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SEQUENTS AND BIVALUATIONS*

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Introduction

At the time when Hilbert's *Beweistheorie* was considered as a dead end after Gödel's results, Gentzen developed a very powerful tool (inspired by some ideas of Paul Hertz) which gave a new birth to proof theory: sequent calculus. This tool is so rich that new features of it are still being discovered, more than sixty years after its creation.

Sequent calculus has shown that proof-theory is not a mere manipulation of meaningless symbols; it allows to carry out some fundamental meta-theorems for many logics such as decidability, consistency, interpolation, etc., in a way which is as intuitive and elegant as model-theoretical methods. In fact sequent calculus breaks the distinction between blind syntax and meaningful semantics and not only at the meta-theoretical level.

In this paper we present some general results which tightly connect sequent rules and bivaluations in such a way that a sequent rule can be seen as a bivalent semantic condition and vice-versa. From these abstract results, not depending on any particular logical language, we can get immediately the completeness theorem for a wide class of logics.

Our main theorem shows that relatively maximal theories respect sequent rules of a certain class of sequent calculus. The concept of relatively maximal theory is a refinement of the concept of maximal theory which has been studied in Poland and in Brazil. It is by combining our theorem with results about this concept, in particular Lindenbaum-Asser theorem, that we get a general form of completeness theorem.

In fact the work presented here is the combination of three lines of research:

(1) The theory of consequence operator as developed by the Polish school of logic from some Tarski's early works (Tarski 1928).
(2) Gentzen's first paper about Hertz's *Satzsysteme* (Genzten 1932).
(3) The idea of generalized bivalent semantics, especially developed by N.C.A. da Costa (da Costa/Béziau 1994).

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The results of this paper were presented in our PhD (Béziau 1995) as part of a general theory of logics that we have called *Universal Logic*. This terminology is used by analogy with the expression "universal algebra": the aim is to develop a conceptual framework and to get some abstract results which turn easier the study of specific logics.

What is involved in this paper is mainly general abstract nonsense. The main difficulty of our results is rather conceptual. But what we show is that when this general abstract nonsense is rightly organized we can get meaningful results with a lot of powerful applications.

1. *The intuitive semantic interpretation of sequent rules*

Consider the two sequent rules for classical implication:

$$\frac{\Sigma, a \to b, \theta}{\Sigma \to a \supset b, \theta} \supset r \qquad \frac{\Sigma \to a, \theta \quad \Sigma', b \to \theta'}{\Sigma, \Sigma', a \supset b \to \theta, \theta'} \supset l$$

These rules, at first sight, may look strange, less "natural" than the so-called natural deduction rules. However they can be easily interpreted in terms of true-false semantics: just forget the contexts Σ, θ, etc., consider that the left side of a sequent is false, the right side is true, $\to$ is or, the empty space between two premises is and, ———— is implication; then you can read the two above rules respectively as follows:

$(B \supset r)$ If a is false or b is true then $a \supset b$ is true

$(B \supset l)$ If a is true and b is false then $a \supset b$ is false

Putting together, these two semantic conditions correspond to the following standard semantic definition of classical implication:

$(B \supset)$ $\quad a \supset b$ is false iff a is true and b is false.

For rules, like rules for conjunction, the comma on the left will be interpreted as an and, and the comma on the right as an or (following Gentzen's original suggestion).

In this paper we will prove a result which justifies this intuitive semantic interpretation in such a way that from this result we can get instantaneously various completeness theorems.

As it is known, such an interpretation is not valid for intuitionistic logic. Our result sets some conditions on systems of sequents for this interpretation to hold; this leads to the notion of SSSS: *structurally standard systems of sequents*. This excludes ill-constructed sequents, such as asymmetric intuitionistic sequents, as well as substructural systems of sequents, such as linear logic.

These logics are quite popular, so one may ask: what is the use of such an out-of-fashion result? Does it have any other applications outside of classical logic?

The answer is yes. For example SSSS include systems for paraconsistent logics, many-valued logics, paracomplete logics, etc. (Examples are given in Section 5).

Let us note that this result does not depend on:

- truth-functionality
- self-extensionality (i.e. replacement theorem)
- the structure of the set of formulas.

For example, with this result we can prove that the above left sequent rule for implication corresponds to the intuitive semantic interpretation of it ($B \supset l$). The connective so-defined is neither truth-functional, nor self-extensional.

That means that our result connects systems of sequents not only with truth-functional bivalent semantics but with any kind of functions from the set of formulas to truth and falsity.

Our result permits to give a proof of the completeness theorem for classical propositional logic which does not depend on algebraic features of this logic. So it is a kind of opposite of Łos's completeness proof (Łos 1951).

As it is known classical bivaluations are at the same time, homomorphisms from the absolute free algebra of formulas to the algebra of truth-functions on $\{0, 1\}$ and characteristic functions of maximal consistent sets.

It happens that one can consider bivalent semantics in which bivaluations are not homomorphisms and our result applies as well to such semantics.

Here what is important therefore is the concept of maximal consistent sets which is generalized in an abstract mood, in such a way that it does not depend on negation or any particular connective.

2. *Bivaluations and Lindenbaum-Asser theorem*

In this section we will recall some definitions and results (without proofs). Most of this material is known for years in Poland and can be found in (Wójcicki 1988). The presentation we follow here is close to one of our recent papers (Béziau 1999b).

