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NAÏVE SET THEORY, PARACONSISTENCY AND INDETERMINACY: PART II

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This is the second part of a two-part paper on naïve set theory. In Part I, I outlined a 'neoclassical' logical framework for naïve set theory and sketched the recapture of classical mathematics in the naïve theory. In the introductory section of this Part, I recap the conclusions of the first paper whilst in section 2 I address the problems which emerge from the neoclassical framework when one tries to define entailment and prove soundness and completeness in the usual ways, arguing in favour of infinitary logic as a resolution of those problems. The soundness and completeness results for the system promised in Part I are then given in section 3. The fourth section looks at how the 'superparadoxes' which can be generated against other non-dialetheic solutions to the paradoxes fail to strike home against this one, finishing in section 5 with an overall conclusion.

1. *Neo-classical set theory*

In Part I I rejected standard accounts of entailment on the grounds that they are unable to give a reasonable account of validity for languages with indeterminate sentences, necessarily indeterminate sentences posing particular difficulties. The account of 'neoclassical' logical entailment I proposed in its place is as follows:

A set of wffs X neoclassically entails a set Y (written X $\models$ Y) iff:

> a) For any wff C in Y, in any *admissible* valuation v in which all wffs in X are true but all in Y but C are false, C is true in v.
> b) For any wff P in X, in any *admissible* valuation v in which all wffs in Y are false but all in X but P are true, P is false in v.

where the admissible valuations are defined relative to a set Δ of atoms and a designated valuation @ which is to be thought of as representing the 'actual' world. Valuation v is admissible iff every atom in Δ which has a truth value in @ has a truth value in v (perhaps the opposite one) and every Δ atom which is gappy in @ is gappy in v.

The idea is that atomic wffs in Δ retain in all possible situations their 'determinacy' status in the actual world, this status being either determinate —having one of the standard values true or false— or else being indeterminate, which in a three-valued logic such as the strong Kleene logic consists in taking the 'gappy' value. The set Δ and valuation @ generate a set of axioms AXDET (which thus has to be thought of as taking Δ and @ as parameters) such that AXDET is the set of all those wffs of the form Det P or $\sim$Det Q which are true in all admissible models. The determinacy operator Det P was then defined by $\Box(\mathrm{P} \vee \sim\mathrm{P})$ with necessity in turn defined in terms of the conditional and the everywhere-true truth constant T by $\Box\varphi \equiv_{\mathrm{df.}} \mathrm{T} \rightarrow \varphi$.

One class of rules which is unproblematically sound neoclassically are 'Minimax' or 'MM' rules, that is to say rules which never permit the minimum value of the premisses to be greater than the maximum value of the conclusion on the ordering true > gap ($\emptyset$) > false (on the strong Kleene, Lukasiewiczian semantics for &, $\vee$ and $\sim$). But certain minimax unsound rules, such as $\sim$E and $\vee$E have the property that individual applications are neoclassically correct. However for these rules, generalised transitivity fails:– the overall premisses may fail to entail neoclassically the overall conclusion. The solution adopted was retain the minimax unsound but neo-classically correct *operational* rules and change the *structural* rules by adding determinacy restrictions. These restrictions were also added to the rules for the conditional $\rightarrow$ whose intended interpretation is as an object language representation of neoclassical entailment itself. For $\rightarrow$E, the restricted rule is:

X	(1) $\mathrm{P} \rightarrow \mathrm{Q}$	Given
Y	(2) P	Given
Z_i	$(3.i)$ Det R_i	$\forall \mathrm{R}_i \in \mathrm{X} \cap \mathrm{Y}$
X, Y, $\bigcup_{i \in I} \mathrm{Z}_i$	(4) Q	1,2 $[3.i]$ $\rightarrow$E

where we also lay down also the disjointness condition: $\bigcup_{i \in I} \mathrm{Z}_i \cap (\mathrm{X} \cup \mathrm{Y}) = \emptyset$. So the idea is that where we have overlap between the antecedent sets of minimax unsound rules such as $\sim$E, $\vee$E, and $\rightarrow$E[1], the overlapping sentences must be determinate.

[1] In the case of $\vee$E the restriction applies to wffs occurring both as antecedents of the major premiss and as an antecedent of one or other of the minor premiss sequents.

Furthermore, in order to represent adequately neoclassical entailment, the $\rightarrow$I rule is restricted in the following way:

X,P	(1) Q	Given
Y_i	(2.i) Det R_i	Given, $\forall \mathrm{R}_i \in \mathrm{X}$
X, $\bigcup\limits_{i \in I} \mathrm{Y}_i$	(3) P $\rightarrow$ Q	1 [2.i] $\rightarrow$ I

subject again to the condition that $\mathrm{X} \cap \bigcup\limits_{i \in I} \mathrm{Y}_i = \emptyset$, whilst to achieve the full power required of an entailment conditional a number of other principles governing the conditional were added such as Transitivity:

X	(1) P $\rightarrow$ Q	Given
Y	(2) Q $\rightarrow$ R	Given
X, Y	(3) P $\rightarrow$ R	1,2 Trans.

to which should have been added Contraposition:

X	(1) P $\rightarrow$ Q	Given
X	(2) $\sim$Q $\rightarrow \sim$P	1, Contrap.

which can be shown to be sound in similar fashion.[2]

In Part I intensional proof-theoretic principles were provided for this modal conditional, in the form of indexing wffs in the antecedent in sequents by an ordinal representing, in effect, the degree of accessibility of the world at which the wff is to be evaluated from the world at which the succeedent wff (we need only consider for present purposes single conclusion logics) is evaluated. It was noted that this proof theory is rather clumsy and that one might wish to dispense with it in the context of mathematical language in which, it would seem, the intensional conditional should be equivalent to the extensional since all mathematical wffs are non-contingent. This is the position adopted in what follows. One consequence is an abandonment of standard model-theoretic semantics:– there is to be only one interpretation of the formulae –we select one arbitrarily as the 'intended' one— and with it an absolute notion of truth in the interpretation, unrelativised to models or possible worlds.

A further group of principles involving the conditional concern the relation between the notion of indeterminacy —$\sim$Det P $\equiv_{\mathrm{df.}} \sim(\mathrm{T} \rightarrow (\mathrm{P} \vee \sim\mathrm{P}))$— and antinomicity: Ant P $\equiv_{\mathrm{df.}}$ (P $\leftrightarrow \sim$P). Whilst these notions can come

[2] Although generalised Cut or transitivity fails neoclassically, the simple principle [if A $\vdash$ B then if B $\vdash$ C then A $\vdash$ C] holds, hence $(\mathrm{A} \rightarrow \mathrm{B} \rightarrow ((\mathrm{B} \rightarrow \mathrm{C}) \rightarrow (\mathrm{A} \rightarrow \mathrm{C})))$ ought to be provable for a conditional representing entailment.

apart in the gappy Lukasiewicz/Kleene framework, I argued that for the non-contingent language of mathematics a sentence which is indeterminate is necessarily indeterminate and so is equivalent, neoclassically, to its negation. Arguing from P to ∼P (or conversely) never takes one from truth to untruth, nor, in the equally important upwards falsity preservation direction, from falsity up to non-falsity. Similarly only an indeterminate sentence can be equivalent to its negation. Hence we can build this equivalence of indeterminacy and antinomicity into the rules governing the language of set theory:

—	(1) $(\mathrm{P} \leftrightarrow \sim\mathrm{P}) \leftrightarrow \sim(\mathrm{T} \rightarrow (\mathrm{P} \vee \sim\mathrm{P}))$	Ant/Det Axiom

and add finally a generalisation of the 'mingle' MiniMax principle that P&∼P entails $\mathrm{Q} \vee \sim\mathrm{Q}$, namely the 'MaxiMingle' principle:

—	(1) $(\mathrm{Ant\ P}\ \&\ \mathrm{Ant\ Q}) \rightarrow (\mathrm{P} \leftrightarrow \mathrm{Q})$	MaxiMin.

This completes the neoclassical propositional logical framework for naïve set theory; the extension to predicate logic consisted solely in incorporating the usual rules modulo a determinacy constraint on ∃E parallel to that on ∨E and a strengthening of the second-order comprehension scheme to the 'naïve':

$$\exists R \forall F_1, ...F_m, \forall x_1, ...x_n(R(F_1, ...F_m, x_1, ...x_n) \leftrightarrow \varphi(F_1, ...F_m, x_1, ...x_n))$$

by allowing that R may occur free in φ.

We need to look finally at the properly mathematical principles. One option is to take both the membership relation ∈ and class brackets {} as the set theoretic primitives; the proper set-theoretic rules can then be taken to be the joint rules for ∈ and {}:

X	(1) t $\in \{x : \varphi x\}$	Given
X	(2) $\varphi x/$t	1 ∈ /{} E
X	(1) $\varphi x/$t	1 Given
X	(2) t $\in \{x : \varphi x\}$	1 ∈ /{} I

together with an Extensionality axiom:

$$\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

These rules yield generalised naïve comprehension:

$$\exists y \forall x(x \in y \leftrightarrow \varphi x)$$

in which y may occur in φ.

An alternative, which I will follow here, is to take the epsilon operator as our fundamental term-forming operator subject to the Hilbert rules:

X	(1) $\exists x \varphi x$	Given.
X	(2) $\varphi \epsilon x \varphi x$	1 ϵI

X	(1) $\forall x(\varphi x \leftrightarrow \psi x)$	Given
X	(2) $\epsilon x \varphi x = \epsilon x \psi x$	$\epsilon =$

For as well as giving us Global Choice (neoclassical naïve set theory yields Global Well-Ordering but not Global Choice) we can define $\{x : \varphi x\}$ by

$$\epsilon x(\forall y(y \in x \leftrightarrow \varphi y)).$$

Michael Potter,[3] defines $\{x : \varphi x\}$ by $\imath x(\forall y(y \in x \leftrightarrow \varphi y))$ where $\imath x \varphi x =_{\text{df.}} \epsilon x \varphi ! x$ and where $\varphi !\mathrm{t}$ is

$$\varphi \mathrm{t} \;\&\; \forall y(\varphi y \rightarrow y = \mathrm{t})$$

for some variable y distinct from t. But we can make do with the simpler definition by building uniqueness into the version of naïve set comprehension we use as our governing principle for $\in$, namely:

$$\forall \mathrm{X} \exists ! y \forall x(x \in y \leftrightarrow \mathrm{X}x)$$

using generalised predicate comprehension to derive *generalised* set comprehension (i.e. y may occur in any instance φ of X) from this. Since

$$\exists ! x(\forall y(y \in x \leftrightarrow \varphi y)) \vdash \exists x(\forall y(y \in x \leftrightarrow \varphi y))$$

we are able to prove each instance of the schema:

$$\forall y(y \in \{x : \varphi x\} \leftrightarrow \varphi y):$$

—	(1) $\exists x(\forall y(y \in x \leftrightarrow \varphi y))$	From Comp.
—	(2) $\forall y(y \in \epsilon x(\forall y(y \in x \leftrightarrow \varphi y)) \leftrightarrow \varphi y)$	1 ϵI
—	(3) $\forall y(y \in \{x : \varphi x\} \leftrightarrow \varphi y)$	2 Def. {}

[3] *Sets: an Introduction.* (Oxford: Clarendon, 1990).

and derive extensionality (using a few easy Lemmas to shorten the proof):

—	(1) $\exists! x(\forall z(z \in x \leftrightarrow z \in \mathrm{u}))$	Comp.
2	(2) $\forall z(z \in \mathrm{a} \leftrightarrow z \in \mathrm{u})$ & $\forall w(\forall z(z \in w \leftrightarrow z \in \mathrm{u}) \rightarrow w = \mathrm{a})$	Hyp.
2	(3) $\forall w(\forall z(z \in w \leftrightarrow z \in \mathrm{u}) \rightarrow w = \mathrm{a})$	2 &E
2	(4) $\forall z(z \in \mathrm{t} \leftrightarrow z \in \mathrm{u}) \rightarrow \mathrm{t} = \mathrm{a}$	3 $\forall$E
2	(5) $\forall z(z \in \mathrm{u} \leftrightarrow z \in \mathrm{u}) \rightarrow \mathrm{u} = \mathrm{a}$	3 &E
—	(6) $\forall z(z \in \mathrm{u} \leftrightarrow z \in \mathrm{u})$	Lemma
2	(7) $\mathrm{u} = \mathrm{a}$	5,6 $\rightarrow$E
2	(8) $\mathrm{t} = \mathrm{a} \rightarrow \mathrm{t} = \mathrm{u}$	7 =E
2	(9) $\forall z(z \in \mathrm{t} \leftrightarrow z \in \mathrm{u}) \rightarrow \mathrm{t} = \mathrm{u}$	4, 8 Trans.
—	(10) $\mathrm{t} = \mathrm{u} \rightarrow \mathrm{u} = \mathrm{t}$	Lemma
2	(11) $\forall z(z \in \mathrm{t} \leftrightarrow z \in \mathrm{u}) \rightarrow \mathrm{u} = \mathrm{t}$	9, 10, Trans.
—	(12) $\forall z(z \in \mathrm{t} \leftrightarrow z \in \mathrm{u}) \rightarrow \mathrm{u} = \mathrm{t}$	1, 11 $\exists$E
—	(13) $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$	12, $\forall$I $\times$2

The conditional at line 13 can be strengthened to a second biconditional since

$$\forall x \forall y(x = y \rightarrow \forall z(z \in x \leftrightarrow z \in y))$$

follows from =E or Leibniz' Law.

