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A GENERAL CHARACTERIZATION OF ADAPTIVE LOGICS*

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Abstract
This paper contains a unified characterization of adaptive logics. The general structure is presented in the simplest possible guise, both for flat and prioritized adaptive logics. The latter are presented as a special case of combined adaptive logics. The aim of the paper is to provide the general framework underlying several other papers in this volume and to prepare the unified metatheory of adaptive logics.

1. *The Dynamics of Reasoning*

Let us start with a loose characterization: a logic is adaptive iff it adapts itself to the specific premises to which it is applied. Two comments are in place here. First, I do not mean to say that the consequence set, determined by the logic, depends on the set of premises. This obviously holds for nearly all logics.[1] What I mean is, first and foremost, that the logic interprets the premise set 'as normally as possible'. What is meant by "normal" depends on the specific adaptive logic, but, given a standard of normality, the effect is always the same. In semantic terms, the effect is that some models of the premises are selected in view of the abnormalities they verify. In proof theoretic terms, the effect is that some rules of inference do not apply in general, but that their instances apply or do not apply to some consequences of

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[1] There are two obvious exceptions: zero logic, according to which nothing is derivable from any premise set (not even the premises themselves), and universal logic, according to which everything is derivable from any premise set. See [7] for a sensible application of zero logic and of an adaptive logic definable from it.

the premises in view of the presence or absence of other consequences. Put differently, that the premises have certain consequences may prevent a rule of inference to be applicable to some other consequences of the premises. Next, I really mean that the logic adapts *itself* to the premises. The reasoner does not interfere in this. The adaptive effect results from the logic — that is, from the semantics as well as from the proof theory — independently of any decision of the human or machine that applies the logic.

In the subsequent sections, I present a more technical characterization. Before doing so, however, I explain what is the use of adaptive logics, which is a simple and straightforward matter.

For a multiplicity of consequence relations that occur in actual and important reasoning, there is no positive test. In other words, there is no systematic procedure that, for every set of premises Γ and for every conclusion A, leads after finitely many steps to a "yes" if A is a consequence of Γ.[2] When I say that such consequence relations are important, I mean that they play a central role, both in everyday reasoning and in scientific reasoning. They may not be required for organizing knowledge in theories. However, they do occur in the reasoning processes that lead to knowledge, such as inductive reasoning, the search for an explanation, handling inconsistent knowledge, and problem solving in general.

As a result of the absence of a positive test, reasoning processes in which such consequence relations are applied display an external as well as an internal dynamics. The *external* dynamics is well known: as new premises become available, consequences derived from the earlier premise set may be withdrawn. In other words, the external dynamics results from the non-monotonic character of the consequence relations — the fact that, for some Γ, Δ and A, $\Gamma \vdash A$ but $\Gamma \cup \Delta \nvdash A$. The *internal* dynamics is very different from the external one. Even if the premise set is constant, certain formulas are considered as derived at some stage of the reasoning process, but are considered as not derived at a later stage. For any consequence relation, insight in the premises is only gained by deriving consequences from them. In the absence of a positive test, this results in the internal dynamics.

The origin of the internal dynamics can easily be understood. Consider some consequence relation $\vdash_{\mathsf{L1}}$ such that $\Gamma \vdash_{\mathsf{L1}} A$ depends on a number of conditions, among which are $\Gamma \vdash_{\mathsf{L2}} B$ and $\Gamma \nvdash_{\mathsf{L3}} C$.[3] Suppose moreover that, for both $\Gamma \vdash_{\mathsf{L2}} B$ and $\Gamma \vdash_{\mathsf{L3}} C$, there is a positive test but no negative test, just as is the case for $\Gamma \vdash_{\mathsf{CL}} A$. It follows immediately that there is

[2] Remark that the consequence relation defined by classical logic is undecidable, but that there is a positive test for it. See [21] for such matters.

[3] The forms of the formulas A, B and C will be related in some specific way, but this need not concern us here.

neither a positive test nor a negative test for $\Gamma \vdash_{\mathsf{L1}} A$. As a result, any reasoning that aims at finding out whether A is L1-derivable from Γ will in general have to be dynamic (that is, except for specific A and Γ). Indeed, while one may apply the positive test to establish that $\Gamma \vdash_{\mathsf{L2}} B$, it will (in general) be impossible to establish that $\Gamma \nvdash_{\mathsf{L3}} C$. How might one react to this situation? The only sensible approach is to presuppose that $\Gamma \nvdash_{\mathsf{L3}} C$ *unless and until* the opposite has been established. In other words, one will take $\Gamma \vdash_{\mathsf{L2}} B$ as a provisional good reason to accept $\Gamma \vdash_{\mathsf{L1}} A$, but be prepared to review this conclusion at a later stage of the reasoning.

For specific A and Γ, there may be a *criterion* to establish $\Gamma \vdash_{\mathsf{L1}} A$. Any sensible person will then apply the criterion.[4] For other A and Γ, a decent heuristics may provide a reliable estimate of whether $\Gamma \vdash_{\mathsf{L1}} A$ — this heuristics may pertain to the way in which the dynamic proof is extended after $\Gamma \vdash_{\mathsf{L2}} B$ has been established. All this, however, changes nothing to the occurrence of an internal dynamics in the reasoning process.

In subsequent sections, I shall present technical characterizations of (different kinds of) adaptive logics. This will include their semantics as well as their dynamic proof theories. The latter are extremely important. Adaptive logics are intended to *explicate actual forms of reasoning* and only their dynamic proofs provide one with such an explication.

I shall distinguish between flat and prioritized adaptive logics. An adaptive logic is called *flat* if all abnormalities are treated on a par. It is called *prioritized* if it avoids some abnormalities before avoiding others (and, where necessary, without avoiding the others). Put differently, a prioritized adaptive logic interprets a premise set as normally as possible with respect to a first set of abnormalities, interprets the result as normally as possible with respect to a second set of abnormalities, and so on. In Section 2, I shall characterize flat adaptive logics, in Section 3 prioritized ones.

After briefly pointing out the role of CL for the description and study of adaptive logics in Section 4, I shall present general characterizations of the semantics and of the dynamic proof theories in Section 5 and 6 respectively. In Section 7, I shall discuss the distinction between corrective and ampliative adaptive logics. Some final warnings are presented in Section 8.

[4] Still, if applying the criterion is extremely 'expensive' in comparison to the expense of a mistaken conclusion, one might decide not to apply it.

2. *Flat Adaptive Logics*

A flat adaptive logic AL is characterized by a triple:

(i) A *lower limit logic*: any monotonic logic.
(ii) A *set of abnormalities*: a set of formulas characterized by a logical form.
(iii) An *adaptive strategy*: this specifies what it means to interpret the premises 'as normally as possible'.