2.1. *Logic, bivalent semantics and closed theories*

Definition 1: *We call a* logic *any structure of type* $\langle \mathbb{L}; \vdash \rangle$ *where* $\mathbb{L}$ *is any set and* $\vdash$ *a binary relation (*deducibility relation*) between sets of objects* *(*theories*, denoted by T, U, ...) of* $\mathbb{L}$ *and objects (*formulas*, denoted by* a*,* b*, ...) of* $\mathbb{L}$*.*

A logic is said normal *when it obeys the following conditions:*

(Reflexivity) *If* $a \in T$*, then* $T \vdash a$

(Monotony) *If* $T \vdash a$ *and* $T \subseteq U$ *then* $U \vdash a$

(Transitivity) *If* $T \vdash a$ *for all* $a \in U$ *and* $U \vdash b$*, then* $T \vdash b$*.*

We say that a logic is a finite normal logic *when it obeys moreover the condition:*

(Finiteness) *If* $T \vdash a$ *then there exists a finite subtheory* To *of* T *such that* $To \vdash a$*.*

Definition 2: *An* adequate bivalent semantics *for a logic* $\mathfrak{L} = \langle \mathbb{L}; \vdash \rangle$ *is a set of functions* BIV *from* $\mathbb{L}$ *to* $\{0, 1\}$ *such that the semantic deducibility relation* $\vDash$ *defined in the usual manner (*$T \vDash a$ *iff for every* $\beta \in \mathsf{BIV}$*, if* $\beta(b) = 1$ *for every* $b \in T$*, then* $\beta(a) = 1$*) by this set is the same as* $\vdash$*. If* $\vdash$ *is included in* $\vDash$*, we say that the semantic is* sound *(for* $\mathfrak{L}$*), and if* $\vDash$ *is included in* $\vdash$ *we say that the semantic is* complete *(for* $\mathfrak{L}$*).*

Definition 3: *A theory* T *such that, if* $T \vdash a$ *then* $a \in T$*, is said to be* closed*.*

A theory can be considered as a bivaluation by taking its characteristic function and a bivaluation can be considered as a theory by taking the set of true formulas under this function. Hereafter we will therefore confuse these two notions, as shows for example the next definition.

Definition 4: *We call* semantics of closed theories *of a logic* $\mathfrak{L}$ *the set of characteristic functions of closed theories of this logic.*

More generally, given a class of theories of a logic $\mathfrak{L}$, say *chic theories*, we call *semantics of chic theories*, the set of characteristic functions of chic theories.

Theorem 5: *The semantics of closed theories of a normal logic is an adequate semantics for it.*

Therefore, for every normal logic, there exists an adequate bivalent semantics, in other words: every normal logic is bivalent (for a discussion on this topic see (Béziau 1997)).

About closed theories we have moreover the following result.

Theorem 6: *A bivalent semantics is sound for a normal logic iff it is included in the semantics of closed theories.*

2.2. *Relatively maximal theories*

Definition 7: *Given a theory T and a formula r such that $T \nvdash r$ and such that for any strict extension U of T, $U \vdash r$, we say that T is* relatively maximal in r.

A theory is said to be relatively maximal *iff there is a formula a such that it is relatively maximal in a.*

Theorem 8: (LINDENBAUM-ASSER) *In a finite monotonic logic, given a formula r and a theory T such that $T \nvdash r$, it is possible to extend T in a theory relatively maximal in r.*

Corollary 9: *The semantics of relatively maximal theories of a finite normal logic is a complete semantics for it.*

It is easy to show that a relatively maximal theory is a closed theory. Joining the above results with the results about closed theories, we have the following.

Theorem 10: *The semantics of relatively maximal theories of a finite normal logic is an adequate semantics for it.*

Corollary 11: *Any class of closed theories of a finite normal logic containing the class of relatively maximal theories is an adequate semantics for this logic.*

To prove Lindenbaum-Asser, it is necessary to use the axiom of choice, in fact (Dzik 1981) has shown that (the statement of) this theorem is equivalent to the axiom of choice.

A *maximal theory* is a theory which is not trivial (i.e. there exists a formula which is not deducible from it), but has no non trivial strict extensions. It is easy to see that a theory is maximal iff it is relatively maximal in every formula not in it.

In classical logic all relatively maximal theories are maximal and therefore these two concepts coincide (such a logic, following Makinson, is said *absolute*), but this not the case of some other logics, like intuitionistic logic.

In (Béziau 1999b), it has been shown that the semantics of relatively maximal theories of a finite normal logic is a minimal adequate semantics for it and that therefore, for a normal logic which is not absolute, the semantics of maximal theories is not complete. This explains why Lindenbaum-Asser theorem is more important than Lindenbaum theorem saying that any non trivial theory of a finite normal logic can be extended in a maximal one, from which it is not possible, in some cases, to get completeness.

3. *Structurally standard systems of sequents (SSSS)*

3.1. *The architecture of sequent systems*

Our result is about a special class of sequent systems[1]: structurally standard systems of sequents (SSSS for short). In order to explain what they exactly are, we will begin by presenting a detailed conceptual analysis of sequent systems which may be interesting to situate our work, at a time when sequent systems has turn into a central methodological tool for the constructions of logics.

One can say that a sequent system is determined by three kinds of concepts: external determinations, structural rules and logical rules, as described in Table 1.