1	(1) $\mathrm{a} = \mathrm{b}$	Hyp.
1	(2) $\mathrm{t} \in \mathrm{a} \rightarrow \mathrm{t} \in \mathrm{b}$	1 =E
1	(3) $\mathrm{t} \in \mathrm{b} \rightarrow \mathrm{t} \in \mathrm{a}$	1 =E
1	(4) $\forall z(z \in \mathrm{a} \leftrightarrow z \in \mathrm{b})$	2,3 &I, $\forall$I
	(5) $\forall x \forall y(x = y \rightarrow \forall z(z \in x \leftrightarrow z \in y))$	4 $\forall$I $\times$2

(The =E rule will be validated later along with neo-classical proofs that identity is an equivalence relation.)

The form of Global Choice we will use is a functional restriction theorem. With R*xz is defined by $z = \epsilon w \mathrm{R} x w$ we have:

FUNCTIONAL RESTRICTION THEOREM:

$$\forall x(\exists y \mathrm{R} xy \rightarrow \exists! z(\mathrm{R}^* xz \,\&\, \mathrm{R} xz)$$

Proof:

1	(1) $\exists y \mathrm{Ra} y$	Hyp.
1	(2) $\mathrm{Ra}(\epsilon w \mathrm{Ra} w)$	1 ϵI

—	(3) $\epsilon w \mathrm{R}aw = \epsilon w \mathrm{R}aw$	=I
—	(4) R*a$(\epsilon w \mathrm{R}aw)$	3 Def R*
5	(5) R*ac & Rac	Hyp.
5	(6) c $= \epsilon w \mathrm{R}aw$	5 &E
—	(7) (R*ac & Rac) $\rightarrow$ c $= \epsilon w \mathrm{R}aw$	6 $\rightarrow$I
—	(8) $\forall v$((R*av & Rav) $\rightarrow v = \epsilon w \mathrm{R}aw$)	7 $\forall$I
1	(9) R*a$(\epsilon w \mathrm{R}aw)$ & Ra$(\epsilon w \mathrm{R}aw)$	4,2 &I
1	(10) $\exists! z$(R*az& Raz)	9,8 &I, $\exists$I
—	(11) $\exists y \mathrm{R}ay \rightarrow \exists! z$(R*a$z$& Ra$z$)	10 $\rightarrow$I
—	(12) $\forall x(\exists y \mathrm{R}xy \rightarrow \exists! z$(R*$xz$& R$xz$)	11 $\forall$I

As well as the operational rules for the logical and mathematical operators, we need to consider the neoclassical restrictions on the structural rules which are essentially determined by the set AXDET of determinacy axioms of the form Det P or ∼Det Q. For φ is neoclassically provable from X if it is provable using the above rules from assumptions which belong either to X or to the axioms which include the fixed set AXDET. In deciding which sentences of set theory belong to AXDET, I urged a bold attitude: adopt a set M which is maximally consistent in the sense that M is (neo-classically) consistent but adding any new wff Det P or ∼Det Q induces inconsistency.

Generalised naïve comprehension yields recursive definition 'for free' and thereby permits the development of the theory of von-Neumann ordinals and Frege/Russell cardinals, the latter including a greatest 'superinfinite' cardinal ∞ with $x \in \infty \leftrightarrow x \cong$ U, where $\cong$ abbreviates the definition of equinumerosity and U is the universal set $\{x : x = x\}$. It will prove fruitful, however, to enrich the von-Neumann definition after the fashion of Frege's definition of the natural numbers, using second-order comprehension to prove the existence of a property Ord x satisfying:

$$\text{Ord } x \leftrightarrow \text{Trans } x \text{ \& WO}(\in)x \text{ \& IND}(\in)x.$$

where the first two conjuncts constrain the ordinals to be transitive sets well-ordered by $\in$ and the last says that x is $\in$-inductive, that is:

$$\forall \mathrm{X}((\mathrm{X}\emptyset \ \& \ \forall y(\forall z(z \in y \rightarrow \mathrm{X}z) \rightarrow \mathrm{X}y)) \rightarrow \mathrm{X}x)$$

From this we get transfinite induction in the following form:

$$\text{Ord a}, (\mathrm{F}\emptyset \ \& \ \forall y(\forall z(z \in y \rightarrow \mathrm{F}z) \rightarrow \mathrm{F}y)) \vdash \ \mathrm{Fa}.$$

Abbreviating the second premiss by IND(F), the proof is:

1	(1) Ord a	Hyp.
2	(2) IND(F)	Hyp.
1	(3) IND($\in$)x/a	Def. Ord, 1, $\leftrightarrow$E, &E
1	(4) IND(F) $\rightarrow$ Fa	3 $\forall$E
1,2	(5) Fa	2,4 $\rightarrow$E

Next we define using second-order relational comprehension a functional relation Vxy (intended informal reading, y is the xth rank V$_x$) by:

$\mathrm{V}xy \leftrightarrow$ Ord x & $y = \{z : \forall w(w \in z \rightarrow \exists x_1 \exists y_1(x_1 < x \,\&\, \mathrm{V}x_1y_1 \,\&\, w \in y_1))\}$

The relation V is functional in the following sense:

$\forall x, y, z(\mathrm{V}xy \rightarrow (\mathrm{V}xz \rightarrow x = z))$

Proof: (here $\mathrm{d} = \{z : \forall w(w \in z \rightarrow \exists x_1 \exists y_1(x_1 < \mathrm{a} \,\&\, \mathrm{V}x_1y_1 \,\&\, w \in y_1))\}$)

1	(1) Vab	Hyp.
2	(2) Vac	Hyp.
1	(3) b = d	1, Def. V $\leftrightarrow$E, &E
2	(4) c = d	2, Def. V $\leftrightarrow$E, &E
—	(5) c = d $\rightarrow$ (d = c)	=Symm $\forall$E $\times 2$
2	(6) d = c	4,5 $\rightarrow$E
—	(7) b = d $\rightarrow$ (d = c $\rightarrow$ b = c)	=Trans. $\forall$E $\times 3$
1,2	(8) b = c	3,6, 7 $\rightarrow$E $\times 2$

The symmetry and transitivity of identity principles are derived in section 2.

The upshot is that we can write Vαx also as V(α) $= x$ or speak as usual of V$_\alpha$. V we then define by V $=_{\mathrm{df.}} \bigcup_{\alpha \in \mathrm{ON}} \mathrm{V}_\alpha$ that is: $\{x : \exists \alpha \in \mathrm{ON}, x \in \mathrm{V}_\alpha\}$ where ON $= \{x :$ Ord $x\}$ the class of all ordinals. From the inductive characterisation of the ordinals we can prove by induction over the ranks that all members of V are themselves $\in$-inductive sets:

THEOREM: $\forall x \in$ V, IND($\in$)x.

Proof, by induction over the V$_\alpha$, so suppose it holds for all members of V$_\beta$, $\beta < \alpha$ and consider $\mathrm{t} \in \mathrm{V}_\alpha$. Take some property F such that:

$$(\mathrm{F}\emptyset \,\&\, \forall y(\forall z(z \in y \rightarrow \mathrm{F}z) \rightarrow \mathrm{F}y));$$

we need to prove Ft. But any member z of t belongs to V_β, for some $\beta < \alpha$, hence by IH, z is $\in$-inductive, hence Fz. From $\forall y(\forall z(z \in y \rightarrow \mathrm{F}z) \rightarrow \mathrm{F}y)$ we conclude $\forall z(z \in \mathrm{t} \rightarrow \mathrm{F}z) \rightarrow \mathrm{Ft}$ hence Ft as required. ❑

As well as our superinfinite greatest cardinal ∞ we have seen that we also have the greatest ordinal ON though of course the Burali-Forti paradox shows us that $\sim$Det (ON $\in$ ON).[4] A more useful ordinal singled out in Part I is Ω defined as the class of all Small ordinals, where a set is Small iff there is no bijection from the set onto the entire universe. If we assume that our favoured maximally consistent AXDET contains Det($\Omega \in \Omega$), then we can reason classically and prove that Ω is itself a Big ordinal (if not it would belong to itself, absurdly) and V_Ω is then the standard cumulative hierarchy. The strategy for classical recapture, that is for the derivation of the results of standard mathematics, is to include in AXDET DetP, for every P $\in L_\alpha$ where α is some Small ordinal and L_α is the language in which all quantifiers are restricted to V_α. Where such a locally maximal AXDET set for L_α is consistent, then let us call L_α classical. If arithmetic is consistent then L_ω is classical and classical reasoning in arithmetic is neoclassically legitimate; more than this, all arithmetic truths are neoclassically provable, as will be shown in section 3. Similarly, if ZFC is consistent, L_κ is classical for some inaccessible κ, classical reasoning about all the sets in V_κ is neoclassically legitimate and all ZFC truths are provable. Hence if standard mathematics is consistent, neoclassical logic not only validates our ordinary classical reasoning concerning it, it also enables us to prove all the truths of standard mathematics. Furthermore, the neoclassical approach also lets us reason using standard operational rules- but with restricted Cut principles- beyond the standard realm, a domain we are ineluctably forced to venture into if we wish to provide a systematic interpretation of standard mathematics.

2. *Infinitary Logic*

Naïve set theory in a neoclassical setting thus promises to provide a powerful framework for mathematics in which standard mathematics is validated whilst the skeletons in the cupboard of classical mathematics —sets such as V_Ω, V, ON or U— whose existence is both required in any reflective account of the theory and at the same time vehemently denied by classical mathematicians,[5] can be shown to exist and results can be established concerning

[4] Assuming DetDet(ON $\in$ ON) $\in$ AXDET.

[5] It is true that analogues of V_Ω, V and ON exist as "proper classes" in NBG theories, and universal sets are countenanced in some deviant classical theories, the most well-known perhaps being NF. For discussion of NF see T.E. Forster, *Set Theory with a Universal Set*, (Oxford: Clarendon, 1992) where there is also discussion of the theories of Church and

them, albeit by reasoning in more restricted non-classical ways. All this on the assumption, of course, that neoclassical naïve set theory is sound.