The lower limit logic LLL is the stable part of the adaptive logic, the part that is not subject to adaptation.[5] From a proof theoretic point of view, the lower limit logic delineates the rules of inference that hold unexceptionally. From a semantic point of view, the adaptive models of Γ are a selection of the lower limit models of Γ. It follows that $Cn_{\mathsf{LLL}}(\Gamma) \subseteq Cn_{\mathsf{AL}}(\Gamma)$.

The set of abnormalities — the term will become clear in the next paragraph — Ω comprises the formulas[6] that are presupposed to be false, unless and until proven otherwise. Ω may comprise those formulas of the logical form that fulfil a certain restriction, *provided* that, for every formula B of the logical form, some $\Delta \subseteq \Omega$ is such that $B \vdash_{\mathsf{LLL}} \bigvee(\Delta)$. Let us consider some examples. In many inconsistency-adaptive logics, Ω is the set of formulas of the form $\exists(A \wedge \sim A)$, in which $\exists A$ abbreviates the existential closure of A. In others, the set is restricted, for example, to formulas in which A is primitive (contains no logical symbols except for identity). In this case, the lower limit logic LLL should warrant that, for every B of the form $\exists(A \wedge \sim A)$, there is a $\Delta \subseteq \Omega$ such that $B \vdash_{\mathsf{LLL}} \bigvee(\Delta)$. In the basic inductive logic from [13], the set of abnormalities consists of all formulas of the form $\exists A \wedge \exists{\sim}A$ in which A is purely functional (no individual constants, quantifiers or sentential letters occur in it). Here the lower limit logic is CL and, for every B of the form $\exists A \wedge \exists{\sim}A$, there is a $\Delta \subseteq \Omega$ such that $B \vdash_{\mathsf{CL}} \bigvee(\Delta)$.[7]

If the lower limit logic is extended with the requirement that no abnormality is logically possible, one obtains a monotonic logic, which is called the

[5] Typically, the lower limit consequences of a set of premises relate to the adaptive consequences as follows: $\Gamma \vdash_{\mathsf{LLL}} A$ iff $\Gamma \cup \Delta \vdash_{\mathsf{AL}} A$ for all sets of formulas Δ.

[6] For some logics, the abnormalities are couples consisting of an open formula with n free variables and of an n-tuple of elements of the domain.

[7] Some flat adaptive logics were described as formula-preferential systems in [23] — see also [1]. It is not clear whether this may be done for all adaptive logics, but the approach is clearly a useful challenge to the Ghent one.

upper limit logic ULL.[8] The effect is easily seen by considering the semantics. An adequate semantics for the upper limit logic is obtained by selecting those lower limit logic models that verify no abnormality. The name "abnormality" refers to the upper limit logic. ULL requires premise sets to be normal, and 'explodes' abnormal premise sets (assigns them the trivial consequence set).

Some examples are useful to clarify the matter. If the lower limit logic is CL and the set of abnormalities comprises the formulas of the form $\exists A \wedge \exists{\sim}A$ (see above), then the upper limit logic is obtained by adding to CL the axiom $\exists A \supset \forall A$.[9] If, as is the case for many inconsistency-adaptive logics, the lower limit logic is a paraconsistent logic[10] PL which contains CL,[11] and the set of abnormalities comprises the formulas of the form $\exists(A \wedge {\sim}A)$, then the upper limit logic is CL.[12] The importance of the set of abnormalities is obvious. If the premise set does not require any abnormality to obtain, the adaptive logic will deliver the same consequences as the upper limit logic. If the premise set requires some abnormalities to obtain, the adaptive logic will still deliver more consequences than the lower limit logic, viz. all upper limit consequences that are not 'blocked' by those abnormalities. In sum, the adaptive logic interprets the set of premises 'as much as possible' in agreement with the upper limit logic; it avoids abnormalities 'in as far as' the premises permit.

Given a lower limit logic and a strategy, different sets of abnormalities may result in the same upper limit logic but still lead to a different adaptive logic. Suppose that the lower limit logic is a rich paraconsistent logic PL, and that the set of abnormalities Ω is either all formulas of the form $\exists(A \wedge {\sim}A)$ or all such formulas that are not PL-equivalent to a disjunction of (less complex)

[8] This is the reason why the set of abnormalities Ω has to be characterized by some logical form.

[9] Semantically, this logic is characterized by those CL-models in which, for each predicate π of adicity i, $v(\pi) \in \{\emptyset, D^i\}$ in which D^i is the i-th Cartesian product of the domain.

[10] A logic is paraconsistent iff it does not validate all instances of $A, {\sim}A \vdash B$.

[11] See Section 4 for the precise meaning of "logic L1 contains logic L2".

[12] This is also the case if the lower limit logic LLL is such that all formulas of the form $\exists(A \wedge {\sim}A)$ are LLL-equivalent to disjunctions of members of Ω. By defining the upper limit logic ULL in such a way that all members of Ω are logically false, one at once warrants that all formulas of the form $\exists(A \wedge {\sim}A)$ are logically false in ULL.

formulas of that form.[13] In both cases, the upper limit logic is the same. Even if combined with the same strategy, however, the resulting adaptive logics are different. If Ω is the restricted set, one obtains a usual and well-behaving adaptive logic. If Ω comprises the formulas of the form $\exists(A\wedge\sim A)$, the rich paraconsistent logic PL will cause the adaptive logic to be a flip-flop logic — one that behaves as the upper limit logic CL if the premise set is $\sim$-consistent and behaves as the lower limit logic PL if the premise set is $\sim$-inconsistent.[14]

Let us consider a further example. Where $\mathcal{W}$ is the set of closed formulas of the usual (non-modal) predictive language, let the lower limit logic be some standard modal logic ML and let Ω be either $\{\Diamond A \wedge \sim A \mid A \in \mathcal{W}\}$ or $\{\Box A \mid A \in \mathcal{W};\ \nvdash_{\mathsf{ML}} \Box A\}$. In both cases, the upper limit is Triv — the system in which A is logically equivalent to $\Box A$ as well as to $\Diamond A$. However, the resulting adaptive logics are very different — see [13] for one involving the first set of abnormalities, and [14] for one involving the second set of abnormalities.

A very important matter has to be brought up at this point. For all that was said before, the reader might think that there is a well-defined set of formulas that need to behave abnormally in view of the premises. This is not the case. The complication derives from the fact that, except in the case of some specific lower limit logics — see below where I mention the Simple strategy — a set of premises may entail a disjunction of abnormalities (members of Ω) without entailing any of its disjuncts.

Disjunctions of abnormalities[15] will be called *Dab-formulas*. In the sequel, any expression of the form $Dab(\Delta)$ will refer to the disjunction of the members of a finite $\Delta \subseteq \Omega$.[16] The Dab-formulas that are derivable by the lower limit logic from the premise set Γ will be called *Dab-consequences* of Γ. If $Dab(\Delta)$ is a Dab-consequence of Γ, then so is $Dab(\Delta \cup \Theta)$ for any

[13] Where the lower limit logic is CluNs — see for example [8] — the restriction comes to selecting those $\exists(A \wedge \sim A)$ in which A is a primitive formula (does not contain a logical symbol, except for identity).