Let us make a few comments to explain Table 1.

(1) Generally associativity is not taken as a structural rule but is externally determined by the definition of the notion of sequent.

(2) In linear logic, structural rules of weakening and contraction are controlled by logical rules for ! (bien sûr) and ? (pourquoi pas). Gentzen's definition of a sequent as a finite sequence together with the structural rules of contraction and permutation gives rise to systems of sequents which can be defined, as it is well-known, simply by considering sequents made of finite sets of formulas and discarding the rules of contraction and permutation, these two rules becoming external determination.

(3) If the logic induced by the system is defined with the first definition of the picture, then this logic is finite and monotonic whatever the rules of the system are, even for example when weakening rules are not rules of the system. These two conditions as we have seen are enough to get Lindenbaum-Asser theorem.

(4) The expression "Substructural logic" is nowadays very fashionable, but what does it mean? One can say that it is a logic in which some of the standard structural rules are not valid, e.g. contraction or weakening (in view of (3) we must be careful with the case of non-monotonic logics). One can also

[1] We use this expression rather than "sequent calculus", due to the ambiguity of the word "calculus". In particular if someone understands "calculus" as meaning "computing", this terminology is misleading in this context, because though sequent systems can provide decision procedures, most of the sequent systems are not decidable. The word "system" seems more appropriate and remembers Hertz's expression *Satzsysteme*, from which Gentzen's work arises. Moreover the result presented here is close in spirit to Hertz's work and Gentzen's first paper about it, being situated at an abstract level, independent of the specificity of the language.

say that intuitionistic logic is substructural because the "structure" of its sequents is not standard. It seems that generally the expression "substructural logic" is used to denote a system of sequents in which some of the standard "structural principles" (cf. the pictures) are violated. Standard meaning here similar to Gentzen's system LK.

	INTERNAL DETERMINATIONS	
	STRUCTURAL RULES	LOGICAL RULES
EXTERNAL DETERMINATIONS		
	e.g.	
SEQUENT	Identity	(Dealing
-monosequent/multisequent		with the
-finite sets	Cut	morphology
-sequences		of the logic)
-structures, e.g.		
idempotent abelian group		
	Weakening	
		e.g.
		Connectives
RULE		
-order on premises		Quantifiers
-cardinality of premises	Permutation	
		Modalities
	Contraction	
PROOF		
-sequence/tree		
-length/depth	Associativity	
(finite or not)		
-order type		
INDUCED LOGIC		
e.g.		
$-T \vdash a \Leftrightarrow$ $\exists\ To$ finite $To \rightarrow a$ is derivable		
$-T \vdash a \Leftrightarrow$ $T \rightarrow a$ is derivable		
STRUCTURAL PRINCIPLES		

Table 1: THE ARCHITECTURE OF SEQUENT SYSTEMS

3.2. *Proto-SSSS and SSSS*

Definition 12: *A* proto-SSSS *is a system of sequents which has the same structural principles of Gentzen's system LK:*

- *structural rules of weakening, permutation, contraction, cut, identity axiom,*
- *sequents are finite sequences of arbitrary length on both sides,*
- *sequent rules are defined as usual (in particular they have finite premises),*
- *the notion of proof is defined as usual (in particular they are of finite length),*
- *the definition of the logic induced by a proto-SSSS is the standard one (the first one in the picture).*

A logic induced by a proto-SSSS is therefore a finite normal logic, as it can easily be seen, and thus the semantics of relatively maximal theories is adequate for it.

Let us recall two well-known basic definitions in order to fix the notations.

Definition 13: *A* sequent σ *on a given set* $\mathbb{L}$ *is a pair of finite sequences (possibly empty) of* $\mathbb{L}$*:* $\sigma = \langle \Sigma; \theta \rangle$*; following Hertz and Gentzen we will write such a sequent as follows:* $\Sigma \to \theta$.

Definition 14: *A (*sequent*)* rule $\mathfrak{R}$ *is a pair* $\mathfrak{R} = \langle PRE; co \rangle$ *where co, the* conclusion *of the rule, is a sequent and where* PRE *is a finite sequence (possibly the empty sequence) of sequents (*premises*).*

A rule such that $\mathfrak{R} = \langle \langle \sigma_1; ...; \sigma_n \rangle; \sigma \rangle$ is symbolically represented as:

$$\frac{\sigma_1 ... \sigma_n}{\sigma} \mathfrak{R}.$$

Definition 15: *Given a rule*

$$\frac{\Sigma 1 \to \theta 1 ... \Sigma n \to \theta n}{\Sigma \to \theta} \mathfrak{R}$$

and sequences $\Sigma 1^+$*, ...,* Σn^+*,* $\theta 1^+$*, ...,* θn^+*, the following rule* $\mathfrak{R}^+$ *is called an* expansion *of* $\mathfrak{R}$*:*

$$\frac{\Sigma 1^+, \Sigma 1 \to \theta 1, \theta 1^+ ... \Sigma n^+, \Sigma n \to \theta n, \theta n^+}{\Sigma 1^+, ..., \Sigma n^+, \Sigma \to \theta, \theta 1^+, ..., \theta n^+} \mathfrak{R}^+.$$

Definition 16: *A* SSSS *is a proto-SSSS which is closed under rule expansion, that is to say if a rule is a member of the system, so is any expansion of this rule.*

The definition of SSSS excludes some rules with conditions of application, like the classical rules for quantifiers. Therefore LK is not a SSSS. But the propositional fragment of LK is a SSSS.

