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THE PROOF THEORY OF COMPARATIVE LOGIC

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1. *Introduction*

Comparative logic, first introduced by Ettore Casari (1987; 1989; 1997) in order to account for some features of natural language comparison within a suitable logical framework, is a paraconsistent logic which can be of interest to several categories of researchers:

- to the *general logician*, for its deep connections with linear and fuzzy logics (Casari, 1989; 1997; Paoli, 1998; 2000);
- to the *paraconsistent logician*, since it has such well-motivated inconsistent extensions as Meyer's and Slaney's *Abelian logic* (the logic of Abelian lattice-ordered groups: Meyer and Slaney, 1989; Casari, 1989; Paoli, 200+*a*);
- to the *algebraist*, as its characteristic models (lattice-ordered pregroups) yield a plausible solution — at least for the Abelian case — to a well-known problem posed by Birkhoff (1940), who suggested to develop a common abstraction of Boolean algebras and lattice-ordered groups (Casari, 1989; 1991; Minari, 200+; Paoli, 2000);
- to the *linguist*, for its applications to the semantics of adjectives and comparison (Casari, 1987; 1997; Paoli, 1999);
- to the *philosopher*, for its applications to the sorites paradox (Paoli, 200+*b*).

Comparative logic has been intensively investigated over the last 15 years (see also Minari, 1988; Paoli, 1996), especially from the semantical viewpoint. On the proof-theoretical side, however, so far the only available system was the Hilbert-style calculus originally devised by Casari (1989), whose postulates resemble rather closely the defining conditions of lattice-ordered pregroups and thus do not give much additional information on the proof-theoretical structure of our logic.

The Gentzen-style calculus hereafter introduced clearly shows that comparative logic is a fully legitimate member of the family of *substructural logics* (see e.g. Restall, 2000; Paoli, 2002). We shall highlight a number of similarities and differences that it bears to other logics belonging to the same class, such as linear or relevance logics, and we shall present Abelian logic as an axiomatic extension of our calculus. As regards the system itself, the only result hitherto obtained is a *negative* one — failure of cut elimination — but the way is now paved for further improvements and investigations on the proof theory of Casari's logic.

We shall proceed as follows. In §2 we shall summarize some of Casari's results on comparative logic, in order to keep the paper self-contained; in §3 we shall introduce and investigate our sequent calculus.

2. *The Hilbert-style calculus for comparative logic*

Let £ be a propositional language containing a denumerable stock VAR of variables $(p_1, p_2, ...)$ and the connectives $\neg, \rightarrow, \wedge, \vee$. To cut down the number of parentheses, we follow the convention according to which $\neg$ binds more strongly than either $\wedge$ or $\vee$, which in turn bind more strongly than $\rightarrow$. The notion of *well-formed formula* (wff) is defined as usual. We shall use lowercase letters $(p, q, ...)$ as metavariables for propositional variables, and uppercase letters $(A, B, ...)$ as metavariables for wffs whatsoever of £. The calculus HC (Casari, 1989) has the following axiom schemata and rules:

A1. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
A2. $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$
A3. $(A \rightarrow A) \rightarrow (B \rightarrow B)$
A4. $\neg\neg A \rightarrow A$
A5. $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$
A6. $\neg(A \rightarrow A) \rightarrow (A \rightarrow A)$
A7. $(\neg(A \rightarrow A) \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A)$
A8.(*i*) $A \wedge B \rightarrow A$ A8.(*ii*) $A \wedge B \rightarrow B$
A9. $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
A10.(*i*) $A \rightarrow A \vee B$ A10.(*ii*) $B \rightarrow A \vee B$
A11. $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$
R1. $A,\ A \rightarrow B \Rightarrow B$
R2. $A,\ B \Rightarrow A \wedge B$

If we delete A3, A6 and A7 from the above list and add: (A12) $A \rightarrow A$ to it, we get a Hilbert-style calculus for the constant-free fragment of subexponential linear logic in the $\{\neg, \wedge, \vee, \rightarrow\}$-vocabulary (cp. Avron, 1988).