[14] Inattentive readers time and again misunderstood all adaptive logics as flip-flops. So, I defined some flip-flop logics, which are indeed adaptive, in order to illustrate the difference with usual adaptive logics. I always considered flip-flops as utterly uninteresting, until some interesting prioritized adaptive logics turned out to be flip-flops — see [13].

[15] I mean *classical* disjunctions of abnormalities — see Section 4.

[16] So, $Dab(\Delta)$ is the classical disjunction of the members of $\Delta \subseteq \Omega$. In many previous papers on specific adaptive logics, $Dab(\Delta)$ functions in a slightly different way — where $p \wedge \sim p \in \Omega$ we (our group) now write $Dab(\{p \wedge \sim p\})$ where we wrote $Dab(\{p\})$ before.

finite Θ. For this reason, it is important to concentrate on the minimal Dab-consequences of the premise set: $Dab(\Delta)$ is a *minimal* Dab-consequence of Γ iff $\Gamma \vdash_{\mathsf{LLL}} Dab(\Delta)$ and there is no $\Theta \subset \Delta$ such that $\Gamma \vdash_{\mathsf{LLL}} Dab(\Theta)$. If $Dab(\Delta)$ is a minimal Dab-consequence of Γ, then Γ determines that some member of Δ behaves abnormally, but fails to determine which member of Δ behaves abnormally. Adaptive logics are obtained by interpreting a set of premises ‘as normally as possible’. As some minimal Dab-consequences of Γ may contain more than one disjunct, this phrase is not unambiguous. It is disambiguated by choosing a specific adaptive strategy.

The oldest known *strategy* is *Reliability* from [3], where it is discussed at the propositional level. Let $U(\Gamma) = \{A \mid A \in \Delta$ for some minimal Dab-consequence $Dab(\Delta)$ of $\Gamma\}$ (the set of formulas that are unreliable on Γ). The Reliability strategy considers a formula as behaving abnormally iff it is a member of $U(\Gamma)$. The effect of this on the semantics and proof theory will be discussed in subsequent sections.

The *Minimal Abnormality* strategy (first presented in [2] for the discussion of the propositional level in semantic terms) delivers some more consequences than the Reliability strategy. Suppose that $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal Dab-consequences of Γ. Roughly — a more precise formulation follows later — the Minimal Abnormality strategy picks one member of each Δ_i as behaving abnormally. Obviously, the strategy does not pick out a specific combination, but considers all of them.

Consider a simple propositional example for an inconsistency-adaptive logic: $\Gamma = \{\sim p, \sim q, p \vee q, p \vee r, q \vee r\}$. If the lower limit logic validates all of full positive logic, $(p \wedge \sim p) \vee (q \wedge \sim q)$ is a minimal Dab-consequence of Γ. On the Reliability strategy, both $p \wedge \sim p$ and $q \wedge \sim q$ are unreliable with respect to Γ — or both p and q are unreliable with respect to Γ if one prefers a different mode of speech (see footnote 26) — and hence r is not an adaptive consequence of Γ. However, if the Minimal Abnormality strategy is chosen, then r is an adaptive consequence of Γ. Indeed, if $p \wedge \sim p$ behaves abnormally (is true), then $q \wedge \sim q$ behaves normally (is false) and hence r is true in view of $\sim q$ and $q \vee r$; if $q \wedge \sim q$ behaves abnormally, then $p \wedge \sim p$ behaves normally and hence r is true in view of $\sim p$ and $p \vee r$. Both strategies are simple and perspicuous from a semantic point of view, but while the Reliability strategy leads to simple dynamic proofs, the dynamic proofs determined by the Minimal Abnormality strategy are rather complicated. Which strategy is adequate in a specific context of application is obviously a very different matter.

Some lower limit logics and sets of abnormalities are such that Δ is a singleton whenever $Dab(\Delta)$ is a minimal Dab-consequence of a premise set. If this is the case, the Reliability and Minimal Abnormality strategies lead to the same result and coincide with what is called the *Simple* strategy: a formula behaves abnormally just in case the abnormality is derivable from the premise set — see [14] , [24] and [25] for examples. Several other strategies have been studied. Most of them were the result of characterizing an existing consequence relation by an adaptive logic. Examples may be found in [9], [12], [22] and [31].

3. *Prioritized Adaptive Logics*

Let us start with combined adaptive logics. If $\mathsf{AL1}$ and $\mathsf{AL2}$ are adaptive logics that have Ω_1 and Ω_2 as their respective sets of abnormalities and share their lower limit logic and strategy, a combined adaptive logic $\mathsf{AL3}$ is obviously defined by the common lower limit logic and strategy and by the set of abnormalities $\Omega_1 \cup \Omega_2$.

In more interesting cases, the lower limit logics or the strategies are also different. Many combinations are possible in this case. In one of them, the consequence set of the combined adaptive logic $\mathsf{L1}$ may be defined by $Cn_{\mathsf{L2}}(Cn_{\mathsf{L3}}(\Gamma))$ in which $\mathsf{L2}$ and $\mathsf{L3}$ are themselves adaptive logics. $\mathsf{L3}$ may, for example, select the consistent consequences[17] of Γ whereas $\mathsf{L2}$ selects all inductive consequences (in the sense of IL^r from [13]) of a premise set. I shall say no more on this as I have no clear idea of the possible combinations that have useful applications.

Prioritized adaptive logics are a specific kind of combination of adaptive logics. The present standard format is as follows. First, one introduces an operator to express the priorities. I shall write this operator as $\Diamond$ because its logical structure turns out to be that of possibility. It may be read, for example, as "it is likely that", where "it is likely that it is likely that" is obviously weaker than "it is likely that". Indefeasible premises are represented by CL-formulas; premises of the next highest priority are represented by a formula of the form $\Diamond A$ in which A is a CL-formula; premises of the next highest priority by a formula of the form $\Diamond\Diamond A$ in which A is a CL-formula; etc.

Just as flat adaptive logics, prioritized adaptive logics are defined from a lower limit logic, a set of abnormalities and a strategy. They differ from flat adaptive logics in two respects. First, the set of abnormalities is defined as a union of sets: $\Omega = \Omega_1 \cup \Omega_2 \cup \ldots$ — there is no need to have a finite bound to this — each of them characterized by some formula of increasing complexity (containing increasing sequences of the symbol $\Diamond$). Typically, Ω_1 will be characterized by such formulas as $\Diamond A \wedge {\sim}A$ or $\Diamond A \wedge \Diamond{\sim}A$, Ω_2 by such formulas as $\Diamond\Diamond A \wedge {\sim}A$ or $\Diamond\Diamond A \wedge \Diamond\Diamond{\sim}A$, etc. The second difference pertains to the strategy. One way to look at the difference is by saying that the

[17] This may for example proceed as follows: Γ is closed by some paraconsistent logic $\mathsf{L4}$, and $\mathsf{L3}$ selects those $A \in Cn_{\mathsf{L4}}(\Gamma)$ for which $\{A\} \cup \Delta$ is consistent whenever $\Delta \subseteq Cn_{\mathsf{L4}}(\Gamma)$ is consistent. In more interesting cases, $\mathsf{L4}$ is itself an inconsistency-adaptive logic.

strategy first interprets the premises 'as normally as possible' with respect to Ω_1, next with respect to Ω_2, and so on.