Remark One can say that conditions of application of rules are structural principles which are hidden in the logical rules (recent works have tried to turn these implicit structural principles into explicit structural rules).

4. *SSSS and bivaluations*

4.1. *Revaluation and soundness*

Definition 17: *We say that a* bivaluation β *from* $\mathbb{L}$ *to* $\{0,1\}$ satisfies a sequent $a_1, ..., a_n \to b_1, ..., b_m$ *on* $\mathbb{L}$ *iff if* $\beta(a_i) = 0$ *for one* i *(*$1 \leq i \leq n$*) or* $\beta(b_i) = 1$ *for one* i *(*$1 \leq i \leq m$*).*

Definition 18: *We say that a* bivaluation β respects a rule $\mathfrak{R}$ *iff it satisfies the conclusion of this rule whenever its satisfies the premises of this rule.*

Theorem 19: *If a bivaluation respects a rule, then it respects any expansion of it.*

Proof. Given a bivaluation β and a rule $\mathfrak{R}$, suppose that β respects $\mathfrak{R}$ and that it does not respect an expansion $\mathfrak{R}^+$ of $\mathfrak{R}$. Therefore β gives the value 1 to each of the premises of $\mathfrak{R}^+$ and the value 0 to its conclusion.

If β gives the value 0 to the conclusion of $\mathfrak{R}^+$, we have (using the same notation as in Definition 15 of the expansion of a rule):

$$\beta(a) = 1 \text{ for every } a \text{ occurring in } \Sigma 1^+, ..., \Sigma n^+, \Sigma$$

and

$$\beta(a) = 0 \text{ for every } a \text{ occurring in } \theta, \theta 1^+, ..., \theta n^+.$$

We see therefore that β gives the value 0 to the conclusion $\Sigma \to \theta$ of $\mathfrak{R}$.

If β gives the value 1 to a premise $\Sigma i^+, \Sigma i \to \theta i, \theta i^+$ of $\mathfrak{R}^+$, as $\beta(a) = 1$ for every a occurring in Σi^+ and $\beta(a) = 0$ for every a occurring in θi^+, there exists b occurring in Σi such that $\beta(b) = 0$ or there exists b occurring in θi such that $\beta(b) = 1$, therefore β gives the values 1 to $\Sigma i \rightarrow \theta i$, as β respects $\mathfrak{R}$, β therefore gives the value 1 to the conclusion $\Sigma \rightarrow \theta$ of $\mathfrak{R}$, which is absurd.

Definition 20: *Given a SSSS* $\mathfrak{S}$*, we call* revaluation *of* $\mathfrak{S}$ *a bivaluation which respects all the rules of* $\mathfrak{S}$*.*

Proposition 21: *A set of revaluations of a SSSS is a sound semantics for the logic induced by this system.*

Proof. Obvious.

Due to Theorem 6, this shows that the characteristic function of a revaluation is a closed theory.

One can wonder if a closed theory respects the rules. This is true for Hilbert-type systems and Gentzen showed in (Gentzen 1932) that it is true for Hertz-type systems, but this is not necessarily true for SSSS.

In the next section we shall prove that characteristic functions of relatively maximal theories respect the rules of a SSSS. From this one can infer that the relatively maximal semantics of a logic induced by a SSSS is sound. But we already know this fact because relatively maximal theories are closed. Thus this result by itself is useless. But combining it with Lindenbaum-Asser theorem we can get completeness, as we will show in Section 5.

Given a bivalent semantics, i.e. a set of bivaluations, if one wants to show that it is a sound semantics for a SSSS, due to the above result, it is enough to show that the bivaluations respect all the rules of the system.

In the case of logical rules, such as for example the two rules for implication presented in Section 1, due to Theorem 19 about rule expansion, it is enough to show that the bivaluations respect theses rules without contexts.

In the case of structural rules, one can show once and for all that any bivaluation respects them. This is obvious for the identity axiom, the permutations rules and the contractions rules. In the cases of the weakening rules and the cut rule, one has just to do a bit of "zerology", to use Fraïsse's expression. That is to say we must verify that the limit cases where empty sequences arise are not problematic.

Consider the following instance of the left weakening rule:

$$\frac{\rightarrow b}{a \rightarrow b}$$

If a bivaluation β gives the value 1 to b then it gives the values 0 to a or the value 1 to b. Therefore β respects this rule.

Now the left weakening rule without context is the following:

$$\frac{\rightarrow}{a \rightarrow}$$

To show that a bivaluation respects this rule, we must show that if it respects the premise it respects also the conclusion. But how can a bivaluation respects an empty sequent? Following Gentzen's idea, the empty sequent means contradiction, therefore the right convention to adopt here is to say that no bivaluations respect the empty sequent. Therefore a bivaluation respects the weakening rule without context, by default.

The case of the cut rule is solved in the same way. One can easily check that a bivaluation respects this rule with some contexts. Now if the rule has no context, no bivaluation can respect simultaneously the two premises of the rule. Therefore bivaluations respect this rule by default.

4.2. *Relatively maximal theories and the main theorem*

In the last section, we have spoken of the satisfaction of a sequent by a bivaluation. Here we will rather speak of the satisfaction of a sequent by a theory (in the case of a closed theory these two definitions are the same).