By deleting A3, A6 and A7 from HC and adding A12 to it, as well as:

A13. $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
A14. $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee C$
A15. $(A \rightarrow \neg A) \rightarrow \neg A$

we get the constant-free fragment of the relevant logic R (see e.g. Dunn, 1986). Notice that A7 can now be recovered as a theorem. Remark also that A6, although not valid in R, is a theorem of the "semi-relevant" logic RM.

By adding: (A16) $A \to (B \to A)$ to HC, we get the constant-free fragment of subexponential affine linear logic, hereafter referred to as HALL. Given A16, the axioms A3, A6 and A7 and the rule R2 become superfluous. Moreover, the axioms A9 and A11 can also be taken in their exported versions.

By adding: (A17) $\neg(A \to A)$ to HC, we get an equivalent reformulation (hereafter referred to as HA) of Meyer's and Slaney's (1989) Abelian logic.

Let us now have a look at the algebraic semantics for HC and some of its extensions. A *lattice-ordered pregroup* (or, for short, an *l-pregroup*) is a structure $\mathcal{P} =< P, +, -, 0, \sqcap, \sqcup >$s.t.:

P1. $< P, + >$ is an Abelian semigroup;
P2. $< P, \sqcap, \sqcup >$ is a lattice;
P3. $- - x = x$;
P4. $0 + 0 = 0$;
P5. $x + -x = 0$;
P6. $x \leq y$ iff $0 \leq -x + y$.

where "$\leq$" denotes the induced lattice ordering of the reduct $< P, \sqcap, \sqcup >$. *L*-pregroups owe their name to the fact that they possess nearly all the properties of Abelian *l*-groups, except that the two roles of "collecting the opposites" ($x + -x = 0$) and of being a neutral element for addition ($x + 0 = x$) are played by two not necessarily identical elements, respectively 0 and its opposite -0. It is easily seen that Abelian *l*-groups are exactly those *l*-pregroups which satisfy: (P7) $0 = -0$.

Classical residuated lattices (Kowalski and Ono, 2001; also called *l-zeroids* by Casari, 1997, and *l*-L_0-*algebras* by Grishin, 1982) are those *l*-pregroups which satisfy: (P8) $x + 0 = 0$, or, equivalently, the bounded *l*-pregroups with 0 as a top element and -0 as a bottom element. It is well-known that Boolean algebras are the idempotent classical residuated lattices; remark, however, that in the literature on bounded algebras the element "0" is usually referred to by "1", and the element "-0" by "0".

LEMMA 1. In every *l*-pregroup: (i) $x \leq x + 0$; (ii) $x \leq y$, $x' \leq y' \Rightarrow x + x' \leq y + y'$.

Proof. See Casari (1989). □

A *model* is an ordered pair $\mathcal{M} =< \mathcal{P}, \rho >$ s.t. $\mathcal{P} =< P, +, -, 0, \sqcap, \sqcup >$ is an *l*-pregroup and ρ is a homomorphism from the free algebra of formulae of £ to $\mathcal{P}$ which extends the arbitrary mapping ρ^* :VAR→ P in such a way that:

$\rho(p) = \rho^*(p)$;
$\rho(\neg A) = -\rho(A)$; $\rho(A \to B) = -\rho(A) + \rho(B)$;
$\rho(A \wedge B) = \rho(A) \sqcap \rho(B)$; $\rho(A \vee B) = \rho(A) \sqcup \rho(B)$.

A wff A is said to be *true in* $\mathcal{M}$ iff $0 \leq \rho(A)$; it is called C*-valid* (in symbols, $\models_C A$) iff it is true in every model. By definition, an A*-valid* formula (in symbols, $\models_A A$) is a formula which is true in any model whose first projection is an Abelian *l*-group, and an ALL*-valid* formula (in symbols, $\models_{ALL} A$) is a formula which is true in any model whose first projection is a classical residuated lattice.