A prioritized adaptive logic AL may be seen as a superposition of adaptive logics AL_1, AL_2, ... that share their lower limit logic and strategy with AL and differ from each other in that their respective sets of abnormalities are Ω_1, Ω_2, ... If $\Diamond^n$ is the longest sequence of diamonds occurring in the modal premise set Γ, a finite characterization is obtained by choosing $\Omega_n \cup \Omega_{n+1} \cup \ldots$ as the set of abnormalities of AL_n. One then has:

$$Cn_{\mathsf{AL}}(\Gamma) = Cn_{\mathsf{AL}_n}(\ldots(Cn_{\mathsf{AL}_2}(Cn_{\mathsf{AL}_1}(\Gamma)))\ldots)\,.$$

While this is nice as a definition, it is essential that the dynamic proof theory of AL does not follow this line — see Section 6.

Obviously, all three components of prioritized adaptive logics may be varied, but even varying the lower limit logic allows one to see the enormous scope of the variation, even for modal logics extending CL. Here are some examples: a predicative version of T is the lower limit logic in [17] and [31], a predicative version of $\mathsf{S5}$ and a very non-standard modal extension of CL, viz. IM, are the lower limit logics of systems studied in [13]. In all these cases, the upper limit logic is the modal system Triv.

In other approaches, the premises are represented by an n-tuple of sets each of which has a different priority. For example, the premises may be written as

$$\Sigma = \langle \Gamma_0, \Gamma_1, \ldots, \Gamma_m \rangle \tag{1}$$

in which the members of Γ_0 (the 'real' premises) receive maximal priority, the members of Γ_1 receive a lower priority, etc. The idea underlying such constructions is to define a consequence set that is the deductive closure in terms of some monotonic logic (usually CL) of Γ_0 plus as much of Γ_1 as may consistently be added to Γ_0, plus as much of Γ_2 as may be consistently added to the union of Γ_0 with the retained subset of Γ_1, etc.

Several different such constructions are possible — see for example [20] for an extensive study. It is fairly simple to characterize consequence relations defined along these lines in terms of prioritized adaptive logics. Where $\Diamond^i A$ abbreviates A preceded by i occurrences of $\Diamond$, a sequence of CL-premise sets such as (1) is 'translated' to the modal premise set

$$\{\Diamond^i A \mid A \in \Gamma_i\} \quad (0 \leq i \leq m) \tag{2}$$

or to something of the same sort.

The characterization of such consequence relations in terms of a prioritized adaptive logic has two major advantages. The first is that the consequence relations are supplied with a proof theory (which explicates actual reasoning according to the consequence relations). The second is that the adaptive characterization immediately leads to interesting variants of consequence relations. While the prioritized consequence relations defined for (1) define the deductive closure by some monotonic logic L of a subset of $\Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_m$, the variants define the L-closure of a subset of $Cn_{\mathsf{L}'}(\Gamma_0) \cup \ldots \cup Cn_{\mathsf{L}'}(\Gamma_m)$, in which L' may be identical to L, but also different from L. For many applications, this leads to a more adequate approach — see for example [31] and [13]. Moreover, even the formulation (2) is superior to (1). Indeed, it is odd that the priorities of the members of the different Γ_i are suppressed to the (informal) metalanguage. If different premises have really a different status, it seems desirable that this shows in the object language.

4. *The Role of Classical Logic*

The Dab-formulas mentioned in the two preceding sections were said to be *classical* disjunctions of abnormalities. If the lower limit logic does not contain classical disjunction, one adds it (using a symbol that is different from the ‘local’ disjunction). It is even advisable to go further. Let us say that a lower limit logic LLL contains CL iff there is a fragment $\mathcal{L}'$ of the language $\mathcal{L}$ of LLL such that, whenever Γ and A are formulas of $\mathcal{L}'$, then $\Gamma \vdash_{\mathsf{LLL}} A$ iff $\Gamma \vdash_{\mathsf{CL}} A$.

Let me clarify this by an example. Let $\sim$ be the standard negation of the standard CL-language $\mathcal{L}'$ and let PL be a paraconsistent logic in which all logical symbols of $\mathcal{L}'$ have the same meaning as in CL, except that $\sim$ is weaker in being paraconsistent. Suppose now that one extends $\mathcal{L}'$ to $\mathcal{L}$ by adding $\neg$ and giving it exactly the meaning that $\sim$ has in CL.[18] In the context of CL, $\neg A$ is equivalent to $\sim A$. In the context of PL, $\sim$ is the paraconsistent negation whereas $\neg$ is classical negation. With respect to the extended language, PL is still weaker than CL because, for example, $p \vee q, \sim p \vdash_{\mathsf{CL}} q$ whereas $p \vee q, \sim p \nvdash_{\mathsf{PL}} q$. However, PL contains CL in the following sense. Let $f(A)$ be the result of replacing every occurrence of $\sim$ in A by $\neg$ and let $f(\Gamma) = \{f(A) \mid A \in \Gamma\}$. Then obviously $\Gamma \vdash_{\mathsf{CL}} A$ iff $f(\Gamma) \vdash_{\mathsf{PL}} f(A)$.

In general, if LLL does not contain CL in this sense, then it is advisable to extend both the language and the lower limit logic with a set of new logical symbols to the effect that LLL contains CL — see [7] for an example in

[18] One way to obtain this result is first extending $\mathcal{L}'$ with $\bot$, characterized by the axiom $\bot \supset A$, and defining $\neg A =_{df} A \supset \bot$.

which all 'original' logical symbols are non-classical but have classical alternatives. This not only makes it possible to characterize the adaptive logic in the standard way described in the two preceding sections. It also greatly simplifies metatheoretic proofs.

Often this move has a merely technical effect, viz. in case the premises are formulated in a fragment of the language that does not contain all classical logic symbols. This is the case for the originally intended application contexts of most adaptive logics. Of course, nothing prevents one, when it makes sense with respect to the application context, to employ the full (extended) language in the proof or even in the premises. In the following two sections, I shall suppose that the lower limit logic contains CL in the sense of the previous paragraph, and that the logical symbols that occur in the text have the meaning they have in CL.