Definition 22: *Given a sequent $a_1, ..., a_n \rightarrow b_1, ..., b_m$ on a set $\mathbb{L}$ and a theory T of $\mathbb{L}$, we say that T* satisfies this sequent *iff ($T \nvdash a_1$ or ... or $T \nvdash a_n$) or ($T \vdash b_1$ or ... or $T \vdash b_m$).*

Lemma 23: *A sequent which is derivable in a SSSS is satisfied by every relatively maximal theories of the logic induced by this system.*

Proof. Suppose that the sequent $a_1, ..., a_n \rightarrow b_1, ..., b_m$ is derivable and suppose $T \vdash a_i$, for every i ($1 \leq i \leq n$) and $T \nvdash b_i$, for every i ($1 \leq i \leq m$). As T is relatively maximal in one r, T, $b_i \vdash r$, for every i ($1 \leq i \leq m$). There are therefore sequences $\Sigma_1, ..., \Sigma_n$ of elements of T such that the sequents $\Sigma_1 \rightarrow a_1, ..., \Sigma_n \rightarrow a_n$ are derivable, and sequences $\theta_1, ..., \theta_m$ of elements of T such that the sequents $\theta_1, b_1 \rightarrow r, ..., \theta_m, b_m \rightarrow r$ are derivable; we have then the following derivation (omitting permutation rules):

$$\frac{\Sigma_1 \rightarrow a_1 \qquad a_1, ..., a_n \rightarrow b_1, ..., b_m}{\Sigma_1, a_2, ..., a_n \rightarrow b_1, ..., b_m} \text{ cut}$$

$$\vdots$$

$$\cfrac{\cfrac{\cfrac{\Sigma_n \to a_n \qquad \Sigma_1, ..., \Sigma_{n-1}, a_n \to b_1, ..., b_m}{\Sigma_1, ..., \Sigma_n \to b_1, ..., b_m}\text{cut} \qquad b_m, \theta_m \to r}{\Sigma_1, ..., \Sigma_n, \theta_m \to b_1, ..., b_{m-1}, r}\text{cut}}{\vdots}$$

$$\cfrac{\Sigma_1, ..., \Sigma_n, \theta_2, ..., \theta_m \to b_1, r \qquad b_1, \theta_1 \to r}{\Sigma_1, ..., \Sigma_n, \theta_1, ..., \theta_m \to r}\text{cut}$$

The sequent $\Sigma_1, ..., \Sigma_n, \theta_1, ..., \theta_m \to r$ is therefore derivable and as the sequences Σ_i $(1 \leq i \leq n)$ and the sequences θ_i $(1 \leq i \leq m)$ are finite sequences of elements T, thus $T \vdash r$, which contradicts the fact that T is relatively maximal in r.

Definition 24: *We say that a* theory respects a rule *iff it satisfies the conclusion of this rule whenever it satisfies its premises.*

Theorem 25: (Main Theorem) *Relatively maximal theories of a logic induced by a SSSS respect the rules of this system.*

Proof. Given a SSSS, we consider a relatively maximal theory T of the logic induced by this SSSS and a rule $\mathfrak{R}$ of this SSSS:

$$\cfrac{\overbrace{a_1^1, \ldots, a_n^1 \to b_1^1, \ldots, b_m^1}^{\sigma^1} \quad \ldots \quad \overbrace{a_1^j, \ldots, a_o^j \to b_1^j, \ldots, b_p^j}^{\sigma^j}}{\underbrace{c_1, \ldots, c_q \to d_1, \ldots, d_s}_{\sigma}}\ \mathfrak{R}$$

Suppose T satisfies all the sequents of the premises of the rule.

Given a premise sequent of $\mathfrak{R}$ $\sigma = a_1, ..., a_n \to b_1, ..., b_m$, there is a i, $(1 \leq i \leq n)$ such that $T \nvdash a_i$ or there is a i, $(1 \leq i \leq m)$ such that $T \vdash b_i$.

In the first case, given r a formula such that T is relatively maximal in r, we have T, $a_i \nvdash r$. There is therefore a sequence Σ of elements of T such that the sequent Σ, $a_i \vdash r$ is derivable.

In the second case, there is a sequence θ of elements of T such that the sequent $\theta \vdash b_i$ is derivable.

In the first case, by a series of applications of the weakening rules, we derive the sequent $\Sigma, a_1, ..., a_n \to b_1, ..., b_m, r$ from the sequent $\Sigma, a_i \vdash r$.

In the second case, by a series of applications of the weakening rules, we derive the sequent $\theta, a_1, ..., a_n \to b_1, ..., b_m$ from the sequent $\theta \vdash b_i$.

We can then use an expansion of the rule $\mathfrak{R}$, by carrying through, according the cases, the sequences of types Σ, θ and the formula r, to derive a sequent of the following type:

$$\sigma' = \Lambda, c_1, ..., c_q \rightarrow d_1, ..., d_s, (r)$$

where Λ is a finite sequence of elements of T. The formula r may appear or not on the right part of the sequent. In both cases, due to the preceding lemma, σ' is satisfied by T.

If r does not appear, the right side of σ' is the same as the right side of σ, therefore T satisfies σ.