To be sure, it was shown in the first part how the classical proofs of antinomy from paradoxes such as Russell's, Burali-Forti's and Curry's breakdown neoclassically. A crucial point here is that the derivation of absurdity $\perp$ (or arbitrary contradiction Q&$\sim$Q) from the antinomicity of some sentence P, i.e. from P $\leftrightarrow\sim$P, fails (here we consider a provably antinomic P, the liar and Russell sentences being of this form):

—	(1) P $\leftrightarrow\sim$P	Given
2	(2 P	Hyp.
2	(3) $\sim$P	1,2, $\leftrightarrow$E
4	(4) Det P	Hyp.
2,4	(5) $\perp$	2,3 [4] $\sim$E
4	(6) $\sim$P	5, $\sim$I
4	(7) P	1, 6 $\leftrightarrow$E
8	(8) Det Det P	Hyp.
4,8	(9) $\perp$	6,7 [8] $\sim$E

the contradiction no longer being derivable outright but rather dependent, neoclassically, on the determinacy assumptions (represented in proof annotations by enclosure in square brackets) of lines 4 and 8. So the fact that P $\leftrightarrow$ $\sim$P does follow neoclassically for many antinomic sentences, such as the Russell sentence $r \in r$, does not show the theory to be inconsistent but merely shows the indeterminacy of e.g. $r \in r$ (granted it is determinate whether or not it is determinate).[6]

But of course this does not rule out the possibility of the emergence of a more subtle antinomy which cannot be blocked neoclassically, unless perhaps by rejecting the determinacy of most of standard mathematics. To rule

Mitchell these, unlike NF, being classical theories known to be consistent relative to standard ZFC. Other examples include Boolos' theory New V, see 'Saving Frege from Contradiction', *Proceedings of the Aristotelian Society* 87 (1986–87) pp. 137–151 and for discussion S. Shapiro and A. Weir, 'New V, ZF and Abstraction', *Philosophia Mathematica*. (3) 7 1999, pp. 293–321 and Arnold Oberschelp's 'Set theory over classes' *Dissertationes Mathematicae*, (Warsaw: 1973). However in these theories proper classes —"classes" (construed differently in the different theories) which fail to satisfy the comprehension principle— are pale shadows of sets and cannot be used, to take one example, to specify the model theory of the set theory which includes them. See my 'Naïve Set Theory is Innocent!' *Mind* 107 1998. pp. 763–798.

[6] Note the difference with standard paraconsistent approaches to naïve set theory: where they accept inconsistency but reject triviality, I reject both.

this out we need a soundness proof for the system. Moreover if neoclassical, not classical, logic is the correct mode of reasoning then the proof ought to be given neoclassically (if not, one cannot expect classicists to consider abandoning their logic for one which can no more validate its own semantics than classical mathematical logic). Indeed there is no hope of a demonstration that the system has models of the intended type, with a domain = U the universal set, for example, except from within naïve set theory itself as meta-theory. One of the main reasons to demur from the orthodox rejection of naïve set theory is the emergence of superparadox when one denies, as the orthodox do, that semantically closed theories exist. So our goal should be to show that the language of naïve set theory is semantically closed — the metalanguage in which soundness (and completeness) theorems are expressed and proved is identical with the object language. But using the naïve theory to prove its own soundness is by no means a trivial matter.

To see this consider a rule which ought to be neoclassically unproblematic: *ex contradictione quodlibet* in the form P, $\sim$P $\vdash$ C. As remarked in Part I, even though $\sim$E is restricted neoclassically to this form:

X	(1) P	Given
Y	(2) $\sim$P	Given
Z_i, $i \in \mathrm{I}$	(3.i) Det Q	Given, $\forall \mathrm{Q} \in \mathrm{X} \cap \mathrm{Y}$
X,Y, $\bigcup\limits_{i \in \mathrm{I}} \mathrm{Z}_i$	(4) C	1,2, [3.i, $i \in$ I], $\sim$E

where $\bigcup\limits_{i\in \mathrm{I}} \mathrm{Z}_i \cap (\mathrm{X} \cup \mathrm{Y}) = \emptyset$, nonetheless in the special case where $\mathrm{X} \cap \mathrm{Y} = \emptyset$ the determinacy restrictions do not bite and the rule takes the classical form. So we have, for example $r \in r$, $r \notin r \vdash$ C by dint of:

1	(1) $r \in r$	Hyp.
2	(2) $r \notin r$	Hyp.
1,2	(3) C	1,2 $\sim$E

This rule was shown (classically) to be sound in the test case approximation of Lukasiewiczian gappy semantics. But now consider how a clause in a standard soundness proof for this particular case would proceed in the full, non-gappy, context of naïve set theory: assume, as the inductive hypothesis, that lines (1) and (2) encapsulate neoclassical entailments and attempt to show that {P,$\sim$P} entails arbitrary C, where, for example, P is the Russell sentence. To demonstrate the latter entailment we would standardly assume that both premisses are true, in some valuation v; thus we might represent this as

①: $\mathrm{True}_v\underline{\mathrm{P}}$ & $\mathrm{True}_v\underline{\sim\mathrm{P}}$

and go on to prove that in that case $\mathrm{True}_v\mathrm{C}$, for our arbitrary valuation v and arbitrary wff C (underlining representing coding of wffs, $\mathrm{True}_v\underline{\mathrm{P}}$ representing the claim that P is true in v). We can try to argue as follows:

1	(1) ①	Hyp.
1	(2) $\mathrm{True}_v\underline{\mathrm{P}}$	1 &E
—	(3) $\mathrm{True}_v\underline{\mathrm{P}} \rightarrow \sim\mathrm{True}_v\underline{\sim\mathrm{P}}$	Semantical $\sim$ clause
1	(4) $\sim\mathrm{True}_v\underline{\sim\mathrm{P}}$	2,3 $\rightarrow$E
1	(5) $\mathrm{True}_v\underline{\sim\mathrm{P}}$	1 &E
6	(6) Det ①	Hyp.
1,6	(7) $\mathrm{True}_v\underline{\mathrm{C}}$	4,5, [6] $\sim$E

but now we have proved that the conclusion is true in any valuation in which the premisses are true *only given the additional assumption* that it is determinate matter whether or not P and its negation are true in valuation v; this will not be the case where P is an antinomic sentence such as the Russell sentence $r \in r$, so we have failed to validate $r \in r$, $r \notin r \vdash$ C.

The problem is that our initial definition of neoclassical entailment is conjunctive. In effect, it conjoins all the premisses, just as the conventional notion does, whereas neoclassically this is illegitimate since for example, the two premisses $r \in r$, $r \notin r$ together entail $\perp$ but the single premiss $(r \in r \mathbin{\&} r \notin r)$ does not.[7] We might try to redefine entailment for this particular instance as: $\forall v, \mathrm{True}_v\underline{\mathrm{P}} \rightarrow (\mathrm{True}_v\underline{\sim\mathrm{P}} \rightarrow \mathrm{True}_v\underline{\mathrm{C}})$. Then we could introduce our assumptions $\mathrm{True}_v\underline{\mathrm{P}}$ and $\mathrm{True}_v\underline{\sim\mathrm{P}}$ separately and by-pass the determinacy restriction on $\sim$E. The "proof" then goes

1	(1) $\mathrm{True}_v\underline{\mathrm{P}}$	Hyp.
2	(2) $\mathrm{True}_v\underline{\sim\mathrm{P}}$	Hyp
—	(3) $\mathrm{True}_v\underline{\mathrm{P}} \rightarrow \sim\mathrm{True}_v\underline{\sim\mathrm{P}}$	Semantical $\sim$ clause
1	(4) $\sim\mathrm{True}_v\underline{\sim\mathrm{P}}$	1,3 $\rightarrow$E
1,2	(5) $\mathrm{True}_v\underline{\mathrm{C}}$	2,4, $\sim$E
1	(6) $2 \rightarrow 5$	5 $\rightarrow$I*
—	(7) $1 \rightarrow (2 \rightarrow 5)$	6 $\rightarrow$I

But the starred $\rightarrow$I application at lines 6 is neo-classically incorrect in the absence of the additional assumption that the extra premiss at line 1 is determinate.

[7] On pain of triviality for then we would quickly get $(r \in r \vee r \notin r)$ as a theorem then $r \in r$ and $r \notin r$ as theorems via two applications of $\vee$E then $\perp$ as a theorem by $\sim$E.

Moreover, with $r \in r$, $r \notin r \models \perp$ interpreted by $\forall v(\text{True}_v \underline{r \in r} \rightarrow (\text{True}_v \underline{r \notin r} \rightarrow \text{True}_v \underline{\perp})$, we get, taking the actual world for v and disquoting, $r \in r \rightarrow (r \notin r \rightarrow \perp)$. But do we want $r \in r$ to entail that $r \notin r$ entails absurdity? Should we not reject that consequent, if we wish to affirm that our system is sound? If we are agnostic on $r \notin r \models \perp$ we must be agnostic on $\models r \in r$, and thus agnostic on the soundness of the system (it will be unsound if $\models r \in r$). And if the consequent is false and $r \in r$ entails it, then $r \in r$ is provably (in neoclassical logic) false too which leads to disaster since then $r \in r$ and therefore also $r \notin r$ are theorems.

Nor is this the least of our problems since we have still to show how to generalise our conditionalising interpretation of $\models$ to handle more than just one specific case with premiss set {P,∼P} indeed more than just some given finite list of cases. The only conceivable way to do this is to define X $\models$ C in some infinitary fashion. Such a move is certainly not unproblematic!

Orthodoxy since the 1930s, indeed, has it that any mathematical representation or idealisation of 'real' logic must be finitary in the sense of allowing only finitely long wffs and proofs. As Gregory Moore and others have shown,[8] this type of proof-theoretic finitism represents a sharp turn away from the position of a great many of the early pioneers of modern logic — Peirce, Schröder, Löwenheim, the early Hilbert, for example— a turn which was strongly resisted by Zermelo, in his reactions to the Gödelian incompleteness theorems. There did survive a 'samizdat' tradition of infinitary logic, with the early pioneers Ramsey, Carnap and Rosser handing the torch on to Tarski's students at Princeton. But the conventional view would appear to be that such systems are of 'merely technical' interest because no one can actually grasp an infinitely long formula. However neither can anyone grasp a wff with more symbols than the estimated number of electrons in the observable universe, yet understanding of such wffs (and virtually all, one might say, wffs of standard formal languages are even longer than this) is said to be "in principle" possible.

What does this phrase "in principle possible", found most often on the lips of constructivist mathematicians, amount to? Arguably to claim that it is in principle possible to grasp or construct a mathematical structure is to claim nothing more than that that structure exists. If so, the phrase can offer no illumination, explanation, or justification for the existence claim with respect to the "in principle possible" structure. And in that case, those who reject

[8] See Gregory H. Moore, 'Beyond First-Order Logic: The Historical Interplay between Mathematical Logic and Axiomatic Set Theory', *History and Philosophy of Logic* 1 (1980) pp. 95–137 and 'The Emergence of First-Order Logic' in *History and Philosophy of Modern Mathematics* (Minnesota Studies in the Philosophy of Science No. 11) (Minneapolis: University of Minnesota Press, 1988) pp. 95–135. See also Stewart Shapiro, *Foundations without Foundationalism: A Case for Second-Order Logic* (Oxford: Clarendon), 1991 Chapter Seven.

finitism have no more reason to eschew infinite structures in proof theory than in model theory; indeed good reason to embrace them if they wish to achieve, as most do, a tight rapprochement between proof theory and model theory.

At any rate the claim that an ontology of infinitary structures is legitimate in the idealised semantics of model theory but not in idealisations of our concrete proofs is sorely in need of justification. Pending a good argument against the legitimacy of infinitary logic, I propose to follow Zermelo in rejecting proof-theoretic finitism but depart from him in the crucial respect of allowing in not only Small infinite strings of expressions or sequents but allowing our syntactic structures to be strings of arbitrary ordinal length, that is to be functions whose domain is any ordinal $\alpha \in$ ON including Big ordinals of size $\geq \Omega$.