Casari (1989) proved a strong completeness result for HC, HA and HALL w.r.t. the above semantics. For our present purposes, however, the following theorem will suffice:

THEOREM 1. (i) $\vdash_{\mathsf{HC}} A$ iff $\models_C A$; (ii) $\vdash_{\mathsf{HA}} A$ iff $\models_A A$; (iii) $\vdash_{\mathsf{HALL}} A$ iff $\models_{ALL} A$.

Proof. See Casari (1989). □

3. *A sequent calculus for comparative logic*

Let $\Gamma, \Delta, ...$ be finite (possibly empty) multisets of wffs of £. The calculus GC is axiomatized by the following postulates:

(Ax) $A \Rightarrow A$

(Cut) $\dfrac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$

(BW) $\dfrac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta, A}$

(BC) $\dfrac{A, A, \Gamma \Rightarrow \Delta, A, A}{A, \Gamma \Rightarrow \Delta, A}$

(L¬) $\dfrac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}$

(R¬) $\dfrac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$

(L→*) $\dfrac{\Gamma \Rightarrow \Delta, A \quad \Pi \Rightarrow \Sigma}{A \to B, \Gamma^*, \Pi^* \Rightarrow \Delta, \Sigma}$

(R→) $\dfrac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$

(L∧) $\dfrac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$ $\dfrac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$

(R∧) $\dfrac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$

(L∨) $\dfrac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}$

(R∨) $\dfrac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}$ $\dfrac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}$

In L→*: (i) $B \in \Gamma \cup \Pi$; (ii) if $B \in \Pi$, then $\Gamma^* = \Gamma$, $\Pi^* = \Pi - \{B\}$; (iii) if $B \notin \Pi$, then $\Gamma^* = \Gamma - \{B\}$, $\Pi^* = \Pi$. (Remember that all of these operations are *multi*set-theoretical, not set-theoretical.)

The most awkward rule in GC is undoubtedly L→*. You can think of it as a generalization of the usual L→ rule:

(L→) $\dfrac{\Gamma \Rightarrow \Delta, A \qquad B, \Pi \Rightarrow \Sigma}{A \to B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$

to the effect that B is allowed to occur in the antecedent of *either* premiss of the inference.

Now consider the following rules:

(BW★) $\dfrac{\Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$ (BC★) $\dfrac{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma, \Sigma \qquad \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$

LEMMA 2. The rules BW★ and BC★ are derivable in GC$-\{$Cut$\}$.

Proof. Easy induction on the length of the proof of $\Pi \Rightarrow \Sigma$ in GC$-\{$Cut$\}$. □

The fact that our structural rules BW and BC have a "balanced" character yields a nice structural property for the $\{\neg, \to\}$-fragment of GC. In order to see it, however, we need an auxiliary notion. The *p-count* $c(p; A)$ of a formula A (cp. van Benthem, 1991) is a function whose arguments are ordered pairs made up by a variable and a wff of £, and whose values are integers. It is defined as follows:

$c(p;p) = 1;$ $\quad c(p;q) = 0$, for $p \neq q$;
$c(p;\neg A) = -c(p;A);$ $\quad c(p;A \to B) = c(p;B) - c(p;A).$

The p-count $c(p;\Gamma)$ of a multiset of formulae Γ is obtained by defining $c(p;A_1,...,A_n)$ as $\sum_{i \leq n} c(p;A_i)$, and by setting $c(p,\Gamma) = 0$ if Γ is empty.

LEMMA 3. If $\Gamma \Rightarrow \Delta$ is provable in the $\{\neg, \to\}$-fragment of GC, then for every variable p in VAR, $c(p,\Gamma) = c(p,\Delta)$.