If r appears, suppose that T does not satisfy σ. Therefore for every i ($1 \leq i \leq q$) we have $T \vdash c_i$ and for every i ($1 \leq i \leq s$) we have $T \nvdash d_i$. An object t of Λ is an element of T, therefore $T \vdash t$. Moreover $T \nvdash r$. Therefore all elements of the left side of σ' are deducible from T and no elements on the right side of σ' are deducible from T, this means that T does not satisfy σ', which is absurd.

5. *Applications*

5.1. *How to apply the main theorem – a basic example*

Completeness

To show how we can apply our main theorem to get instantaneously the completeness theorem we will take our starting example: the logic of pure classical implication.

Consider the SSSS $\mathfrak{S} \rightarrow$ constituted by structural rules and the two rules $(\supset r)$ and $(\supset l)$ for implication and the logic induced by this system: $\mathfrak{L} \rightarrow = \langle \mathbb{F}; \vdash \rangle$.

Given a relatively maximal theory T of this logic, due to the Main Theorem 25, it preserves respectively the two rules $(\supset r)$ and $(\supset l)$, and in particular it preserves these two rules without contexts, therefore it is easy to see through Definitions 22/24 that the characteristic function of T obeys respectively the conditions $(B \supset r)$ and $(B \supset l)$.

This means that the class of characteristic functions of relatively maximal theories of $\mathfrak{L} \rightarrow$ is included into the set of bivaluations $\mathsf{BIV} \rightarrow$ defined by conditions $(B \supset r)$ and $(B \supset l)$. Therefore, due to Lindenbaum-Asser theorem (a logic induced by a SSSS is a finite normal logic), more precisely to

Corollary 11 according to which any semantics including the class of relatively maximal theories is complete, we get completeness:

$$\text{If } T \vDash a \text{ then } T \vdash a$$

where $\vDash$ is the semantic deducibility relation induced in the usual way by conditions $(B \supset r)$ and $(B \supset l)$.

This shows how, with our main theorem, we can get completeness for the implicative fragment of the classical propositional logic. The same reasoning can be applied for each pair of sequent rules defining each classical connective ($\wedge$, $\vee$, $\neg$). Therefore this permits to give a very elegant proof of the completeness theorem which treats each connective independently. We can apply this method to other classical connectives like bi-conditional ($\leftrightarrow$), Sheffer's stroke, etc. In fact using this method we can construct sequent rules for connectives by translating their truth-tables into sequent rules according to the intuitive semantic interpretation of sequent rules given by Definitions 17/18.

More than this, in the above example of implication, we can prove completeness independently for each of the two rules for implication and their corresponding semantic conditions. This shows clearly that our result can be applied to non-truth functional connectives who have non-truth functional bivalent semantics. Before giving other examples, let us see how we can get also soundness.

Soundness

As we have seen (Proposition 21), any set of bivaluations which respect the rules of a SSSS is a sound semantics for the logic induced by this SSSS. Moreover (Theorem 19), it is sufficient to show that bivaluations respects the rules without contexts.

A bivaluation obeying the conditions $(B \supset l)$ and $(B \supset r)$ obviously respects the two sequent rules $(\supset l)$ and $(\supset r)$.

What our general results permit to show without any difficulty is that the bivalent semantics $\mathsf{BIV} \rightarrow$ is included into the set of characteristic functions of closed theories and contains the set of characteristic functions of relatively maximal theories. Any such intermediate set of bivaluations is adequate (Theorem 6, Corollary 11). An additional result one may want to prove is that $\mathsf{BIV} \rightarrow$ is exactly the semantics of relatively maximal theories, and that therefore due to the result mentioned in Section 2 it is a minimal semantics for the above SSSS.

This is not difficult to prove. One can also prove that in this case all relatively maximal theories are maximal. These results are proved in (da Costa/Béziau 1994).

Consider now the system $\mathfrak{S} \to'$ which is the system $\mathfrak{S} \to$ without the cut rule. By Gentzen's cut-elimination theorem for this fragment of LK we know that $\mathfrak{S} \to'$ induces the same logic as the logic induced by $\mathfrak{S} \to$ and that therefore $\mathsf{BIV} \to$ is an adequate semantics for it.

But our results cannot be applied directly to $\mathfrak{S} \to'$ because $\mathfrak{S} \to'$ is not a SSSS.

5.2. *Sequent-rules and classical truth-tables*

Given one of the sixteen truth-tables for one classical connective, it is very easy to transform it into sequent rules, using Definitions 16 and 18. These sequent rules together with structural rules will be, according to our results, an axiomatization of the truth table for the connective.

Let us take the example of Sheffer's stroke. Its truth-table is the following:

a	b	$a \mid b$
1	1	0
0	1	1
1	0	1
0	0	1

With our method we can transform each line of this table into a sequent rule. We have therefore the four following sequent rules (that we write without contexts, but one must keep in mind that we are in a SSSS):

$$\frac{\to a \qquad \to b}{a \mid b \to} \qquad \frac{a \to \qquad \to b}{\to a \mid b}$$

$$\frac{\to a \qquad b \to}{\to a \mid b} \qquad \frac{a \to \qquad b \to}{\to a \mid b}$$

Of course we can find a simpler sequent systems for this connective. The data given in the truth-tables can be simplify in order to get two conditions corresponding to two rules with the subformula properties, such there is one with aşb on the left side of the conclusion sequent and there is another one with $a \mid b$ on the right side of the conclusion sequent. Therefore we get two rules who have the same central features as the standard pairs of rules of Gentzen's system. The left rule is the first one above and the right one is the following (which is equivalent to the three other one above):

$$\frac{a, b \to}{\to a \mid b}$$

which corresponds to the following semantic condition:

$$\text{If } \beta(a) = 0 \text{ or } \beta(b) = 0 \text{ then } \beta(a \mid b) = 1.$$

As it is known Sheffer's stroke is enough to express all classical propositional connectives. Therefore the SSSS with the two sequent rules above and structural rules is a axiomatization of classical propositional logic.