5. *The Semantics*

The semantics of all adaptive logics is defined in the same way in terms of the lower limit logic LLL, the set of abnormalities Ω and the strategy — let ULL be the upper limit logic defined by LLL and Ω. I shall start with *flat* adaptive logics.

$M \models A$ will denote that M assigns a designated value to A, in other words that M verifies A. If the semantics is two-valued — and it is shown in [27] that any semantic system may be rephrased in two-valued terms — then $M \models A$ comes to $v_M(A) = 1$. $M \models \Gamma$ will denote that M verifies all members of Γ.

For any LLL-model, we define its abnormal part:

Definition 1: $Ab(M) = \{A \in \Omega \mid M \models A\}$

Where $Dab(\Delta_1),\ Dab(\Delta_2),\ \ldots$ are the minimal Dab-consequences of the premise set Γ,[19] $U(\Gamma) = \Delta_1 \cup \Delta_2 \cup \ldots$ is the set of formulas that are unreliable with respect to Γ. Let AL^r and AL^m be the adaptive logics defined from LLL and Ω by the Reliability strategy and the Minimal Abnormality strategy respectively.

Definition 2: *A* LLL*-model M of Γ is* reliable *iff* $Ab(M) \subseteq U(\Gamma)$.

[19] The minimal Dab-consequences of Γ may be semantically defined in view of the soundness and completeness of LLL with respect to its semantics.

In other words, M is a reliable model of Γ iff all abnormalities verified by M are unreliable with respect to Γ. Intuitively, $U(\Gamma)$ comprises the abnormalities that, in view of the Reliability strategy, cannot be avoided if all members of Γ are supposed to be true. A reliable model is one that verifies no other abnormalities.

Definition 3: $\Gamma \vDash_{\mathsf{AL}^r} A$ *iff* A *is verified by all reliable models of* Γ.

Definition 4: *A* LLL*-model* M *of* Γ *is* minimally abnormal *iff there is no* LLL*-model* M' *of* Γ *such that* $Ab(M') \subset Ab(M)$.

In other words, M is a minimally abnormal model of Γ iff no other lower limit model of Γ is (set theoretically) less abnormal than M.[20]

Definition 5: $\Gamma \vDash_{\mathsf{AL}^m} A$ *iff* A *is verified by all minimally abnormal models of* Γ.

The special status of adaptive logics appears from their semantics. It does not make sense to say that a model is reliable or minimally abnormal, but only to say that a LLL-model is a reliable model (or a minimally abnormal model) *of* Γ.[21]

Let us now turn to *prioritized* adaptive logics. First remember that their set of abnormalities is a union of sets: $\Omega = \Omega_1 \cup \Omega_2 \cup \ldots$. Next, the lower limit logic LLL and the upper limit logic ULL are only different with respect to modal premises (premises of the form $\Diamond^i A$). In other words, if A and all members of Γ are closed formulas of the standard CL-language, then $\Gamma \vdash_{\mathsf{LLL}} A$ iff $\Gamma \vdash_{\mathsf{ULL}} A$. Finally, for every LLL-model M and for every priority level $i \in \mathbb{N} - \{0\}$, we define $Ab^i(M) =_{df} \{A \in \Omega^i \mid M \models A\}$.

There is a simple way to characterize the semantics. For any priority level $i \in \mathbb{N}$ (remark that zero is included here), we define a set $\mathcal{M}_i$ of selected LLL-models of Γ. Let $Dab^i(\Delta)$ $(i \geq 1)$ denote a Dab-formula $Dab(\Delta)$ with $\Delta \subseteq \Omega_i$. For each priority level $i \in \mathbb{N}$, $\mathcal{M}_i$ determines a set of *minimal* Dab^{i+1}*-consequences* of Γ. These are the minimal Dab^{i+1}-formulas that are verified by all members of $\mathcal{M}_i$. Where $Dab^i(\Delta_1)$, $Dab^i(\Delta_2), \ldots$ are the

[20] As I see it, this definition makes only sense if it can be demonstrated that, for any model M that is not minimally abnormal, there is a minimally abnormal model M' such that $Ab(M') \subset Ab(M)$ — see [8] and, for a generalization, [10].

[21] For stubborn readers: no LLL-model is reliable or minimally abnormal with respect to all premise sets, and only the ULL-models are reliable and minimally abnormal with respect to the empty set. For the same reason, adaptive logics have no valid formulas and no theorems of their own: $Cn_{\mathsf{AL}}(\emptyset) = Cn_{\mathsf{ULL}}(\emptyset)$ and the intersection of $Cn_{\mathsf{AL}}(\Gamma)$ for all Γ is $Cn_{\mathsf{LLL}}(\emptyset)$.

minimal Dab^i-consequences of Γ, $U^i(\Gamma) =_{df} \Delta_1 \cup \Delta_2 \cup \ldots$. The *reliable* models of Γ are obtained as follows, M being a variable for LLL-models:

$$\mathcal{M}_0^r =_{df} \{M \mid M \models \Gamma\}$$

and

$$\mathcal{M}_{i+1}^r =_{df} \{M \in \mathcal{M}_i^r \mid Ab^{i+1}(M) \subseteq U^{i+1}(\Gamma)\}\,.$$

M is a reliable model of Γ iff $M \in \mathcal{M}_0^r \cap \mathcal{M}_1^r \cap \ldots$. If there is a longest sequence of symbols $\Diamond$ in Γ and its length is n, then $U^i(\Gamma) = \emptyset$ for all $i > n$. Finally, $\Gamma \models_{\mathsf{AL}^r} A$ iff A is verified by every reliable model of Γ.

It is even simpler to obtain the *minimally abnormal* models of Γ:

$$\mathcal{M}_0^m =_{df} \{M \mid M \models \Gamma\}$$

and

$$\mathcal{M}_{i+1}^m =_{df} \{M \in \mathcal{M}_i^m \mid \text{for no } M' \in \mathcal{M}_i^m,\ Ab^{i+1}(M') \subset Ab^{i+1}(M)\}$$

M is a minimally abnormal model of Γ iff $M \in \mathcal{M}_0^r \cap \mathcal{M}_1^r \cap \ldots$. $\Gamma \models_{\mathsf{AL}^m} A$ iff A is verified by every minimally abnormal model of Γ.