Proof. Induction on the proof of $\Gamma \Rightarrow \Delta$ in the $\{\neg, \to\}$-fragment of GC. Just a few cases of the induction step:
(*Ad* BW). $c(p;A,\Gamma) = c(p;A) + c(p;\Gamma)$ =(IH) $c(p;A) + c(p;\Delta) = c(p;A,\Delta)$.
(*Ad* BC). $c(p;A,\Gamma) = c(p;A)+c(p;\Gamma)+c(p;A)-c(p;A)$ = (IH) $c(p;A)+c(p;\Delta) = c(p;A,\Delta)$.
(*Ad* L→*). $c(p;A \to B,\Gamma^*,\Pi^*) = c(p;A \to B)+c(p;\Gamma^*)+c(p;\Pi^*) = c(p;B)-c(p;A)+c(p;\Gamma^*)+c(p;\Pi^*)$ = (IH) $c(p;\Delta)+c(p;\Sigma) = c(p;\Delta,\Sigma)$.
□

Now, let us examine two extensions of GC: GALL, which is obtained by adding to GC the two rules of weakening:

(LW) $\dfrac{\Gamma \Rightarrow \Delta}{A,\Gamma \Rightarrow \Delta}$ (RW) $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$

and GA, which is obtained by adding to GC the empty sequent:

(Λ) $\Rightarrow$

A more compact formulation of GA can be found in Paoli (200+a).

It can be easily seen that GALL coincides with the sequent calculus for subexponential affine linear logic. In fact:

LEMMA 4. In GALL: (i) the rule L→* can be replaced by the usual rule L→ of left introduction for implication; (ii) the rules BW and BC are derivable.

Proof. (i) In the presence of the weakening rules,

$$\frac{B,\Gamma \Rightarrow \Delta, A \quad \Pi \Rightarrow \Sigma}{A \to B, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \ (\text{L}\to^*) \qquad \text{becomes}$$

$$\frac{\Pi \Rightarrow \Sigma}{A \to B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\quad \text{(several LW, RW)}$$

and thus L→* collapses onto L→.
(ii) BW is trivially derivable in GALL. As far as BC is concerned,

$$\frac{A, A, \Gamma \Rightarrow \Delta, A, A}{A, \Gamma \Rightarrow \Delta, A}\ \text{(BC)} \quad \text{becomes} \quad \frac{A \Rightarrow A}{A, \Gamma \Rightarrow \Delta, A}\quad \text{(several LW, RW)}$$

□

LEMMA 5. In GA: (i) the rule BC is derivable; (ii) the postulates Λ, L→* can be replaced by the rule:

$$\frac{B, \Gamma \Rightarrow \Delta, A}{A \to B, \Gamma \Rightarrow \Delta}\ (\text{L}{\to}{**})$$

Proof. (i) In fact, the inference:

$$\frac{A, A, \Gamma \Rightarrow \Delta, A, A}{A, \Gamma \Rightarrow \Delta, A}\ \text{(BC)} \qquad \text{becomes:}$$

$$\frac{\dfrac{A \Rightarrow A}{\Rightarrow A \to A}\ (\text{R} \to) \qquad \dfrac{A, A, \Gamma \Rightarrow \Delta, A, A \qquad \Rightarrow}{A \to A, A, \Gamma \Rightarrow \Delta, A}\ (\text{L} \to *)}{A, \Gamma \Rightarrow \Delta, A}\ \text{(Cut)}$$

(ii) The rule L→** is derivable in GA:

$$\frac{B, \Gamma \Rightarrow \Delta, A \qquad \Rightarrow}{A \to B, \Gamma \Rightarrow \Delta}\ (\text{L}{\to}{*})$$