5.3. *Applications to non truth-functional bivalent semantics*

5.3.1. *A paracomplete logic in which classical logic is translatable*

Let us consider the SSSS with the standard structural rules and the classical rules for implication and just the left rule for negation:

$$\frac{\rightarrow a}{\neg a \rightarrow}$$

It is easy to show that cut-elimination holds for this systems, following an observation made in (Raggio 1968) for a similar system. This shows that this system is decidable and consistent.

If we define $\neg^* = a \supset \neg a$, it is easy to show that this connective obeys the two classical rule for negation and that therefore it behaves like a classical negation. Therefore we can easily translate classical logic in the logic induced by this system.

The semantics for this system is given by the two conditions for classical implication and the following condition for negation:

If $\beta(a) = 1$ then $\beta(\neg a) = 0$.

This logic is paracomplete, that is to say, a formula and its negation can both be false.

5.3.2. *Paraconsistent logic*

I was led to develop the results of this paper by the study of paraconsistent logic, in particular da Costa's paraconsistent logic C1 (da Costa 1963). When I started to work on this logic, an open problem was to find a sequent system for it. There had been an aborted tentative by Raggio (Raggio 1968). To build a sequent system, I worked with the bivalent non-truth functional semantics of this logic (da Costa/Alves 1977) and I transformed the intuitive conditions for bivaluations into sequent rules. At the time I didn't have the general results presented here, but I show indirectly that it was working by proving the completeness theorem.

Let us see how easy it is with the general results to build sequent rules for C1.

The semantic conditions for C1 are the usual ones for conjunction, disjunction and implication; here are the conditions for negation:

(1) If $\beta(a) = 0$ then $\beta(\neg a) = 1$
(2) If $\beta(a \wedge \neg a) = 1$, then $\beta(\neg(a \wedge \neg a)) = 0$
(3) If $\beta(\neg\neg a) = 1$ then $\beta(a) = 1$
(4) If $\beta(a * b) = 1$, and if $\beta(a) = 0$ or $\beta(\neg a) = 0$ and $\beta(b) = 0$ or $\beta(\neg b) = 0$, then $\beta(\neg(a * b)) = 0$, where $*$ is $\wedge$, $\vee$ or $\supset$.

Conditions (1) and (2) are obviously translated into the two following sequent rules:

$$\frac{a \rightarrow}{\rightarrow \neg a} \qquad \frac{\rightarrow a \wedge \neg a}{\neg(a \wedge \neg a) \rightarrow}$$

In order to get a rule with the subformula property we transform the condition (3) into the equivalent condition

(3') If $\beta(a) = 0$ then $\beta(\neg\neg a) = 0$

which is easily translated into the following rule:

$$\frac{a \rightarrow}{\neg\neg a \rightarrow}$$

Concerning the condition (4) we get the following rule:

$$\frac{\rightarrow (a * b) \qquad a, \neg a \rightarrow \quad b, \neg b \rightarrow}{\neg(a * b) \rightarrow}$$

This rule is a bit strange. In particular it does have obey the subformula property. However it turns out that decidability and cut-elimination can be obtained without much difficulty for this system (Béziau 1993).

These methods can been used for constructing sequent systems for logics which are both paraconsistent and paracomplete as the ones presented in (Béziau 1989) and (Béziau 1990).

5.4. *Remarks about cut-elimination and decidability*

When one thinks about a system of sequents, one has in mind Gentzen's systems with rules with the subformula property, systems in which all the rules except the cut rule have this property and also in which all connectives are defined independently. Our definition of SSSS and our methods to get completeness do not require these restrictions.

For example the sequent version of natural deduction for classical logic (Gentzen's M-system) is an SSSS.

With our methods we can prove easily that the following rule (we omit the contexts, but we take it as a SSSS rule) for negation together with structural rules is an axiomatization of classical negation:

$$\frac{\neg a \to b \qquad \neg a \to \neg b}{\to a}$$

Given some conditions for bivaluations, we can translate them into rules which have not the subformula property and are mixed rule (defining simultaneously various connectives). We don't know if the system so-obtained is consistent. But our methods can be used to get consistency, trying to get an equivalent system in which we can show than the empty sequent is not derivable, by the method of cut-elimination. Our method can also be used to show that a logic defined by a given set of bivaluations is decidable, using a similar method.

5.5. *Applications to many-valued logic*

There are two ways of applying our methods to many-valued logics. First, given a many-valued logic, we can define a bivalent non truth-functional semantics for it and then translate conditions of this semantics into sequent rules. Secondly we can develop the concept of many-valued sequents and generalize our methods to this concept (this generalization was suggested to us by Matthias Baaz).