The following are easily seen to obtain for flat as well as for prioritized adaptive logics. In general,

$$Cn_{\mathsf{LLL}}(\Gamma) \subseteq Cn_{\mathsf{AL}^r}(\Gamma) \subseteq Cn_{\mathsf{AL}^m}(\Gamma) \subseteq Cn_{\mathsf{ULL}}(\Gamma)\,.$$

If Γ is normal, then

$$Cn_{\mathsf{LLL}}(\Gamma) \subset Cn_{\mathsf{AL}^r}(\Gamma) = Cn_{\mathsf{AL}^m}(\Gamma) = Cn_{\mathsf{ULL}}(\Gamma)\,;$$

if Γ is abnormal, then, in all interesting cases,[22]

$$Cn_{\mathsf{LLL}}(\Gamma) \subset Cn_{\mathsf{AL}^r}(\Gamma) \subset Cn_{\mathsf{AL}^m}(\Gamma) \subset Cn_{\mathsf{ULL}}(\Gamma)\,.$$

[22] As Γ is abnormal, $Cn_{\mathsf{ULL}}(\Gamma)$ is the trivial set (the set of all formulas). The other consequence sets are non-trivial, except if $Cn_{\mathsf{LLL}}(\Gamma)$ is trivial (Reassurance Theorem), which for example is the case if Γ is itself the trivial set.

6. *The Dynamic Proof Theory*

Dynamic proofs differ from usual proofs in two respects. The first pertains to annotated proofs.[23] Apart from (i) a line number, (ii) a formula, (iii) the line numbers of the formulas from which the formula is derived, and (iv) the rule by which the formula is derived (the latter two are the justification of the line), dynamic proofs also contain (v) a *condition*. If, as our research group now prefers, the condition is a set of abnormalities, then the meaning of the condition may be understood as follows: the formula (second element of the line) is derived unless one of the elements of the condition is true — that is, provided the elements of the condition may be taken to be false on the premises.

The second difference with respect to usual proofs is that dynamic proofs are not only defined by a set of deduction rules, but also by a marking definition. The deduction rules allow one to add lines to the proof, the marking definition defines which lines are marked at a stage of the proof — I discuss "a stage of a proof" in detail later, but its meaning is intuitively clear: adding a new line brings the proof to its next stage.

The second element of a line is considered as derived from the premises at some stage of the proof iff the line is unmarked at that stage. A formula may be derived at one stage, not derived at a later one, and again derived at a still later one. In view of this, I later introduce a 'more stable' kind of derivability: final derivability.[24] Intuitively, this is reached when the dynamics of a proof has stopped. $\Gamma \vdash_{\mathsf{AL}} A$ will indicate that A is finally derivable from Γ. This notion is provably sound and complete with respect to the semantics and is proof-independent: in whichever way a specific proof from Γ starts off, if $\Gamma \vdash_{\mathsf{AL}} A$, then A can be finally derived in that specific proof.

Let us first have a look at the deduction rules. If they are formulated in generic format, they are identical for all adaptive logics in standard format. Let Γ be the set of premises as before.[25]

[23] Non-annotated dynamic proofs are sequences of formulas as usual, except that some formulas may be marked as the proof proceeds. The marking definitions for such proofs are more complex than those for annotated proofs, which I present below.

[24] Of course, "derivability at a stage" is itself a stable notion, but it is not proof-independent: if A is "derivable at a stage" from Γ, there is a proof from Γ and a stage s such that A is derived at stage s of that proof. Still, A need not be derivable at a stage in any extension of a different proof from Γ.

[25] The only rule that introduces non-empty conditions is RC. In other words, before RC is applied in a proof, the condition of every line will be $\emptyset$.

PREM If $A \in \Gamma$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) A, (iii) $-$, (iv) PREM, and (v) $\emptyset$.

RU If $A_1, \ldots, A_n \vdash_{\mathsf{LLL}} B$ and each of $A_1, \ldots, A_n$ occur in the proof on lines $i_1, \ldots, i_n$ that have conditions $\Delta_1, \ldots, \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, \ldots, i_n$, (iv) RU, and (v) $\Delta_1 \cup \ldots \cup \Delta_n$.

RC If $A_1, \ldots, A_n \vdash_{\mathsf{LLL}} B \vee Dab(\Theta)$ and each of $A_1, \ldots, A_n$ occur in the proof on lines $i_1, \ldots, i_n$ that have conditions $\Delta_1, \ldots, \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, \ldots, i_n$, (iv) RC, and (v) $\Delta_1 \cup \ldots \cup \Delta_n \cup \Theta$.

Where

$$A \quad \Delta$$

abbreviates that A occurs in the proof on the condition Δ, the rules may be phrased more transparently as follows:

PREM If $A \in \Gamma$:

$$\begin{array}{ll} \ldots & \ldots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \ldots, A_n \vdash_{\mathsf{LLL}} B$:

$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$$

RC If $A_1, \ldots, A_n \vdash_{\mathsf{LLL}} B \vee Dab(\Theta)$

$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

There is a striking correspondence between dynamic proofs and LLL-proofs. Suppose that one transforms each line

$$A \quad \Delta$$

from a dynamic proof into

$$A \vee Dab(\Delta)\,,$$

where "$\vee Dab(\emptyset)$" is defined as the empty string. It is easy enough to establish, by an obvious induction on the length of the proof, that the resulting sequence of formulas is a LLL-proof obtained by applications of PREM and

RU only. This result is extremely useful from a metatheoretic point of view. It also clarifies what is going on in a dynamic proof. The whole point of the proof format lies in the marking definition and its interpretation, viz. a line i at which A has been derived on the condition Δ justifies one in considering A as derived from the premises unless and until line i is marked.

We now turn to the *marking definitions*. I shall start with the definitions for *flat adaptive logics* — prioritized adaptive logics need special treatment. At any stage of the proof, zero or more Dab-formulas are derived on the empty condition. Given a dynamic proof, we shall say that $Dab(\Delta)$ is a *minimal* Dab*-formula* at stage s of the proof if, at that stage, $Dab(\Delta)$ occurs in the proof on the empty condition and, for any $\Delta' \subset \Delta$, $Dab(\Delta')$ does not occur in the proof on the empty condition. Where $Dab(\Delta_1)$, …, $Dab(\Delta_n)$ are the minimal Dab-formulas at stage s of the proof, $U_s(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$ is the set of unreliable formulas at stage s.[26] The underlying idea of the marking definition is clear. The present stage of the proof provides a certain understanding of the premises. If this understanding is correct, the formulas derived at unmarked lines are finally derived from the premises.

Definition 6: *Marking for Reliability: Line i is marked at stage s iff, where Δ is its condition, $\Delta \cap U_s(\Gamma) \neq \emptyset$.*

Marking for the Minimal Abnormality strategy is slightly more tiresome. Let $\Phi^\circ_s(\Gamma)$ be the set of all sets that contain one disjunct out of each minimal Dab-formula at stage s. Let $\Phi^\star_s(\Gamma)$ contain, for any $\varphi \in \Phi^\circ_s(\Gamma)$, the set $Cn_{\mathsf{LLL}}(\varphi) \cap \Omega$. Finally let $\Phi_s(\Gamma)$ contain those members of $\Phi^\star_s(\Gamma)$ that are not proper supersets of other members of $\Phi^\star_s(\Gamma)$.[27] The underlying idea of the marking definition is as before. If the proof at its present stage offers a correct understanding of the premises, then the unmarked formulas are final consequences of the premises whereas the marked formulas are not.