Thus we will have infinitary conjunctions and disjunctions of arbitrary infinite length, including Big conjunctions and disjunctions. Universal and existential quantification over the individuals can then be represented by wffs such $\&\mathrm{Ft}_{i\ i<\Omega}$ and $\vee\mathrm{Ft}_{i\ i<\Omega}$ where the singular terms t_i range over all the simple terms of the language (here $\ulcorner i\ i < \Omega\urcorner$, is metalinguistic notation indicating that there is a function with domain Ω indexing the conjuncts and disjuncts of the generalisation[9]). I will also introduce an infinitary conditional $\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}$ with any number of antecedents, indexed by some set I, which we will use to represent entailment for our language in the language itself. As remarked, the intuition that mathematical language is non-contingent is taken seriously so that there is only one ‘intended’ model for the sentences of the language. That being so we can represent entailments by conditionals of the form True $\underline{\mathrm{A}_{i\ i\in\mathrm{I}}} \rightarrow$ True $\underline{\mathrm{B}}$. In fact, since we will be able to prove all Tarskian biconditionals (for items which are definitely wffs, at any rate), we can simply allow $\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}$ to represent the entailment of B from the premisses A_i and try to link this notion with provability.

In more detail then, the construction of our infinitary language L —a construction which takes place, of course, in natural language augmented with some mathematical notions— proceeds as follows. I presuppose as the background logical framework the legitimacy of informal use of the neoclassical logic outlined above and in Part I. It will be shown that we can represent all the strings of L in L and as above, underlining will be used as our informal metalinguistic representation of the coding of strings. So where $\ulcorner\mathrm{t}\urcorner$ is a metalinguistic term whose referents are expressions of L, $\ulcorner\underline{\mathrm{t}}\urcorner$ is a metalinguistic term which, for a given assignment to any parameters in $\ulcorner\mathrm{t}\urcorner$, stands, on the

[9] Corner quoting being used here as quasi-quotation as in Quine’s *Mathematical Logic* (Cambridge Mass., Harvard University Press, 1940 §6) that is as combinations, representing concatenation, of metalinguistic names of expressions and variables ranging over all non-logical or mathematical constants.

intended interpretation, for the canonical representation of the referent of $\ulcorner t \urcorner$ on that assignment.

We need first of all disjoint sets of basic categories: logico-mathematical operators, individual constants, one-place predicates and two-place predicates. All but the first set must be of superinfinite size ∞:– we could use the set of Small ordinals, the set of ordered pairs of Small ordinals and the set of ordered triples of Small ordinals for the latter three sets. The logical operators are conjunction, disjunction, negation and the conditional named in the meta-language by:– &, $\vee$, $\sim$ and $\rightarrow$ (the notation is officially reverse Polish but I will use parentheses informally) whilst the mathematical terms are the epsilon operator ϵ and the membership predicate '$\in$' —(set-theoretic braces and other parentheses are informal metatheoretic devices of abbreviation). Each primitive is to be distinct from every other syntactic primitive, of course. The expressions are 'superstrings', that is sequences of any ordinal length, whose terms are themselves expressions or sets of expressions. We can interpret such strings as functions from the ordinal which gives its length into (a subset of) V; similarly proofs will be construed as strings of sequents, where sequents are ordered pairs $\langle X, A\rangle$ with A a wff and X a set of wffs, so proofs are members of V also.

The well-formed expressions (wfes) we then define by recursion. The individual constants are singular terms of each level. The atomic sentences of level zero are all sentences of the form Ft or Rtu, where t and u are individual constants, F and R one-place and two-place predicate constants respectively. The level zero sentences are then all sentences of the form:

$$\sim \mathrm{A}_0,\ \&\mathrm{A}_{i\ i\in \mathrm{I}}, \vee \mathrm{A}_{i\ i\in \mathrm{I}} \text{ and } \mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}$$

Here the string $\&\mathrm{A}_{i\ i\in \mathrm{I}}$ is an ordered pair whose first term is & and whose second term is a set of wffs, these being the conjuncts. likewise for $\vee \mathrm{A}_{i\ i\in \mathrm{I}}$ with the second term being the set of disjuncts. The notation $\ulcorner_{i\ i\in \mathrm{I}}\urcorner$ is thus meta-linguistic notation with $\mathrm{A}_{i\ i\in \mathrm{I}}$ a metalinguistic parameter whose intended referents are sets of wffs indexed by some set I. We could, indeed, insist that I be an ordinal so that the conjunctions and disjunctions are strings in which the conjuncts appear in the particular order given by I. But granted commutativity and associativity of conjunction and disjunction we save ourselves adding in those rules by letting conjuncts and disjuncts occur unordered in their compound sentences.

The string $\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}$ is a triple whose first term is $\rightarrow$, second term the (unordered set) of the A_i and whose last term is B. It is an infinitary generalisation of a right-bracketing conditional of the form $\mathrm{A} \rightarrow (\mathrm{B} \rightarrow (\mathrm{C} \rightarrow \mathrm{D}))$ except that, as with conjunction and disjunction, we do not bother with the ordering of the antecedents. Since the conditional is to represent entailment in the object language this corresponds to our taking sequents as relating sets, not sequences, of wffs to succeedent wffs.

So much for level zero sentences. For level α, we define the set of level α singular terms to consist in the individual constants plus all epsilon terms of the form $\epsilon x\varphi x$ for φ a sentence of level $\beta < \alpha$. We can take these to be strings whose first term is the epsilon operator, second term a simple individual constant t and third term a sentence φ of level $\beta < \alpha$ the constant playing the role of a variable in finitary logic so that the epsilon term binds all occurrences of t in φ. The atoms are then defined as before except that the singular terms may be of any level $< \alpha$ and the sentences of level α are defined in the same way as at level zero except again the constituent sentences may be of any level less than α.

Finally we turn to the truth and absurdity constants T and $\perp$. To define these we have to look slightly ahead to the semantics and discuss the notion of the 'intended' interpretation. To play this role we simply select any function f which maps all singular terms onto U, the universal set, all predicate terms onto its powerset P(U), all relational terms onto $[\mathrm{P(U)}]^2$. By the Bigness of Ω, Ω^2 and Ω^3 there are functions f_1, f_2, and f_3 from the singular terms, monadic predicates and relational terms respectively, onto U. Restrict the latter two to functions f_2^* onto P(U) and f_3^* onto $[\mathrm{P(U)}]^2$ respectively and our function f is $f_1 \cup f_2^* \cup f_3^*$ which is a function by the disjointness of the three syntactic categories. An atom Ft or Rtu is then true just in case $f(\mathrm{t}) \in f(\mathrm{F})$ in the first case or just in case $\langle f(\mathrm{t}), f(\mathrm{u})\rangle \in f(\mathrm{R})$ in the second.

Having selected an interpretation f to play the role of 'the intended one', we then define T as the infinite conjunction of all atomic sentences which are definitely true, relative to f where ⌜Definitely P⌝ is defined by ⌜necessarily P⌝ i.e. ⌜P is entailed by the (metatheoretic) truth constant⌝.[10] Similarly, $\perp$ is the disjunction of all atoms which are definitely not true[11] in the intended interpretation. The idea here is that the atomic sentences of our pure mathematical language are "necessarily true or necessarily false"; we might, perhaps, gloss this as "provable or refutable by use of primitive atomic rules". The notion of an atomic wff is, indeed, a "fuzzy" one since it can be indeterminate whether an item belongs to L: for example, since it is indeterminate whether ON $\in$ ON, it is indeterminate whether atoms of level ON are wffs. Nonetheless this does not induce any indeterminacy in T or $\perp$ since they comprise only items which are definitely true or untrue in the intended interpretation, and nothing which is not determinately a wff can be definitely true

[10] The notion of definite truth will be considered in more detail later.

[11] Equivalently false: in the non-gappy framework of naïve set theory, untruth = falsity.

or untrue. Since T and $\perp$ are definitely true and definitely false respectively, Det T (and Det Det T etc.) ought to belong to AXDET, likewise Det $\perp$, Det Det $\perp$ etc. so that we can make it a requirement on an admissible AXDET set that it contain arbitrary determinations of these constants.

Next we need to look at the proof theory of L for clearly we have to augment our proof theory somewhat, to accommodate L's increased resources. Turning to the MiniMax rules, the commutative and associative rules are redundant now that conjunctions and disjunctions are construed as unordered sets whilst some of the others have to be generalised to infinitary forms. Thus the De Morgan transformations:–

X	(1) $\sim(\mathrm{A}\&\mathrm{B})$	Given
X	(2) $(\sim\mathrm{A} \vee \sim\mathrm{B})$	1 MM

X	(2) $(\sim\mathrm{A} \vee \sim\mathrm{B})$	Given
X	(2) $\sim(\mathrm{A}\&\mathrm{B})$	1 MM

become in their infinitary form:

X	(1) $\sim\&\mathrm{A}_{i\ i\in\mathrm{I}}$	Given
X	(2) $\vee[\sim\mathrm{A}_{i\ i\in\mathrm{I}}]$	1 MM

X	(2) $\vee[\sim\mathrm{A}_{i\ i\in\mathrm{I}}]$	Given
X	(2) $\sim\&\mathrm{A}_{i\ i\in\mathrm{I}}$	1 MM

(the square brackets here are informal meta-theoretic devices for improving readability). Similarly the distributivity rules legitimising interchange of, for example, $\mathrm{A}\&(\mathrm{B} \vee \mathrm{C})$ with $(\mathrm{A}\&\mathrm{B}) \vee (\mathrm{A}\&\mathrm{C})$ are generalised, in the case given, to:

X	(1) $\&(\vee(\mathrm{A}_{ij})_{j\in f(i)})_{i\in\mathrm{I}}$	Given
X	(2) $\vee(\&(\mathrm{A}_{ij})_{i\in\mathrm{I},j\in k^*})_{k\in\mathrm{K}}$	1 MM

X	(2) $\vee(\&(\mathrm{A}_{ij})_{j\in f(i)})_{i\in\mathrm{I}}$	Given
X	(2) $\vee[\&\mathrm{A}_{ij\ i\in\mathrm{I},j\in k^*}]_{k\in\mathrm{K}}$	1 MM

where, for each $i \in \mathrm{I}$, $f(i)$, is an index set indexing the disjuncts of the ith conjunct of $\&[\vee\mathrm{A}_{ij\ j\in f(i)}]_{i\in\mathrm{I}}$ (likewise for disjuncts of $\vee(\&(\mathrm{A}_{ij})_{j\in f(i)})_{i\in\mathrm{I}})$; Here K is $\Pi[f(i) : i \in \mathrm{I}]$, the product of the family $f(i) : i \in \mathrm{I}$[12] and k^* is

[12] This exists in naïve set theory, for every indexed set, since naïve comprehension gives us the set of all ordered sequences such that the ith term is a member of the set indexed by i and this set is by definition the product.

the set of terms of the sequence $k \in \Pi[f(i) : i \in \mathrm{I}]$. The first inference is therefore an infinitary generalisation of the inference from e.g.

$(\mathrm{A}_{11} \vee \mathrm{A}_{12})$ & $(\mathrm{A}_{21} \vee \mathrm{A}_{22} \vee \mathrm{A}_{23})$ & $(\mathrm{A}_{31} \vee \mathrm{A}_{32})$ to
$(\mathrm{A}_{11}\&\mathrm{A}_{21}\&\mathrm{A}_{31}) \vee (\mathrm{A}_{11}\&\mathrm{A}_{22}\&\mathrm{A}_{31}) \vee (\mathrm{A}_{11}\&\mathrm{A}_{23}\&\mathrm{A}_{31}) \vee$
$(\mathrm{A}_{11}\&\mathrm{A}_{21}\&\mathrm{A}_{32}) \vee (\mathrm{A}_{11}\&\mathrm{A}_{22}\&\mathrm{A}_{32}) \vee (\mathrm{A}_{11}\&\mathrm{A}_{23}\&\mathrm{A}_{32}) \vee$
$(\mathrm{A}_{12}\&\mathrm{A}_{21}\&\mathrm{A}_{31}) \vee (\mathrm{A}_{12}\&\mathrm{A}_{22}\&\mathrm{A}_{31}) \vee (\mathrm{A}_{12}\&\mathrm{A}_{23}\&\mathrm{A}_{31}) \vee$
$(\mathrm{A}_{12}\&\mathrm{A}_{21}\&\mathrm{A}_{32}) \vee (\mathrm{A}_{12}\&\mathrm{A}_{22}\&\mathrm{A}_{32}) \vee (\mathrm{A}_{12}\&\mathrm{A}_{23}\&\mathrm{A}_{32})$.