Conversely, it is easy to see that Lemma 2 holds for GA too. Hence, Λ and L→* are derivable given L→**:

$$\frac{\dfrac{\Gamma \Rightarrow \Delta, A \qquad \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, A}\ (\text{BW}^\star)}{A \to B, \Gamma^*, \Pi^* \Rightarrow \Delta, \Sigma}\ (\text{L} \to **)$$

$$\cfrac{\cfrac{A \Rightarrow A}{\Rightarrow A \to A}\,(\mathrm{R}\to) \qquad \cfrac{A \Rightarrow A}{A \to A \Rightarrow}\,(\mathrm{L}\to **)}{\Rightarrow}\,(\mathrm{Cut}) \qquad \Box$$

In many relevance logics, the addition of the *Mingle* rules (Dunn, 1986):

$$(\mathrm{LM}) \quad \frac{A, \Gamma \Rightarrow \Delta}{A, A, \Gamma \Rightarrow \Delta} \qquad\qquad (\mathrm{RM}) \quad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A, A}$$

sometimes also called *anticontraction* rules (Avron, 1991) or *duplication* rules (Dosen, 1988) yields weaker systems than those obtained by adding unrestricted weakening rules. This is not the case for GC. In fact:

LEMMA 6. GC + {LM, RM} $= \mathsf{GALL}$.

Proof. The left-to-right inclusion is trivial. From right to left:

$$\cfrac{\cfrac{A \Rightarrow A}{\Rightarrow A \to A}\,(\mathrm{R}\to) \qquad \cfrac{\cfrac{A \Rightarrow A}{A, A \Rightarrow A}\,(\mathrm{LM}) \qquad \Gamma \Rightarrow \Delta}{A \to A, A, \Gamma \Rightarrow \Delta}\,(\mathrm{L}\to *)}{A, \Gamma \Rightarrow \Delta}\,(\mathrm{Cut})$$

and similarly for RW. □

The next thing to do is proving the equivalence of our sequent calculi with the Hilbert-style formalisms of §2.

THEOREM 2. (i) If $\vdash_{\mathsf{HC}} A$, then $\vdash_{\mathsf{GC}} \Rightarrow A$; (ii) if $\vdash_{\mathsf{HA}} A$, then $\vdash_{\mathsf{GA}} \Rightarrow A$; (iii) if $\vdash_{\mathsf{HALL}} A$, then $\vdash_{\mathsf{GALL}} \Rightarrow A$.

Proof. (i) Induction on the length of the proof of A in HC. Some examples:

$$\cfrac{\cfrac{A \Rightarrow A \qquad B \Rightarrow B}{A \to A, B \Rightarrow B}\,(\mathrm{L}\to *)}{\Rightarrow (A \to A) \to (B \to B)}\,(\text{several R}\to)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, A \Rightarrow A, A}\,(\mathrm{BW})}{A \Rightarrow A \rightarrow A, A}\,(\mathrm{R} \rightarrow)}{\neg(A \rightarrow A), A \Rightarrow A}\,(\mathrm{L}\neg)}{\Rightarrow \neg(A \rightarrow A) \rightarrow (A \rightarrow A)}\,(\text{several R} \rightarrow)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A \rightarrow A, A \Rightarrow A}\,(\mathrm{L} \rightarrow *) \quad A \Rightarrow A}{A \rightarrow A, A \rightarrow A, A \Rightarrow A}\,(\mathrm{L} \rightarrow *)}{A \rightarrow A, A \Rightarrow A, \neg(A \rightarrow A)}\,(\mathrm{R}\neg) \quad A \Rightarrow A}{\neg(A \rightarrow A) \rightarrow (A \rightarrow A), A, A \Rightarrow A, A}\,(\mathrm{L} \rightarrow *)}{\neg(A \rightarrow A) \rightarrow (A \rightarrow A), A \Rightarrow A}\,(\mathrm{BC})}{\Rightarrow (\neg(A \rightarrow A) \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A)}\,(\text{several R} \rightarrow)$$