[26] So, where Ω is, for example, characterized by the form $\exists(A \wedge {\sim}A)$ and $\exists(A \wedge {\sim}A) \in U_s(\Gamma)$, $\exists(A \wedge {\sim}A)$ is said to be unreliable. Intuitively it is more attractive to say that A is unreliable in this case, and this way of speech was followed in most older papers. In line with this, it was there said that $A \in U_s(\Gamma)$, and this turns out less transparent for the general approach of the present paper.

[27] The underlying idea is most easily seen if one replaces the minimal Dab-formulas at stage s by the minimal Dab-consequences of Γ — this is the situation that is reached in a proof when its dynamics comes to an end. Define $\Phi(\Gamma)$ from these in the same way as $\Phi_s(\Gamma)$ is defined in the text from the minimal Dab-formulas at stage s. It can be proved that $\varphi \in \Phi(\Gamma)$ iff there is a minimally abnormal model M of Γ such that $Ab(M) = \varphi$ — see [6] for two specific cases and forthcoming work for the generalization of the proof.

Definition 7: *Marking for Minimal Abnormality: Line i is marked at stage s iff, where A derived on the condition Δ at line i, (i) there is no $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$, or (ii) for some $\varphi \in \Phi_s(\Gamma)$, there is no line at which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.*

In other words, a line at which A is derived on the condition Δ is *not* marked iff (i) there is a $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$ and (ii) for any $\varphi \in \Phi_s(\Gamma)$, there is a line at which A has been derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.[28]

The formulas *derived* from Γ *at a stage* of the proof are those that are derived at a line that is unmarked at that stage. As the proof proceeds, unmarked lines may be marked and vice versa. So, it is important that one defines a different, stable, kind of derivability:

Definition 8: *A is* finally derived *from Γ on line i of a proof at stage s iff (i) A is the second element of line i, (ii) line i is not marked at stage s, and (iii) any extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.*

Definition 9: $\Gamma \vdash_{\mathsf{AL}} A$ *(A is* finally AL-derivable *from Γ) iff A is finally derived on a line of a proof from Γ.*

Remark that these are definitions, and that they are not intended to be of any direct computational use — see below.

Let us now turn to *prioritized* adaptive logics. As mentioned before, the deduction rules PREM, RU and RC are as for flat adaptive logics. Their set of abnormalities is a union of sets: $\Omega = \Omega_1, \Omega_2, \ldots$. The easiest way to characterize their marking definitions is as follows. Suppose that the following line occurs unmarked in a proof at stage s

$$i \qquad Dab^i(\Delta) \qquad \ldots \quad \ldots \quad \Theta$$

and that the level of each abnormality in Θ is lower than i (that the length of sequences of the symbol $\lozenge$ is less than i). We shall then say that $Dab^i(\Delta)$ is a *minimal Dab^i-formula* at stage s of the proof.[29] For each priority level

[28] Suppose that $\Phi_s(\Gamma) = \Phi(\Gamma)$ — see the previous footnote. It follows that line i is *not* marked iff two conditions are fulfilled. The first is that all members of the condition of line i are falsified by some minimally abnormal model of Γ. This entails that A is verified by that minimally abnormal model. The second condition is that, for every minimally abnormal model M of Γ, the proof contains a line which shows that M verifies A.

[29] The underlying idea is that, as the line is unmarked, $Dab^i(\Delta)$ is considered as derived at stage s of the proof.

$i \in \mathbb{N} - \{0\}$, one defines $U^i_s(\Gamma)$ and $\Phi^i_s(\Gamma)$ from the minimal Dab^i-formulas at stage s in the same way as $U_s(\Gamma)$ and $\Phi_s(\Gamma)$ were defined for the flat case. Marking is governed by Definitions 6 and 7, *except that it proceeds stepwise*: at any stage, the definitions are first applied to mark lines in view of $U^1_s(\Gamma)$ (or $\Phi^1_s(\Gamma)$), next to mark lines in view of $U^2_s(\Gamma)$ (or $\Phi^2_s(\Gamma)$), and so on.[30] Final derivability is governed by Definitions 8 and 9 as for the flat case.

For flat adaptive logics as well as for prioritized adaptive logics, it can be shown that final derivability is sound and complete with respect to the adaptive semantics (for Reliability and Minimal Abnormality respectively).

Several points deserve some further clarification. Let us start with the notions of a proof and of a stage of a proof. By a proof[31] I mean a sequence of lines written according to a set of instructions, viz. the deduction rules.[32] As for usual proofs, these rules express permissions to add lines to the proof.

There is a 'deeper' account of the notion of a proof. On this account, a stage of a proof is a sequence S of lines and a proof is a sequence or chain Σ of stages. In all cases that interest us here, proofs start from stage zero, which is the empty sequence, in other words from the empty chain. The inference rules describe which chains are permissible. In all cases that interest us here, a stage is obtained by extending the previous stage, but possibly with the marks of its lines changed, with exactly one line.[33] Remark that the inference rules may also be seen as describing the permissible transformations on a chain Σ, and that a proof at a stage may be seen as a specific sequence of transformations, the first on stage zero, the second on the result of the first, and so on.

A chain Σ (or its last S) establishes that certain formulas are derivable at a stage from certain premises. For the dynamic proofs discussed in this paper,

[30] As any proof is finite, even the actual process by which the marks are added ends after finitely many steps.

[31] Some readers may be surprised that this notion needs any discussion at all. They should remember that the 'traditional' (Hilbertian) definition of a proof excludes proofs for non-monotonic logics and hence is bound to be too restrictive.

[32] There is no need to make the notion of a proof parasitic on the notion of a proof of a formula from a set of formulas. In this paper, I phrased the premise rule in such a way that it refers to a given set of premises. One may just as well phrase it as a rule permitting to introduce any formula. Of course, all the proof *establishes* is that some formulas are derivable from some premise set. This is the set Γ of formulas that are introduced in the proof by the premise rule. In the specific case of monotonic logics, the proof moreover establishes that those formulas are derivable from all supersets of Γ.

[33] If the number of lines of the previous stage is finite, it is natural to append the new line. However, there are some weird cases where we have to consider infinite proofs — see [6, §7] — in which case the new line is inserted.

these are the second elements of unmarked members of S. For Hilbertian proofs, they are the second elements of all members of S. That A is finally derived from Γ in S, or by the chain Σ that ends with S, is defined with respect to all permissible transformations on Σ, in other words with respect to all chains Σ' that extend the Σ.[34] Again, Hilbertian proofs are a border case in that one need not refer to the permissible transformations on Σ, but only to Σ itself, in order to establish that A is finally derived from Γ by the chain Σ.

So, final derivability is well defined. But is it possible to establish that some formula has been finally derived from a premise set? Next, in cases where it is impossible to establish this, what is the use of derivability at a stage?