We need also infinitary forms of the introduction and elimination rules for & and $\vee$. The rule &I, for example, becomes:

X_i	$(1.i)$ A_i	Given $i \in \mathrm{I}$
$\bigcup_{i\in\mathrm{I}} \mathrm{X}_i$	(2) $\&\mathrm{A}_{i\ i\in\mathrm{I}}$	$1.i$ $i \in \mathrm{I}$, &I

whilst $\vee$E is:

X	(1) $\vee\mathrm{A}_{i\ i\in\mathrm{I}}$	Given
$\mathrm{Y}_{i\ i\in\mathrm{I}}$, A_i	$(2.i)$ C	Given $i \in \mathrm{I}$
$\mathrm{W}_{j\ j\in\mathrm{J}}$	$(3.j)$ Det R	Given, $\forall \mathrm{R} \in \mathrm{X} \cap \bigcup_{i\in\mathrm{I}} \mathrm{Y}_i$
X, $\bigcup_{i\in\mathrm{I}} \mathrm{Y}_i$, $\bigcup_{j\in\mathrm{J}} \mathrm{W}_j$	(4) C	$1,2.i$ $i \in \mathrm{I}$, [$3.j$ $j \in \mathrm{J}$], $\vee$E

Here, as in part I, the square brackets highlight the premisses which ensure the determinacy of all wffs occurring both among the antecedents of the major premiss and among the antecedents of a minor premiss; as with finitary $\vee$E, we require that $(\mathrm{X} \cap \bigcup_{i\in\mathrm{I}} \mathrm{Y}_i) \cap \bigcup_{j\in\mathrm{J}} \mathrm{W}_j = \emptyset$.[13]

The introduction and elimination rules for the infinitary conditional are fairly obvious generalisations from the finitary case, namely $\rightarrow$I:

X,$\mathrm{A}_{i\ i\in\mathrm{I}}$	(1) B	Given
$\mathrm{Y}_{j\ j\in\mathrm{J}}$	$(2.j)$ Det C	for all $\mathrm{C} \in \mathrm{X}$
X, $\bigcup_{j\in\mathrm{J}} \mathrm{Y}_j$	(2) $\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}$	$1 \rightarrow \mathrm{I}$

where $\mathrm{X} \cap \bigcup_{j\in\mathrm{J}} \mathrm{Y}_j = \emptyset$, and $\rightarrow$E:

[13] This is actually weaker than the rule specified in Part I the determinacy restriction there (and for the other rules) being the unnecessarily strong $(\mathrm{X} \cup \bigcup_{i\in\mathrm{I}} \mathrm{Y}_i) \cap \bigcup_{j\in\mathrm{J}} \mathrm{W}_j = \emptyset$.

X_i	(1.i) A_i	Given, $i \in \mathrm{I}$
Y	(2) $\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}$	Given
Z_j	(3.j) Det C_j	$\forall \mathrm{C}_j \in \Sigma$, $j \in \mathrm{J}$
$\bigcup\limits_{i\in\mathrm{I}}\mathrm{X}_i$, Y, $\bigcup\limits_{j\in\mathrm{J}}\mathrm{Z}_j$	(4) B	1.i $i \in \mathrm{I}$, 2, [3.j, $j \in \mathrm{J}$], →E

Here, at line (3), Σ is the set of all wffs which either belong to $\bigcup\limits_{i\in\mathrm{I}}\mathrm{X}_i \cap \mathrm{Y}$ or else belong to $\mathrm{X}_i \cap \mathrm{X}_k$ for some $i \neq k$. We also require, generalising from the finitary case, that $\bigcup\limits_{j\in\mathrm{J}}\mathrm{Z}_j \cap (\bigcup\limits_{i\in\mathrm{I}}\mathrm{X}_i \cup \mathrm{Y}) = \emptyset$ and that $\bigcup\limits_{j\in\mathrm{J}}\mathrm{Z}_j \cap (\mathrm{X}_i \cup \mathrm{X}_k) = \emptyset$ for $i \neq k$.

To check for soundness of these rules we look as before to the Lukasiewiczian framework as an approximation to the naïve case. In order that the semantics for the infinitary → mirrors the definition of the neoclassical entailment relation it is intended to represent in the object language (except that in dealing with the conditional as it occurs in set theory we treat it extensionally and omit reference to accessible models or possible worlds) we need these clauses, where truth is truth in the interpretation we have selected to play the role of the actual mathematical interpretation:

1) $\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}$ is true iff a) if all of the A_i are true then B is true and b) if B is false and all the $\mathrm{A}_{i\ i\in\mathrm{I}}$ but A_j are true, then A_j is false.
2) Otherwise, $\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}$ is false.

These rules can then be proven sound, that is can be shown to preserve neoclassical entailment:

Proof: →I: *Truth-preservation*: Suppose all of X, $\bigcup\limits_{j\in\mathrm{J}}\mathrm{Y}_j$, hence all of X are true. We have two clauses to check in verifying the truth of the conditional succeedent:– a) firstly suppose all of the $\mathrm{A}_{i\ i\in\mathrm{I}}$ are true; then by the correctness of the premiss (1), B is true so $\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}$ is true. Suppose on the other hand that b) B is false, all of $\mathrm{A}_{i\ i\in\mathrm{I}}$ but A_j are true; once again the correctness of premiss sequent (1) ensures that A_j is false hence $\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}$ is true.

Falsity Preservation: Suppose $\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}$ is false and all of X, $\bigcup\limits_{j\in\mathrm{J}}\mathrm{Y}_j$ but P are true.[14] Since $\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}$ is not true, either i) all of the A_i are true but B is not or ii) B is false, all of the A_i but A_j are true but A_j is not false. Now

[14] Unless otherwise indicated, phrases such as "all wffs except P are true" or "all wffs but P are true" are shorthand for "if a wff is not P, it is true" so that the question of the truth of P is left open.

since $P \notin X \cap \bigcup_{j \in J} Y_j$, $P \notin \bigcup_{j \in J} Y_j$. For if P did belong to this union then all of X are true. But then if case i) holds all of the premisses of line (1) are true but the conclusion is not, contrary to the soundness of (1); and if case ii) holds, the conclusion of (1) is false, all its premisses except A_j are true but A_j is not false, which is impossible. Hence $P \in X$ and by the truth of all of $\bigcup_{j \in J} Y_j$ is either true or false. But it cannot be true else all of X are true which we have seen is not possible.

→E: *Truth-preservation*:
Suppose all of $\bigcup_{i \in I} X_i$, Y, $\bigcup_{k \in K} Z_k$ are true; then by the $1.i$ premisses all of the A_i are true and by line (2) $A_{i\ i \in I} \rightarrow B$ is true hence, by the semantics for →, B is true.

Falsity Preservation. Suppose B is false and that all wffs in $\bigcup_{i \in I} X_i$, Y, $\bigcup_{k \in K} Z_k$ except P are true.
There are three cases:– i) P occurs in exactly one X_i but not in Y; ii) P occurs in Y but not in $\bigcup_{i \in I} X_i$; iii) P occurs in $\bigcup_{i \in I} X_i \cap Y$ or in $(X_i \cap X_j)$ for some $i \neq j$. In the latter case, P has to be determinate by one of the $3.j$ premisses and the disjointness condition on $\bigcup_{j \in J} Z_j$ but it cannot be true else B is true by truth-preservation hence it is false. For the first two cases:

Case i); By the correctness of line (2) and the truth of all of Y, $A_{i\ i \in I} \rightarrow B$ is true hence since B is false at least one A_l is not true. Since P occurs in only one of the X sets, $P \in X_l$ and all the other X sets contain only truths; thus all the other A_i are true and by the semantics for →, A_l is false hence P is false by the correctness of $1.l$.

Case ii): By the correctness of lines $(1.i)$ and the truth of all of $\bigcup_{i \in I} X_i$, each A_i is true; since B is false, $A_{i\ i \in I} \rightarrow B$ is false hence by the correctness of line (2), P is false. ❑

The transitivity rule for the single antecedent conditional is also generalisable to:

X_i	(1.i) $P_i \rightarrow Q_i[i \in I]$	Given $i \in I$
Y	(2) $Q_{i\ i \in I} \rightarrow R$	Given
$\bigcup_{i \in I} X_i$, Y	(4) $P_{i\ i \in I} \rightarrow R$	1,2 Trans.

where all the P_i are distinct or else all the P_i wffs but at most one are the truth constant T.

To see that this rule is sound (in the Lukasiewiczian approximation), suppose all of $\bigcup_{i \in I} X_i$, Y are true (in some given model) and suppose further, on the one hand, that all of the P_i are true. Then by the $1.i$ premisses, all of the Q_i are true hence R is true by line (2); suppose, on the other hand, that R is false, all of the P_i bar P_j are true. By lines $1.i$, all of the Q_i are true, $i \neq j$, hence Q_j is false —by line (2)— hence P_j is false, by $1.j$. For the falsity preservation direction, suppose all of $\bigcup_{i \in I} X_i$, Y except A are true and that $P_{i\ i \in I} \rightarrow R$ is false. There are two possibilities: a) all of the P_i are true but R is untrue; b) R is false, all of the P_i but P_j are true and P_j is not false. In case a), suppose firstly that all the Q_i are true. Then $Q_{i\ i \in I} \rightarrow R$ is false, so by line (2), A $\in$ Y and is false. So suppose at least one Q_k is not true. Then $P_k \rightarrow Q_k$ is false and by line $1.k$, A $\in X_k$ and is false. In case b) suppose all the Q_i but Q_k are true, but Q_k is not false. Then $Q_{i\ i \in I} \rightarrow R$ is false, A $\in$ Y and is false. Suppose next that all the Q_i but Q_k are true, and Q_k is false. Then $P_k \rightarrow Q_k$ is false (whether or not $i = j$) A $\in X_k$ and is false. Suppose finally that two distinct antecedents Q_m and Q_n are not true. Then since all the P_i are distinct or all but one are T, at least one of P_m, P_n is true so at least one of $P_m \rightarrow Q_m$, $P_n \rightarrow Q_n$ is false; hence A $\in X_m$ or A $\in X_n$ and either way is false.

The interpretation of the truth (falsity) constant as a conjunction of all definitely true atomic sentences (disjunction of all definitely untrue ones) necessitates, however, that we drop the substitution rule of part I:

X	(1) $P \rightarrow Q$	Hyp
X	(2) $(P \rightarrow Q)^*$	$1 \rightarrow_{\text{SUB}}$

where * is an admissible substitution of wffs for atoms. For take three atomic sentences Ft, Gu, and Hv with Ft and Gu definitely true in the intended interpretation and Hv definitely false there. Then the substitution rule would generate inconsistency as follows:

T	(1) T	Hyp.
T, Ft	(2) Gu	1 &E. Exp.
—	(3) Ft $\rightarrow$ Gu	2 $\rightarrow$I
—	(4) Ft $\rightarrow$ Hv	3 Sub.

—	(5) Ft	1 &E
—	(6) Hv	4,5 →E
—	(8) ⊥	6 ∨I

We could restrict the substitution rule so that only definitely true atoms may be substituted for definitely true atoms, definitely false for definitely false but this would be to cripple the rule, and moreover the rule would then be redundant. Furthermore, its role in the completeness proof of Part I can, as we shall see, be replaced by the proof-theoretic power afforded by the interpretation of T and $\perp$.