(ii) As to A17,

$$\dfrac{\dfrac{A \Rightarrow A \qquad \Rightarrow}{A \rightarrow A \Rightarrow}\,(\mathrm{L} \rightarrow *)}{\Rightarrow \neg(A \rightarrow A)}\,(\mathrm{R}\neg)$$

(iii) Well-known, in the light of Lemma 4. □

THEOREM 3. (i) If $\vdash_{\mathsf{GC}} \Rightarrow A$, then $\vdash_{\mathsf{HC}} A$; (ii) if $\vdash_{\mathsf{GA}} \Rightarrow A$, then $\vdash_{\mathsf{HA}} A$; (iii) if $\vdash_{\mathsf{GALL}} \Rightarrow A$, then $\vdash_{\mathsf{HALL}} A$.

Proof. By Theorem 1, it is enough to prove: (i′) if $\vdash_{\mathsf{GC}} \Rightarrow A$, then $\models_C A$; (ii′) if $\vdash_{\mathsf{GA}} \Rightarrow A$, then $\models_A A$; (iii′) if $\vdash_{\mathsf{GALL}} \Rightarrow A$, then $\models_{ALL} A$.

To this purpose, we introduce the concept of *extended model*. An extended model is an ordered pair $\mathcal{M} = < \mathcal{P}, \rho >$ which is exactly like a model, except for the fact that the definition of ρ is extended to the case of sequents as follows:

$\rho(A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m) = -\rho(A_1) + \ldots + -\rho(A_n) + \rho(B_1) + \ldots + \rho(B_m)$;
$\rho(\Rightarrow) = -0$.

The sequent $\Gamma \Rightarrow \Delta$ is said to be *true in* $\mathcal{M}$ iff $0 \leq \rho(\Gamma \Rightarrow \Delta)$; it is called *C-valid* (in symbols, $\models_C \Gamma \Rightarrow \Delta$) iff it is true in every extended model. The concepts of A-validity and ALL-validity are adapted accordingly.

Now we can prove that: (i″) if $\vdash_{\mathsf{GC}} \Gamma \Rightarrow \Delta$, then $\models_C \Gamma \Rightarrow \Delta$; (ii′) if $\vdash_{\mathsf{GA}} \Gamma \Rightarrow \Delta$, then $\models_A \Gamma \Rightarrow \Delta$; (iii′) if $\vdash_{\mathsf{GALL}} \Gamma \Rightarrow \Delta$, then $\models_{ALL} \Gamma \Rightarrow \Delta$.

Once this is done, (i′), (ii′) and (iii′) trivially follow for $\Gamma = \emptyset$, $\Delta = \{A\}$. The proof of (iii″) is well-known. (i″) and (ii″) are proved by induction on the length of the proof of $\Gamma \Rightarrow \Delta$ in GL (resp. GA). Some cases:

(*Ad* Λ). By definition, $\rho(\Rightarrow) = -0 = 0$ in any Abelian l-group.

(*Ad* BW). By induction hypothesis $0 \leq \rho(\Gamma \Rightarrow \Delta)$, whence by P5, P2 and Lemma 1.(i), $0 \leq \rho(\Gamma \Rightarrow \Delta) + 0 = \rho(\Gamma \Rightarrow \Delta) + -\rho(A) + \rho(A) = \rho(A, \Gamma \Rightarrow \Delta, A)$.

(*Ad* BC). Let $\rho(\Gamma \Rightarrow \Delta) = x, \rho(A) = y$. By induction hypothesis, P4 and P5, $0 \leq x + (-y + y) + (-y + y) = x + 0 + 0 = x + 0 = x + -y + y = \rho(A, \Gamma \Rightarrow \Delta, A)$.