As the consequence relations lack a positive test, there is no algorithm for establishing in general that A is finally derivable from Γ even if it is. Still, this does not prevent the existence of *criteria* that enable one to establish, for specific A and Γ that A has been finally derived from Γ in a given proof. Some criteria were presented in [5], [15] and [16], and more criteria may be derived from results presented in those papers. Unfortunately, most of these criteria are awfully complex and only transparent for people that are well acquainted with the dynamic proofs. Recently, work was started in terms of goal directed proofs. The idea is not to formulate a specific criterion, but rather to articulate a proof procedure that functions as a criterion. The proof procedure is applied to $\Gamma \vdash_{\mathsf{AL}} A$. Whenever the proof procedure stops, it establishes that $\Gamma \vdash_{\mathsf{AL}} A$ or that $\Gamma \nvdash_{\mathsf{AL}} A$. Preparatory work on the propositional fragment of CL is presented in [18] and some first results on the proof procedure for inconsistency-adaptive logics are presented in [11]. An interesting aspect of the proof procedure is that one may start 'exploring' the premise set Γ in terms of the proof format described in this paper and, after some formula A was derived on an unmarked line, switch to the dynamic proofs in order to establish whether A is finally derivable from Γ.

Let us now turn to the second question. What if no criterion enables one to conclude from a proof whether some formula is or is not finally derivable from the premise set? The answer or rather the answers to this question are presented in [5]. Roughly, the answers go as follows. First, there is a characteristic semantics for derivability at a stage. Next, it can be shown that, as a dynamic proof proceeds, the insight in the premises provided by the proof never decreases and may increase. In other words, derivability at a stage provides an estimate for final derivability, and, as the proof proceeds, this estimate may become better, and never becomes worse. In view of all this,

[34] One might define a proof A from Γ as this set of permissible transformations on Σ. However, I will continue to reserve "proof" for a proof at a stage.

derivability at a stage gives one exactly what one might expect, viz. a fallible but sensible estimate of final derivability. At any stage of the proof, one has to decide (obviously on the basis of pragmatic considerations) whether one will continue the proof or will act on the basis of present insights. This is fully in line with present-day views on rationality. A different matter is whether the proof is carried out in an efficient way, that is: efficient with respect to obtaining a reliable (but fallible) estimate of final derivability. The goal directed proofs mentioned in the previous paragraph offer means to obtain efficient proofs, but clearly more research on this problem is desirable.

Some purists will prefer to read the dynamic proofs of adaptive logics as lower limit proofs in disguise, and will at best accept that an adaptive consequence is derivable from the premises if its derivability is warranted by a criterion as meant two paragraphs ago. In doing so, they clearly recognize the sense of adaptive logics and depart from traditional lines that purer purists would stick to. Nevertheless, they are still mistaken. Humans do not postpone judgement indefinitely, not even in logical matters. Humans decide on provisional and fallible insights, even in logical matters. And, in view of the facts and of present best insights, humans cannot afford to behave differently. That fallible forms of reasoning can be caught in a formally stringent frame and that their properties can be established by formally decent means — see [6] and many other papers — are facts that no purist can deny.

Before ending this section, let me briefly mention *direct dynamic proofs*. Such proofs were first presented in [19]. They do not follow the standard adaptive format, but proceed in terms of more common formulations — in general in terms of the CL-language. An example of a direct dynamic proof format is presented and compared to the connected official adaptive format in [25] and [31].

7. *Corrective and Ampliative Adaptive Logics*

Many logicians believe that there is a standard of deduction — for most CL, for others a relevant logic, a paraconsistent logic, or perhaps intuitionistic logic. Even those who (like me) think that logics are basically instruments that are more or less suitable in specific circumstances, will adopt some logic as the standard of deduction in a specific situation.

Suppose that we are dealing with a context in which CL is taken as the standard of deduction. If the lower limit logic of AL is CL (or, for example, a modal extension of CL), it is said that AL is *ampliative*. This is the case for inductive generalization (without background knowledge), for compatibility, etc. If the lower limit logic is weaker than CL (in the sence of Section 4), as is the case for inconsistency-adaptive logics, the adaptive logic is called *corrective* — the theory was intended to be interpreted in terms of CL, but turned out to be inconsistent and hence is interpreted as consistently as possible.

Not all adaptive logics that are corrective (with respect to CL) are also inconsistency-adaptive. Paraconsistent logics allow for negation-gluts and inconsistency-adaptive logics interpret a premise set as consistently as possible — that is, they reduce the gluts in as far as logical means allow for such a reduction. It is just as sensible, given certain premises, to start from a lower limit logic that allows for gaps or for gluts as well as gaps with respect to negation, or that allows for gluts or gaps or both with respect to any other logical symbol. In some cases, it is more suitable to apply a lower limit logic that allows for ambiguities in the non-logical symbols. The corresponding adaptive logics will interpret premise sets as normally as possible with respect to those abnormalities — see [28] for an ambiguity-adaptive logic and [7] for combinations, including the extreme lower limit zero logic CL0.

From a technical viewpoint, the distinction between corrective and ampliative adaptive logics is immaterial. The formal characterization is always the same, as may be seen from the fact that the distinction was not even mentioned in the previous sections. In other words, the distinction reduces to the user's choice of a standard of deduction (in a specific situation). Of course, this choice is quite important as it has effects on the justification for applying a specific adaptive logic in given circumstances.

8. *Some Warnings*

As more adaptive logics were studied, their systematization had to be modified several times. There is no warrant that the characterizations presented in this paper are final.

The second warning is that, although the properties of adaptive logics are very different from those of more common logics, there is no reason not to apply the usual metatheoretic means to study these properties. This is what the whole adaptive programme is about: to arrive at a formally decent characterization of a specific non-standard kind of logics and of their metatheoretic properties.

There is no general proof, at this moment, that all consequence relations for which there is no positive test may be turned into an adaptive logic and hence may be given a dynamic proof theory. However, as the study of adaptive logics proceeded, more and more such consequence relations were mastered — see [4], [9], [12], [19], [22], [29], [30], [32] — and we did not come across a sensible consequence relation that appeared beyond reach. Apart from defining and studying further adaptive logics, the most urgent matter at this moment seems to be the generalization of the metatheory in terms of the general characterizations that are presented in this paper.

The idea behind the adaptive logic programme is to return to the original aim that underlies the study of logic: to explicate human reasoning. The last hundred and fifty years of the history of logic were a success story. Still, time and again, people pursuing that original aim — modal logicians, relevant logicians and paraconsistent logicians are just three examples — had to fight those who, impressed by the established results, wanted to restrict logic to them. The empirical study of forms of reasoning is useful, relevant and important. This, however, is not what the fight is about. The fight is about the separation between sound and fallacious reasoning. Nevertheless, once this separation will be correctly made, a much larger part of actual reasoning will presumably turn out to be sound.[35]

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