5.5.1. *Łukasiewicz's three valued logic*

Suszko in (Suszko 1975) defined an adequate non-truth functional bivalent semantics for Łukasiewicz logic L3 (with implication and negation) with the following conditions:

(a) $\beta(a) = 0$ or $\beta(\neg a) = 0$
(b) If $\beta(b) = 1$, then $\beta(a \supset b) = 1$
(c) if $\beta(a) = 1$ and $\beta(b) = 0$, then $\beta(a \supset b) = 0$
(d) if $\beta(a) = \beta(b)$ and $\beta(\neg a) = \beta(\neg b)$, then $\beta(a \supset b) = 1$
(e) if $\beta(a) = \beta(b) = 0$ and $\beta(\neg a) \neq \beta(\neg b)$, then $\beta(a \supset b) = \beta(\neg a)$
(f) if $\beta(\neg a) = 0$, then $\beta(\neg\neg a) = \beta(a)$
(g) if $\beta(a) = 1$ and $\beta(b) = 0$, then $\beta(\neg(a \supset b)) = \beta(\neg b)$

(h) if $\beta(a) = \beta(\neg a) = \beta(b)$ and $\beta(\neg b) = 1$, then $\beta(\neg(a \supset b)) = 0$.

There are various ways to transform these conditions into sequent rules. In order to transform them into rules we first transform this set of conditions into an equivalent set of conditions presented, except condition (f), in the following table (for more details see (Béziau 1999a)):

a	$\neg a$	b	$\neg b$	$a \supset b$	$\neg(a \supset b)$
0	0	0	0	1	0
0	0	0	1	0	0
0	0	1	0	1	0
0	1	0	0	1	0
0	1	0	1	1	0
0	1	1	0	1	0
1	0	0	0	0	0
1	0	0	1	0	1
1	0	1	0	1	0

It is then easy to see that the system S3 consisting of structural rules and the following logical rules is an axiomatization of L3.

$$\frac{\to a}{\neg a \to}\neg l \qquad \frac{a \to}{\neg\neg a \to}\neg\neg l \qquad \frac{\to a}{\to \neg\neg a}\neg\neg r$$

$$\frac{\neg a \to \qquad \to \neg b}{a \supset b \to} \supset l1 \qquad \frac{\to a \qquad b \to}{a \supset b \to} \supset l2$$

$$\frac{\to \neg a, b}{\to a \supset b} \supset r1 \qquad \frac{a \to \qquad \neg b \to}{\to a \supset b} \supset r2$$

$$\frac{a \supset b \to \qquad \to a, \neg a \qquad \to b, \neg b}{\to \neg(a \supset b)} \neg \supset r$$

$$\frac{a \supset b \to \quad a \to \quad \neg a \to}{\neg(a \supset b) \to} \neg \supset l1 \qquad \frac{a \supset b \to \quad b \to \quad \neg b \to}{\neg(a \supset b) \to} \neg \supset l2$$

S3 has not the subformula property, but has a property which is quite close. The main formulas of the premises of the rules are either proper subformulas of the main formula of the conclusion, or negations of proper subformulas of it. It is easy to see that this property is enough to entail the decidability of the system S3 without cut, which can be shown to be equivalent to S3 via cut-elimination.

5.5.2. *Many-valued sequents*

The bivalence of standard sequent rules is obviously related to the fact that sequent are two sided and that the left side can be considered as falsity and the right side as truth.

In many-valued logic the distinction between designated and undesignated elements is also a kind of bivalence as stressed by (Malinowski 1993), indeed one of the reason why it is possible to provide bivalent semantics for many-valued logics.

To generalize the notion of sequents to the case of many- valued semantics, we will keep the two fundamental sides of the sequents: the left side will correspond to undesignated values and the right side to designated values. But these sides will be divided in several parts corresponding to the various designated and undesignated elements. For example, in the case of Łukasiewicz's logic L3, the left side will have two parts corresponding to the undesignated value 0 and 1/2 and the right side only one part corresponding to the designated value 1. Let us divide the left part of such three-valued sequent by the symbol “$<$”: the left side of this symbol will correspond to the value 0 and the right side to the value 1/2:

$$0 < 1/2 \to 1$$

The three conditions of the three-valued semantics for negation given by the following table

a	$\neg a$
0	1
1/2	1/2
1	0

can therefore be translated respectively into the three following three-valued sequent rules:

$$\frac{< \quad \to a}{\neg a < \quad \to} \qquad \frac{< \quad a \to}{< \ \neg a \to} \qquad \frac{a < \quad \to}{< \quad \to \neg a}$$

The following step consists in generalizing the concept of relatively maximal theory into a concept of many-valued relatively maximal theory, to prove a generalized version of Lindenbaum-Asser for it as well as a generalized version of our main theorem.

It has been proved that there are some finite matrices which are not finitely axiomatizable by an Hilbert's system (Urquhart 1977) (Wojtylak 1984), but with the notion of many-valued SSSS we can finitely axiomatize any finite matrix. Once again sequent systems show here that the difference between

Hilbert's style proof-theoretical methods and semantic methods is due to the weakness of the former and not to the weakness of proof-theoretical methods in general. Indeed proof-theoretical methods and semantic methods appear in this context to be equivalent, being intertranslatable.

In the same way that our result in the case of bivalence does not depend on truth-functionality, the generalization of the methods to many-valuedness does not depend on truth-functionality. Therefore these generalized methods apply not only to matrix semantics but to any many-valued semantics.

Non truth-functional many-valued semantics and the related many-valued sequent systems can be use for the logic C1 in order to get back the subformula property.

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