(*Ad* L → *). Let $\rho(\Gamma \Rightarrow \Delta) = x, \rho(\Pi \Rightarrow \Sigma) = y, \rho(A) = z, \rho(B) = w$. We must distinguish two cases. If the application of L → * under consideration is:

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Pi \Rightarrow \Sigma}{A \to B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

then by induction hypothesis $0 \leq x + z$, i.e. (P3, P6) $-z \leq x$, and $0 \leq -w + y$, i.e. (P6) $w \leq y$. By Lemma 1.(ii), $-z + w \leq x + y$, whence by P6 $0 \leq x + y + -(-z + w) = \rho(A \to B, \Gamma, \Pi \Rightarrow \Delta, \Sigma)$.

On the other hand, if the application of L → * has the form:

$$\frac{B, \Gamma \Rightarrow \Delta, A \quad \Pi \Rightarrow \Sigma}{A \to B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

then by induction hypothesis $0 \leq -w + x + z$, i.e. (P6) $w \leq x + z$, and $0 \leq y$. By Lemma 1.(ii) and P5, we get $x + 0 \leq x + y$ from the latter disequality and $w + -z \leq x + z + -z = x + 0$ from the former. By transitivity, then, $w + -z \leq x + y$, i.e. (P6) $0 \leq x + y + -(-z + w) = \rho(A \to B, \Gamma, \Pi \Rightarrow \Delta, \Sigma)$. □

THEOREM 4. GC is not cut-free.

Proof. First of all, recall the following fact. Were the Cut rule eliminable in GC, the system GC itself would be a conservative extension of its own $\{\neg, \to\}$-fragment, for suppose otherwise. Then, since the rules L ∧, L ∨ introduce into any proof where they are used at least a formula which doesn't belong to the $\{\neg, \to\}$-fragment, GC would not have the subformula property and thus would not be cut-free.

Now, consider the proof in Table 1.

Is the sequent $q \Rightarrow p \vee \neg p, q$ provable without using Cut? To answer this question, we introduce an auxiliary concept. If $\mathcal{D}$ is a proof-tree in GC and $\phi =< S_1, \ldots, S_n >$ is any of its branches, then for $1 \leq i \leq j \leq n$ the

$$
\dfrac{\dfrac{p \vee (p \to p) \Rightarrow p \vee (p \to p)}{\Rightarrow p \vee (p \to p) \to p \vee (p \to p)}\ (\mathrm{R}\to)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{p, p \Rightarrow p, p}\ (\mathrm{BW})}{p \Rightarrow p, p \to p}\ (\mathrm{R}\to)}{p \Rightarrow p \vee \neg p, p \to p}\ (\mathrm{R}\vee)}{p \Rightarrow p \vee \neg p, p \vee (p \to p)}\ (\mathrm{R}\vee)}{q, p \Rightarrow p \vee \neg p, p \vee (p \to p), q}\ (\mathrm{BW})
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{p, p \Rightarrow p, p}\ (\mathrm{BW})}{p \Rightarrow \neg p, p, p}\ (\mathrm{R}\neg)}{p \Rightarrow p \vee \neg p, p, p}\ (\mathrm{R}\vee)}{p \Rightarrow p \vee \neg p, p \vee (p \to p), p}\ (\mathrm{R}\vee) \quad q \Rightarrow q}{q, p \to p \Rightarrow p \vee \neg p, p \vee (p \to p), q}\ (\mathrm{L}\to *)}{q, p \vee (p \to p) \Rightarrow p \vee \neg p, p \vee (p \to p), q}\ (\mathrm{L}\vee) \quad q \Rightarrow q}{q, q, p \vee (p \to p) \to p \vee (p \to p) \Rightarrow p \vee \neg p, q, q}\ (\mathrm{L}\to *)}{\dfrac{q, q \Rightarrow p \vee \neg p, q, q}{q \Rightarrow p \vee \neg p, q}\ (\mathrm{BC})}\ (\mathrm{Cut})
$$

Table 1.

sequence $\phi' =< S_i, S_{i+1}, \ldots, S_{j-1}, S_j >$ is called a *maximal contraction segment* iff:

(i) for $i \leq p < j$, the sequent S_{p+1} is obtained from S_p by an application of BC;

(ii) S_i (and, if $j \neq n, S_{j+1}$) is obtained by application of a rule which is not BC (possibly the 0-premiss rule Ax).