The plural antecedent form of $\rightarrow$ (for which we include the case in which there are finitely many antecedents) behaves like the right associative iterated conditionals, e.g. $(\mathrm{A}_0 \rightarrow (\mathrm{A}_1 \rightarrow (\mathrm{A}_2 \rightarrow \mathrm{B})))$ in one respect, namely that from both the latter wff and also $((\mathrm{A}_0, \mathrm{A}_1, \mathrm{A}_2) \rightarrow \mathrm{B})$ we can conclude to B from A_0, A_1, A_2. However the iterated and plural forms are distinct, the distinction being clearer in Polish notation: the plural is $\langle \rightarrow, \{\mathrm{A}_0, \mathrm{A}_1, \mathrm{A}_2\}, \mathrm{B}\rangle$, whilst the iterated form is $\langle \rightarrow, \mathrm{A}_0, \rightarrow, \mathrm{A}_1, \rightarrow, \mathrm{A}_2, \mathrm{B}\rangle$. This enables us to distinguish e.g. $(\mathrm{A}, \mathrm{B}) \rightarrow \mathrm{A\&B}$ from $\mathrm{A} \rightarrow (\mathrm{B} \rightarrow \mathrm{A\&B})$. The first is true and indeed provable since A, B $\vdash$ A&B, whilst the latter fails where A and B are not both 'non-contingent'. In this way we resolve some of our problems over entailment. We have $r \in r$, $r \notin r \vdash \perp$; the determinacy restriction on →I blocks $r \in r \vdash r \notin r \rightarrow \perp$ (permitting us to reject $r \notin r \rightarrow \perp$ without commitment to $r \notin r$) whilst the neoclassical constraint blocks $(r \in r \,\&\, r \notin r) \vdash \perp$;[15] but we are able to represent the *ex contradictione* entailment in the object language by $((r \in r, r \notin r) \rightarrow \perp))$.

Turning to quantification now, this is effected in L by means of certain types of regular infinitary conjunction and disjunction where we index the class of substitution terms by Ω. For individual quantification, we abbreviate meta-linguistically $\&(\varphi x/\mathrm{t}_i)_{i<\Omega}$ and $\vee(\varphi x/\mathrm{t}_i)_{i<\Omega}$ by the usual $\forall x \varphi x$ and $\exists x \varphi x$.[16] As remarked the $i\ i < \Omega$ and x/t_i notation is metalinguistic and indicates the existence of a function from Ω indexing the simple terms of the language and generating a set of instances each of which results from the replacement of specific occurrences of a term in a fixed wff φ with a 'variable', an item outside our language, then uniform replacement of the variable

[15] The constraint on infinitary transitivity also blocks this derivation even though we have $(r \in r \& r \notin r) \vdash r \in r$, $(r \in r \& r \notin r) \vdash r \notin r$ and $r \in r, r \notin r \vdash \perp$.

[16] More restricted quantifications, over the natural numbers, small ordinals etc. can be expressed by regular conjunctions whose instances are all of the form φa but where substitutions for a are the canonical singular terms from the appropriate class, numerals, terms for small ordinals etc.

with a simple term of the language, each simple term occurring in an instance in this way. Similarly the first order monadic quantifications $\forall X \Phi X$ or $\exists X \Phi X$ are represented by infinitary conjunctions and disjunctions of the form: $\&(\Phi X/\mathrm{F}_i)_{i<\Omega}$ and $\vee(\Phi X/\mathrm{F}_i)_{i<\Omega}$, where now the function from Ω indexes all the simple monadic predicates of the language and the second order quantifications $\forall R \Phi R$ or $\exists R \Phi R$ are represented by similar conjunctions and disjunctions this time with the indexing function indexing all the simple relational terms. However to get the full power of standard quantification from our infinitary conjunction and disjunction rules we need to extend the second-order predicate comprehension schemata to an analogous schema for individual terms.

$$\exists x\ x = \mathrm{a} \text{ (that is, } \vee(\mathrm{t}_i = \mathrm{a})_{i<\Omega})$$

for all singular terms, simple or complex, a of the language. In this way we achieve the effect of ∀E and ∃I, for all our different quantifiers. Thus for any singular term, simple or complex, u we can prove $\varphi x/\mathrm{u} \rightarrow \exists x \varphi x$:

1	(1) $\varphi x/\mathrm{u}$	Hyp.
—	(2) $\exists x\ x = \mathrm{u}$	Axiom
3.i	(3.i) $\mathrm{t}_i = \mathrm{u}$	Hyp.
1, 3.i	(3.i.1) $\varphi x/\mathrm{t}_i$	1, 3.i =E →E[17]
1, 3.i	(3.i.2) $\vee(\varphi x/\mathrm{t}_i)_{i<\Omega}$	3.i.1 ∨I
1	(4) $\vee(\varphi x/\mathrm{t}_i)_{i<\Omega}$	2, 3.i.2 $<\Omega$, ∨E
—	(5) $\varphi x/\mathrm{u} \rightarrow \exists x \varphi x$	4 →I

This yields ∃I; ∨E, where the disjunction is of the form $\vee(\varphi x/\mathrm{t}_i)_{i<\Omega}$, will be classed as an application of ∃E, likewise for the appropriate forms of &I and &E (using the singular comprehension schema) with respect to ∀I and ∀E.

All this assumes, of course, that we have an identity predicate in the language and rules to match; but since we have second-order quantification we can define $\mathrm{t} = \mathrm{u}$ by $\forall X(X\mathrm{t} \rightarrow X\mathrm{u})$. The following rules are then derivable neoclassically:

1.i	(1.i) $\mathrm{F}_i\mathrm{t}$	Hyp. $i < \Omega$
—	(2.i) $\mathrm{F}_i\mathrm{t} \rightarrow \mathrm{F}_i\mathrm{t}$	1.i →I
—	(3) $\mathrm{t} = \mathrm{t}$	2.i $i < \Omega$ &I

[17] See below on the =E rule. If $\varphi x/\mathrm{u}$ is $\mathrm{t}_i = \mathrm{u}$ then →E is illegitimate but then we can prove $\exists x\ \mathrm{t}_i = x$ directly from =I and ∨I hence the conditional at line (5) by vacuous →I.

Δ	(1) $\mathrm{t} = \mathrm{u}$	Given
Δ	(2) $\forall X\ X\mathrm{t} \to X\mathrm{u}$	1 Def. $=$
—	(3) $\exists X \forall y(Xy \leftrightarrow \varphi y)$	Comp.
4	(4) $\forall y(\mathrm{F}y \leftrightarrow \varphi y)$	Hyp.
4	(5) $\mathrm{Ft} \leftrightarrow \varphi\mathrm{t}$	4 $\forall$E
Δ	(6) $\mathrm{Ft} \to \mathrm{Fu}$	2 $\forall$E
Δ, 4	(7) $\varphi\mathrm{t} \to \mathrm{Fu}$	5,6 Trans.
4	(8) $\mathrm{Fu} \leftrightarrow \varphi\mathrm{u}$	4 $\forall$E
Δ, 4	(9) $\varphi\mathrm{t} \to \varphi\mathrm{u}$	7,8 Trans.
Δ	(10) $\varphi\mathrm{t} \to \varphi\mathrm{u}$	3,9 $\exists$E

This rule then enables us to prove by $\to$E an $=$E rule of the form:

Γ	(1) $\varphi\mathrm{t}$	Given
Δ	(2) $\mathrm{t} = \mathrm{u}$	Given
Σ_i	(3) Det P_i	$\forall \mathrm{P}_i \in \Gamma \cap \Delta, i \in \mathrm{I}$
$\Gamma, \Delta, \bigcup_{i \in \mathrm{I}} \Sigma_i$	(4) $\varphi\mathrm{u}$	1,2 [3.i] $=$E

We can also derive the symmetry (=Symm.) —$\forall x \forall y(x = y \to y = x)$— and transitivity (=Trans.) of identity under the given definition. Symmetry is direct from the =E rule as is Transitivity in the form:

$$\forall x \forall y \forall z(x = y \to (y = z \to x = z)).$$

One last proof-theoretic matter which should be looked at is mathematical induction. One might think that there will be no need for induction in an infinitary language like L. Thus instead of inductive proofs that e.g. commutativity holds for each natural number, with respect to all others, one simply has brute proofs for each number, with the property of being a number characterised by an infinite disjunction of the form $x = 0 \vee x = 1$...[18] However this is too sweeping a response. To be sure, if there are proofs for each number that it has some property φ, this will give us, by infinitary &I, $\forall n \varphi$ that is $\& \varphi n_{i, i \in \omega}$ where the n_i are all the canonical numerals. But that is not quite what we usually want, we want an explicit statement of the form $\forall x(\mathrm{Num}\ x \to \varphi x)$, where x ranges over everything, i.e. we need a regular conjunction (of conditionals) over all simple terms and where Num x is the

[18] See Gödel's comments on infinitary languages in 'Russell's Mathematical Logic' in Paul Schilpp (ed.) *The Philosophy of Bertrand Russell*, Third Edition (London: Harper & Row, 1963).

standard inductive characterisation of being a number:

$$\forall \mathrm{F}((\mathrm{F}0 \mathbin{\&} \forall y(\mathrm{F}y \rightarrow \mathrm{F}y')) \rightarrow \mathrm{F}x)$$

Now we can get that if we can prove that

$$(i)\forall x(\text{Num } x \leftrightarrow x = 0 \vee x = 1...);$$

indeed the left to right direction is enough for then we need only assume Num t and use $\vee$E on the r.h.s. taken together with the proof of $\&\varphi n_{i,i\in\omega}$. But the obvious proof here of (i) will be an infinitary version of a standard inductive proof. Moreover the typical basis we will have for supposing that there exists an infinitary &I proof of $\&\varphi n_{i,i\in\omega}$ and other formulae of that type will be quasi-inductive in the sense that we will be able to show informally that for each term numeral n (to take the simple number-theoretic case as an example), we can extend a proof of $\varphi\mathrm{t}$ to one of $\varphi\mathrm{t}'$.[19] There will, however, be one important difference. Where in a standard inductive proof one has an inductive step or steps in which one supposes for the sake of argument that some inductive hypothesis is true —that φ holds of n, n a parameter in the above example— in the infinitary case one will have rather an *Induction Theorem*. That is, for each numeral there will be a *proof* of φn and that proof, one informally sees, will be a proper sub-proof of an extended proof of φ for n', the extension pretty much the same as the move from φn to $\varphi n'$ in a standard inductive step in finitary logic.

Finally, then, the definition of proofhood. As remarked proofs are to be identified with strings of sequents, that is with functions from some initial segment of the ordinals whose images are sequents. These are ordered pairs consisting of a premiss set and a wff and are represented by X $\Rightarrow$ P. We characterise the subset of strings which are genuine proofs recursively by a clause of this sort:

$$\begin{aligned}&\pi \in \text{PROOF} \leftrightarrow \exists\alpha \text{ Ord } \alpha \mathbin{\&} {\sim}\text{Limit } \alpha \mathbin{\&} \forall\beta \leq \alpha\\ &\quad \pi(\beta) = \mathrm{A} \Rightarrow \mathrm{A}\vee\\ &\quad \vee\pi(\beta) = \mathrm{T} \Rightarrow \mathrm{A} \text{ for A an axiom}^{20}\end{aligned}$$

[19] But it would be wrong to respond that, if so, why do we not simply make do with the indubitably finite, informal proof. For this is an informal proof of a mathematical existence claim, here of the existence of the abstract structure of some particular formal proof. And it will often be that this claim is only correct if the proof system and language are infinitary.

[20] That is, an instance of one of the axiom schemata of singular term or predicate comprehension, or of naïve set comprehension or an instance of Ant/Det or Maximin or a member of AXDET.

