *and* Bdnr2

Given that Bdn, Bdnr1 and Bdnr2 share the same definition of consistency (and a different one to that of B+,F) we do a completeness proof for these three logics. We define the Bdn (Bdnr1, Bdnr2) canonical model exactly as the B+,F canonical model but with this difference: now any theory a is consistent iff the negation of a theorem does not belong to $\underline{a}$.
To begin with, clearly an analogue of Lemma 7.1 for Bdn (Bdnr1, Bdnr2) is immediate. Next, we prove:

Lemma 14.1: F ∈ a *iff* a *is inconsistent.*
Proof. Suppose F ∈ a. By A7, (F → F) → F ∈ a. Thus, a is inconsistent. Assume now a is inconsistent. Then, A → F ∈ a for Bdn theorem A. By A7 and Modus Ponens, (A → F) → F is a theorem. So, F ∈ a.

Lemma 14.2: *Let* a,b,c ∈ K^T *with consistent* c *and* R^Tabc. *Then, there is some* x ∈ S^C *such that* c ⊆ x *and* R^Tbax.
Proof. Suppose R^Tabc; a,b,c ∈ K^T and c consistent. Define (Cfr. Lemma 4.5) the theory y = {B : ∃A(A → B ∈ b and A ∈ a)} such that R^Tbay. We first prove that y is consistent. Suppose it is not. Then, (cfr. Lemma 10.1) F ∈ y. By definition of y, A → F ∈ b, A ∈ a. By A7, (A → F) → F ∈ a. Given R^Tabc, F ∈ c, which is impossible c being consistent. So, we have a consistent theory y such that R^Tbay, F ∉ y, whence Lemma 4.1 applies and there is some x ∈ K^C such that y ⊆ x and F ∉ x. So, x is consistent, i.e., x ∈ S^C. Given that y ⊆ x and R^Tbay, we conclude R^Tbax, which was to be proved.

Lemma 14.3: *The canonical* P3*, i.e.,* R^Cabc *and* c ∈ S^C ⇒ ∃x(x ∈ S^C and R^Cbax) *holds in the* Bdn *canonical model.*
Proof. Lemma 10.2.

Lemma 14.4: *Let* a ∈ S^C. *Then, there is some* x ∈ S^C *such that* R^Caax.
Proof. Suppose a ∈ S^C. Define (cfr. Lemma 4.5) the theory y = {B : ∃A(A → B ∈ a and A ∈ a)} such that R^Taay. We first prove that y is consistent. If by reductio hypothesis it were not, then ¬A ∈ y (A a theorem). By definition of y, B → ¬A ∈ a and B ∈ a. Now, by T16, ¬B ∈ a. Thus, B ∧ ¬B

$\in$ a and, by double negation, $\neg\neg(B \wedge \neg B) \in a$. But $\neg\neg(B \wedge \neg B)$ is the negation of a theorem (T14) and in consequence a is inconsistent contradicting the reductio hypothesis.

Therefore, we have a consistent y in K^T such that $R^T aay$. Since Lemma 4.1 applies, we have some $x \in K^C$ such that $y \subseteq x$ and $F \notin x$. Given $R^T aay$, $y \subseteq x$ and definitions, we deduce $R^T aax$ with $x \in S^C$, as required.

So, the canonical P4 holds in the canonical model.

Lemma 14.5: *Let* a,b,c $\in K^T$ *with* c *consistent and* $R^T abc$. *Then, there is some* $x \in K^T$ *and some* $y \in S^C$ *such that* $R^T abx$ *and* $R^T xby$.

Proof. Grant the premisses of the theorem and define (cfr. Lemma 4.5) the theory $u = \{B : \exists A(A \rightarrow B \in a \text{ and } A \in b)\}$ satisfying $R^T abu$. Define also the theory $w = \{B : \exists A(A \rightarrow B) \in u \text{ and } A \in b)\}$ such that $R^T ubw$. Next, we prove w consistent. Suppose it is not, then $F \notin w$ (Lemma 10.1). By definition of w, $B \rightarrow F \in u$, $B \in b$. By definition of u, $A \wedge (B \rightarrow F) \in a$, $A \in b$. By T19, $(A \wedge B) \rightarrow F \in a$. Given that $R^T abc$ and $A \wedge B \in b$ (since A, $B \in b$), $F \in c$. This contradicts the hypothesis. Therefore, u, $w \in K^T$, w is consistent, $R^T abu$ and $R^T ubw$. As $F \notin w$, Lemma 4.1 is applicable and there is some y in S^C such that $w \subseteq y$ and $R^T uby$. Now, by Lemma 4.2, there is some x in K^C such that $u \subseteq x$ and $R^T xby$. As $R^T abu$, $R^T abx$ as required.

Lemma 14.6: *The canonical* P5 *holds in the* Bdnr2 *canonical model.*

Proof. Lemma 14.5.

Lemma 14.7: *Clause (v) holds in the* Bdn *canonical model.*

Proof. Lemma 10.1.

Lemma 14.8: *The* Bdn (Bdnr1, Bdnr2) *canonical model is indeed a* Bdn (Bdnr1, Bdnr2) *model.*

Proof. Lemmas 4.9, 14.3 and 14.7 (14.3, 14.4 and 14.7; and 14.3, 14.6 and 14.7).

Finally,

Theorem 14.1: (Completeness of Bdn, Bdnr1, Bdnr2) *If* $\vDash_{Bdn} A$, *then* $\vdash_{Bdn} A$ *(If* $\vDash_{Bdnr1} A$, *then* $\vdash_{Bdnr1} A$*; if* $\vDash_{Bdnr2} A$, *then* $\vdash_{Bdnr2} A$*).*

Proof. An analogue of lemma 4.4 and 14.8.

15. *Final remark*

Bdn, Bdnr1 and Bdnr2 can be defined with a negation connective instead of the falsity constant F. See [3] for a general strategy.

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