Inspecting the previous definition, it is easily seen that for any sequent $\mathcal{S}$ in the proof-tree $\mathcal{D}$ there is exactly one maximal contraction segment ending with $\mathcal{S}$ in $\mathcal{D}$. Now we shall derive a contradiction from the assumption that there exists a proof $\mathcal{D}$ of $q \Rightarrow p \vee \neg p, q$ in GC – {Cut}, arguing by induction on the number m of applications of BC in $\mathcal{D}$.

($m = 0$). The sequent $q \Rightarrow p \vee \neg p, q$ is not an axiom. Hence it was obtained either by BW from $\Rightarrow p \vee \neg p$ of by R $\vee$ from $q \Rightarrow p, q$ (or, for that matter, $q \Rightarrow \neg p, q$). In the former case, $\Rightarrow p \vee \neg p$ could have been obtained only by R $\vee$ from $\Rightarrow p$ or $\Rightarrow \neg p$, which is impossible. In the latter case, $q \Rightarrow p, q$ could have been obtained only by BW from $\Rightarrow p$, which is impossible as well.

($m > 0$). If the sequent $q \Rightarrow p \vee \neg p, q$ was not obtained by BC, we argue as in the basis case or as in Case 2 below. Otherwise, let ϕ be the maximal contraction segment ending with $q \Rightarrow p \vee \neg p, q$ in $\mathcal{D}$ (by $(k)q$ we denote the multiset $q, \ldots, q$ (k times)):

$$\cfrac{\begin{array}{c}(k)q \Rightarrow p \vee \neg p, (k)q \\ \vdots \\ q, q \Rightarrow p \vee \neg p, q, q\end{array}}{q \Rightarrow p \vee \neg p, q}$$

ϕ must necessarily look like this: in fact, any application of BC in ϕ can only have q as a principal formula, since $p \vee \neg p$ only occurs on the right side of the arrow.

As ϕ is maximal, $(k)q \Rightarrow p \vee \neg p, (k)q$ could have been obtained either by BW from $(k-1)q \Rightarrow p \vee \neg p, (k-1)q$ (Case 1) or by R $\vee$ from either $(k)q \Rightarrow p, (k)q$ or $(k)q \Rightarrow \neg p, (k)q$ (Case 2). In the former case, just cancel from $\mathcal{D}$ the two redundant sequents and apply the induction hypothesis to the proof-tree $\mathcal{D}'$ thus obtained, which contains $m - 1$ applications of BC. In the latter case, there are two further subcases. If GC is not a conservative extension of its own $\{\neg, \rightarrow\}$-fragment, we are done by our initial remark. If it is, then the sequent $(k)q \Rightarrow p, (k)q$, which contains no occurrences of "$\wedge$" or "$\vee$", must be provable in the $\{\neg, \rightarrow\}$-fragment. But such a case is ruled out by Lemma 3, since $c(p; (k)q) = 0$ and $c(p; p, (k)q) = 1$. If

$(k)q \Rightarrow p \vee \neg p, (k)q$ was obtained from $(k)q \Rightarrow \neg p, (k)q$, we argue similarly. □

As to the other two systems, it is well-known that GALL is cut-free (Grishin, 1982), whereas (an equivalent reformulation of) GA is provably not so (Paoli, 200+a): the sequents corresponding to the excluded third and to the law of distribution are not provable without Cut. It would be interesting to find cut-free variants of GC and GA, or else to identify "deep" proof-theoretical reasons for the failure of cut-elimination. We leave this open problem up to the interested reader.*

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* ADDED IN PROOF. Since this paper was submitted, the problem has been solved. George Metcalfe (personal communication), in fact, provided cut-free hypersequent calculi for both comparative and Abelian logic.

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