$\vee\exists \mathrm{I}\pi(\beta) = \mathrm{X} \Rightarrow \mathrm{A}_i, i \in \mathrm{I}\ \&\ \exists\gamma < \beta$ s.t. $\pi(\gamma) = \mathrm{X} \Rightarrow\ \&\mathrm{A}_{i\ i\in\mathrm{I}} \vee ...$[21]
$\vee\exists \mathrm{I}\pi(\beta) = \mathrm{U}_{i\in\mathrm{I}}\mathrm{X}_i \Rightarrow \&\mathrm{A}_{i\ i\in\mathrm{I}}\ \&\ \forall i \in \mathrm{I}, \exists\gamma < \beta$ s.t. $\pi(\gamma) = \mathrm{X}_i \Rightarrow \mathrm{A}_i$
$\vee\exists \mathrm{I,J}(\pi(\beta) = \mathrm{X}, \mathrm{U}_{i\in\mathrm{I}}\mathrm{Y}_i, \mathrm{U}_{j\in\mathrm{J}}\mathrm{Z}_j \Rightarrow \mathrm{C}$
$\quad \&\ \exists\gamma < \beta$ s.t. $\pi(\gamma) = \mathrm{X} \Rightarrow \vee\mathrm{A}_{i\ i\in\mathrm{I}}$
$\quad \&\ \forall i \in \mathrm{I}, \exists\delta < \beta$ s.t. $\pi(\delta) = \mathrm{Y}_i, \mathrm{A}_i \Rightarrow \mathrm{C}$
$\quad \&\ \forall \mathrm{B} \in \mathrm{X} \cap \mathrm{U}_{i\in\mathrm{I}}\mathrm{Y}_i, \exists\epsilon < \beta\ \exists j \in \mathrm{J}, \pi(\epsilon) = \mathrm{Z}_j \Rightarrow \mathrm{Det\ B})$
$\quad \&\ (\mathrm{X} \cup \mathrm{U}_{i\in\mathrm{I}}\mathrm{Y}_i) \cap \mathrm{U}_{j\in\mathrm{J}}\mathrm{Z}_j = \emptyset\vee$
$\exists \mathrm{I}(\pi(\beta) = \mathrm{X,Y}\ \mathrm{U}_{i\in\mathrm{I}}\mathrm{Z}_i \Rightarrow\ \mathrm{C}\ \&$
$\quad \&\ \exists\gamma, \delta < \beta$ s.t. $\pi(\gamma) = \mathrm{X} \Rightarrow\ \mathrm{A}\ \&\ \pi(\delta) = \mathrm{Y} \Rightarrow\sim\mathrm{A}$
$\quad \&\ \forall \mathrm{B} \in \mathrm{X} \cap \mathrm{Y}, \exists\delta < \beta\ \exists i \in \mathrm{I}$ s.t. $\pi(\delta) = \mathrm{Z}_i \Rightarrow \mathrm{Det\ B})$[22]
$\quad \&\ (\mathrm{X} \cup \mathrm{Y}) \cap \mathrm{U}_{i\in\mathrm{I}}\mathrm{Z}_i = \emptyset\vee$
$\pi(\beta) = \mathrm{X}, \mathrm{U}_{j\in\mathrm{J}}\mathrm{Y}_j \Rightarrow \mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}\ \&\ \exists\gamma < \beta\ \pi(\gamma) =$
$\quad \mathrm{X}, \{\mathrm{A}_{i\ i\in\mathrm{I}}\} \Rightarrow\ \mathrm{B}\ \&\ \forall\mathrm{C} \in \mathrm{X}, \exists\delta < \beta\ \exists j \in \mathrm{J}$ s.t.
$\quad \pi(\delta) = \mathrm{Y}_j \Rightarrow\ \mathrm{Det\ C}\ \&\ \mathrm{X}\ \cap \mathrm{U}_{j\in\mathrm{J}}\mathrm{Y}_j = \emptyset\vee$
$\exists \mathrm{I,J,K}\ (\pi(\beta) = \mathrm{U}_{i\in\mathrm{I}}\mathrm{X}_i, \mathrm{Y}, \mathrm{U}_{j\in\mathrm{J}}\mathrm{Z}_j, \Rightarrow \mathrm{B}\ \&$
$\quad \exists\gamma < \beta$ s.t. $\pi(\gamma) = \mathrm{Y} \Rightarrow \mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow\ \mathrm{B}\ \&\ \forall i \in \mathrm{I}, \exists\delta < \beta$
$\quad (\pi(\delta) = \mathrm{X}_i \Rightarrow \mathrm{A}_i\ \&\ \forall\mathrm{C} \in (\mathrm{Y} \cap \mathrm{U}_{i\in\mathrm{I}}\mathrm{X}_i) \vee (\mathrm{X}_i \cap \mathrm{X}_k), k \neq j$
$\quad \exists\epsilon < \beta\ \exists j \in \mathrm{J}(\pi(\epsilon) = \mathrm{Z}_j \Rightarrow\ \mathrm{Det\ C}\ \&\ \mathrm{U}_{j\in\mathrm{J}}\mathrm{Z}_j \cap (\mathrm{U}_{i\in\mathrm{I}}\mathrm{X}_i \cup \mathrm{Y}) =$
$\quad \emptyset\ \&\ \mathrm{U}_{j\in\mathrm{J}}\mathrm{Z}_j \cap (\mathrm{X}_i \cup \mathrm{X}_k), i \neq k, = \emptyset)))$

3. *Soundness and Completeness*

The next task is to prove soundness and completeness theorems for the formalised system L which is our regimentation of naïve set theory. The theory[23] of L is to be a semantically closed theory since resort to ascent up hierarchies is firmly eschewed; what is required, then, is an informal demonstration that there are, in L, soundness and completeness proofs for L. First, however, we need to develop a semantics for L in order to pair semantics notions with those from proof theory. Our highly impredicative form of second-order comprehension permits a recursive definition of truth by ensuring that there is some property P —for familiarity let us instantiate the variable with 'True'— satisfying:

True $x \leftrightarrow x = \underline{\mathrm{Ft}}\ \&\ \mathrm{Ft} \vee x = \underline{\mathrm{Rtu}}\ \&\ \mathrm{Rtu}\vee$ ... [through all the atoms]

[21] Similarly for the other 'singular premiss' rules:– $\vee$I, MM, $\forall$E, $\exists$I, $\sim$I, Expansion, ϵI, $\epsilon =$ and Contraposition.

[22] With a similar clause for Transitivity.

[23] I will treat L ambiguously as both the language of the theory and the theory itself, the ambiguity being harmless in this context.

$\vee$ Neg x & $\forall$ [or $\exists$] y in x, $\sim$True y
$\vee$ Conj x & $\forall y(y$ in $x \rightarrow$ True $y)$
$\vee$ Disj x & $\exists y(y$ in x & True $y)$
$\vee$ Cond x &
&((((&(Ant $\mathrm{u}_m\ x \leftrightarrow \vee(\mathrm{u}_m = \mathrm{t}_i)[i \in \mathrm{I}])[m \in \Omega]$
& True $\mathrm{t}_i)_{[i\in \mathrm{I}]} \rightarrow \forall z($Cons $z\ x \rightarrow$ True $z)))\mathrm{I} \in \mathrm{P}(\Omega)$

Some explanation is in order. The first clause deals in 'brute' fashion with truth for atoms.[24] Of course, we could have replaced conjuncts such as $x = \underline{\mathrm{F}}\mathrm{t}$ & Ft with $x = \underline{\mathrm{Ft}}$ & $f(\mathrm{t}) \in f(\mathrm{F})$ (or indeed $g(\mathrm{t}) \in g(\mathrm{F})$, for any interpretation function g, thus generalising to a notion of truth relative to a possibly 'unintended' interpretation function g). A fair amount of obvious detail has been skipped here —in particular definitions of syntactic notions such as x is a conjunction (Conj x), y is an immediate constituent of x (y in x), x_i is an antecedent of conditional x (Ant $x_i x$), y is a consequent of conditional x (Cons yx) and so on. These definitions will ensure we have such theorems as:

$$\vdash \ \text{Ant tu} \rightarrow \exists x, y\ \mathrm{u} = \langle \rightarrow, x, y\rangle \ \&\ \mathrm{t} \in x$$

The right conjunct of the final clause

&((((&(Ant $\mathrm{u}_m\ x \leftrightarrow \vee(\mathrm{u}_m = \mathrm{t}_i)[i \in \mathrm{I}])[m \in \Omega]$ & True $\mathrm{t}_i)_{[i\in \mathrm{I}]} \rightarrow$

$\forall z($Cons $z\ x \rightarrow$ True $z)))\mathrm{I} \in \mathrm{P}(\Omega)$

is a conjunction in which we go through all subsets of the set of simple terms.[25] Each conjunct, indexed by some given subset, is an infinitary conditional whose antecedent contains for each term t_i in the subset a conjunction one conjunct of which is the claim that t_i is true, the other being an

[24] Note that if we tried to define truth in general in brute fashion, e.g. by True $x \leftrightarrow \vee(x = \underline{\varphi}_i$ & $\varphi_i)_{i\in \mathrm{I}}$ where I is a meta-linguistic set indexing all the canonical names of L wffs, then we run into the problem that the defining formula must have rank ON in V since the component canonical terms are of unbounded rank, and indeed must include the truth formula itself. Hence the formula is not definitely a wff since it is not determinate whether $\mathrm{ON} \in \mathrm{V}_{\mathrm{ON}}$.

[25] Hitherto the metalinguistic notation &$(\varphi_i)[i \in \mathrm{I}]$ has represented an infinitary conjunction any two conjuncts of which differ solely by substitution of one term indexed by a member of I for another. Here we have a slightly more complex case —the notation &$(\psi_{\mathrm{I}})[\mathrm{I} \in \mathrm{P}(\Omega)]$ represents a conjunction whose conjuncts differ by virtue of being themselves regular conjunctions, as in the previous sentence, but with the variation occurring across different sets of simple terms. But in this case as before the metalinguistic expression picks out a well-formed formula of L.

infinitary conjunction (the same in each case) expressing the claim that the referents of the simple terms in the given subset are exactly the antecedents of x. The consequent of the infinitary conditional states (relative to the hypothesis given in the antecedent) that if all the antecedent wffs of x are true then the consequent of x is true.

Call a wff φ syntactically definite, or *s-definite*, iff $\psi(\underline{\theta})$ is definitely true or definitely false, in the intended interpretation, where ψ is any syntactic claim and θ any sub-formula of φ (this will be tightened up later). Then every instance of the naïve Tarskian schema:

$$\text{True } \underline{\varphi}_i \leftrightarrow \varphi_i$$

is provable for all s-definite wffs in L (see the Appendix I: Tarskian equivalences). So we could define $\models \varphi$ by True $\underline{\varphi}$ at least for s-definite (mathematical) φ. However granted the Tarskian equivalences we can simply use φ itself to express $\ulcorner\varphi\urcorner$ is valid (for mathematical φ) and we will not go wrong, granted those equivalences, over s-definite wffs. Moreover the infinitary conditional enables us to define X $\models$ B by $\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow$ B where the levels of the A_i in X have some upper bound so that $\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow$ B is definitely a wff; in this way, the weak completeness result:– if $\vdash \varphi$ then $\models \varphi$ —is in fact equivalent to an apparently stronger result:– if X $\vdash \varphi$ then X $\models \varphi$ —for very many sets of wffs X.

Given the above semantics, and with lower case 'true' representing truth in the interpretation selected as 'intended' (and in terms of which the constants T and $\perp$ are defined) we have restricted meta-theoretic soundness and completeness results:

THEOREM:

$$\forall\varphi \in \Delta, \varphi \text{ is true iff } \vdash \varphi$$

Here Δ is the class of all sentences such that every sub-formulae of a member is either definitely true or definitely not true.

Completeness: by induction on wff complexity we prove that if φ is definitely true then $\vdash \varphi$ and if it is definitely not true then $\vdash\sim\varphi$.

Atomic case: If A is definitely true then A is a conjunct of T then $\vdash \varphi$ by &E. If A is definitely not true, then A is a disjunct of $\perp$ so that $\sim$A is provable by:

1	(1) A	Hyp.
1	(2) $\perp$	1 $\vee$I
—	(3) $\sim$A	2, $\sim$I

Inductive clauses:– consider the case for $\rightarrow$. If $\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}$ is definitely true and in Δ then either some A_i is definitely untrue or B is definitely true. In the first case we have:

—	(1) $\sim\mathrm{A}_i$	IH
2	(2) A_i	Hyp.
$\mathrm{A}_{i\ i\in \mathrm{I}}$	(3) B	1,2 $\sim$E, Exp.
—	(4) $\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}$	3 $\rightarrow$I

whilst the second is similar using the provability of B. If, on the other hand, $\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}$ is definitely untrue then since every sub-formula is either definitely true or definitely untrue, all of the A_i are definitely true and B is definitely not true. Hence the conditional is refutable via:

1	(1) $\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}$	Hyp.
—	(2.i) A_i	IH, $i \in \mathrm{I}$
1	(3) B	1,2.i $i \in \mathrm{I}$ $\rightarrow$E
—	(4) $\sim$B	IH
1	(5) $\perp$	3,4 $\sim$E
—	(6) $\sim (A_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B})$	5, $\sim$I

Corollary: For $\varphi \in \Delta$, $\vdash \varphi$ or $\vdash\sim\varphi$. Proof: Every φ in Δ is either definitely true or definitely not true. ❑

Hence if the language of arithmetic is in Δ, as seems plausible, then all arithmetic truths are neoclassically provable in naïve set theory; if the language of ZFC is in Δ then all truths of ZFC set theory are provable.[26]

Soundness: by induction on proof length. Since the key sentences belong to Δ, the proof can use the classical reasoning of the soundness proofs for Lukasiewiczian semantics (but with the simplification that untruth and falsity are equated). Again consider the case for the basic $\rightarrow$ rules, in particular the trickier falsity preservation directions:

[26] Of course many theorems of ZFC, e.g. the Axiom of Foundation, will fail as usually expressed. But they will hold relativised to V_κ, for inaccessible κ.

→I:

$\mathrm{X},\mathrm{A}_{i\ i\in\mathrm{I}}$	(1) B	Given
$\mathrm{Y}_{j\ j\in\mathrm{J}}$	(2.j) Det C	for all $\mathrm{C}\in\mathrm{X}$
$\mathrm{X}, \bigcup\limits_{j\in\mathrm{J}} Y_j$	(3) $\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}$	1, [2.j] $j\in\mathrm{J}$ →I

where $\mathrm{X}\cap\bigcup\limits_{j\in\mathrm{J}}\mathrm{Y}_j=\emptyset$.

Falsity Preservation: Suppose $\mathrm{A}_{i\ i\in\mathrm{I}}\rightarrow\mathrm{B}$ is false (untrue) and all of X, but P are true. Since $\mathrm{A}_{i\ i\in\mathrm{I}}\rightarrow\mathrm{B}$ belongs to Δ, it follows from the fact that it is untrue that all of the A_i are (definitely) true but B is (definitely) untrue. The inference here is from

> Not: [if all of the A_i are true then B is true and if B is untrue and all of the A_i but A_j are true then A_j is untrue]

to

> Either: Not [if all of the A_i are true then B is true] or Not: [if B is untrue and all of the A_i but A_j are true then A_j is untrue]

For the first disjunct it follows (here we require that $\mathrm{A}_{i\ i\in\mathrm{I}}\rightarrow\mathrm{B}$ is in Δ so questions of truth for all its components are determinate) that all of the A_i are true but B is untrue, whence, by the correctness of line 1, P is false.

For the second disjunct, it follows in like fashion that B is untrue, all of the A_i but A_j are true and A_j is also true so the conclusion is the same.

→E:

X_i	(1.i) A_i	Given, $i\in\mathrm{I}$
Y	(2) $\mathrm{A}_{i\ i\in\mathrm{I}}\rightarrow\mathrm{B}$	Given
Z_k	(3.k) Det C_j	$\forall\mathrm{C}_j\in\Sigma$
$\bigcup\limits_{i\in\mathrm{I}}\mathrm{X}_i$, Y, $\bigcup\limits_{k\in\mathrm{K}}\mathrm{Z}_k$	(4) B	1.i $i\in\mathrm{I}$, 2, 3.k, $k\in\mathrm{K}$, →E

where Σ is the set of all wffs which either belong to $\bigcup\limits_{i\in\mathrm{I}}\mathrm{X}_i\cap\mathrm{Y}$ or else belong to $\mathrm{X}_i\cap\mathrm{X}_k$ for some $i\neq k$ and where $\bigcup\limits_{j\in\mathrm{J}}\mathrm{Z}_j\cap(\bigcup\limits_{i\in\mathrm{I}}\mathrm{X}_i\cup\mathrm{Y})=\emptyset$ and $\bigcup\limits_{j\in\mathrm{J}}\mathrm{Z}_j\cap(\mathrm{X}_i\cup\mathrm{X}_k)=\emptyset$ for $i\neq k$.

Falsity Preservation: Suppose B is untrue and all wffs in $\bigcup\limits_{i\in\mathrm{I}}\mathrm{X}_i$, Y, $\bigcup\limits_{k\in\mathrm{K}}\mathrm{Z}_k$ other than P are true. Since $\mathrm{A}_{i\ i\in\mathrm{I}}\rightarrow\mathrm{B}\in\Delta$, either all of the A_i are true or one at least is untrue. If the first holds, $\mathrm{A}_{i\ i\in\mathrm{I}}\rightarrow\mathrm{B}$ is untrue, $\mathrm{P}\in\mathrm{Y}$ and is untrue by line (2). If the second disjunct is the case and A_j is untrue then $\mathrm{P}\in\mathrm{X}_j$ and is untrue by line 1.j. ❏

But these soundness and completeness results are restricted to the classical sector of the language Δ where all wffs are determinate. Since our reasoning in the naïve theory deals with indeterminate sentences and, particularly when interpreting the theory, with classes such as U, ON and V concerning which many matters are indeterminate, we need to extend the soundness result. We need also to show how soundness and completeness results are provable for L from within L, if L is to have the semantic closure property which any stable theory ought to have.[27] In fact, as our troubles at the beginning of section 2 showed, soundness from within is the only way we will be able to generalise the result. It is necessary, first of all, to show that L can express its own syntax and proof theory. We note firstly that every member x of the universal set U has a ‘primitive’ name $\ulcorner x \urcorner$, this being the converse image of f_1 the restriction of our ‘intended’ interpretation f to the simple terms. Moreover the informal specifications of syntactic properties and relations — e.g. being a wff, being an λ-sized conjunction each of whose conjuncts is a relational predication one of whose terms occurs in the set X, and so on— can clearly be formalised themselves in L so that all our syntactic notions are definable in L.

Let the empty set be canonically named by some fixed simple term $\ulcorner \emptyset \urcorner$ which designates it in the intended interpretation; write this CAN $\emptyset \ulcorner \emptyset \urcorner$ ($\emptyset$ is canonically named by $\ulcorner \emptyset \urcorner$)and extend the definition of CAN via the recursive definition (instance of set comprehension)

$$\begin{aligned} \mathrm{CAN}xy \leftrightarrow \quad & x = \emptyset \;\&\; y = \ulcorner \emptyset \urcorner \vee \\ & \vee(\exists \mathrm{I}\; y = \ulcorner \epsilon v(\forall w(w \in v \leftrightarrow \vee(w = z_i))) \urcorner [i \in \mathrm{I}], \;\& \\ & \&\; \forall i \in \mathrm{I}, \exists t \in x\; \mathrm{CAN}tz_i \;\& \\ & \forall t \in x, \exists i \in \mathrm{I}\; \mathrm{CAN}tz_i).^{28} \end{aligned}$$

Thus a class term canonically names a class if it lists (perhaps infinitely), by means of a disjunction of identity statements, all its members each designated by one of its canonical names. There will be infinitely many such terms for each x which is canonically named, of course, but using epsilon terms we can specify a unique canonical name as $\epsilon w(\mathrm{CAN}xw)$ and represent the canonical name of x meta-theoretically by $\underline{x}$. Every member of V has a canonical name.

[27] See again my ‘Naïve Set Theory is Innocent!’ op. cit.

[28] Here logical symbols such as “$=$” and “$\rightarrow$” are used ambiguously as both as parts of the mathematical English of the metalanguage and as names of the corresponding operators in the object language.

Proof. Suppose for all $\beta < \alpha$, $\forall x \in \mathrm{V}_\beta$, $\exists y$ CNxy. Let a be an arbitrary member of V_α; from the definition of V_α, each member y of a belongs to some V_β as above, (if a has no members then CN$\mathrm{a}\ulcorner\emptyset\urcorner$). Hence by IH for each such y there is a z with CNyz so that $\underline{y}$ exists for each such y and hence $\ulcorner w = \underline{y}\urcorner$ exists for each such y. Letting the index class I be just the class of all y in a, hence I = a, we get the existence of an infinitary disjunction φ of the form: $\vee\ulcorner w = \underline{y}\urcorner$, $\forall y \in$ I. The class term $\{w : \varphi\}$ is thus an expression which lists all and only the canonical names of members of a in an infinite disjunction of the type specified in the definition of CAN. For any y in I, there is a t such that CAN$t\underline{y}$ namely y and for any t in a there is y in I with CAN$t\underline{y}$ namely y once again. ❏

As promised in section 2, I will extend the use of the metalinguistic device of underlining so that where $\ulcorner\varphi\urcorner$ is a (perhaps complex) metalinguistic term referring (under assignments to any component parameters) to an object language expression x, $\ulcorner\underline{\varphi}\urcorner$ refers (under the same assignment to parameters) to an object language canonical name ϵw(CAN$\mathrm{t}w$) where t refers to x in the intended interpretation (here we presuppose some choice function over canonical names). Thus, e.g. $\underline{\text{'A} \rightarrow \text{B'}}$, refers, given an assignment of wff x to 'A' and wff y to 'B' to some object language term ϵw(CAN$\mathrm{t}w$) where t refers in the intended interpretation to the string whose first term is the conditional, second term is x and third term is y.

V *completeness*.

Our first completeness theorem shows us that naïve set theory in L decides all 'positive' questions of identity and membership in V.

THEOREM: $\forall y \in$ V, if $x \in y$ then $\vdash \underline{x \in y}$; if $x = y$ then $\vdash \underline{x = y}$;

Proof: If $x \in y$ then $v = \underline{x}$ is one of the disjuncts in the class term which is $\underline{y}$ hence by =I, $\vee$I and set comprehension $\vdash \underline{x \in y}$. If $x = y$ then by the uniqueness of canonical names $\underline{x} = \underline{y}$ hence $\vdash \underline{x = y}$ by =I. ❏

We do not have the negative halves of these principles and a consequence of this is that our language L is a 'fuzzy' set with it being indeterminate whether certain strings, ON-long strings, for example, are wffs. But this does not leave us incapable of proving negatives, e.g. that string s is not a wff or not a proof. Thus we can prove by induction that no wff starts with two predicate letters, none however long is of the form FF ... . Similarly we can prove by induction that every wff and every proof string is a member of V.

Moreover consider any property φx. It may well be indeterminate, for many x, whether or not φx applies to x but we can narrow φx down to a 'slimmer', more precise property ⌜Definitely φx⌝ defined, as remarked earlier, by $\Box\varphi x$ (i.e. $\mathrm{T} \rightarrow \varphi x$). For $\varphi \mathrm{t}$ is true (in the mathematical sector) just when $\Box\varphi \mathrm{t}$ is but when $\varphi \mathrm{t}$ is indeterminate $\Box\varphi \mathrm{t}$ is false:–

1	(1) $\Box\psi$	Hyp.
2	(2) ψ	Hyp.
2	(3) $\psi \vee \sim \psi$	2 $\vee$I
—	(4) $\psi \rightarrow (\psi \vee \sim \psi)$	3 $\rightarrow$I
1	(5) Det ψ	1,4 Trans.
—	(6) $\Box\psi \rightarrow$ Det ψ	5 $\rightarrow$I

so by contraposition, $\sim$Detψ $\rightarrow\sim\Box\psi$. By moving from φx to $\Box\varphi x$ we extrude the indeterminate elements which are now part of the negative extension of the concept ⌜Definitely φ⌝.

Of course higher-order indeterminacy means we may never be able, for a particular φ, to expunge all indeterminacy. Let us define Det^1 P $\equiv_{\mathrm{df.}}$ Det P, $\mathrm{Det}^{\beta+1}\mathrm{P} \equiv_{\mathrm{df.}}$ Det (Det^{β}P) and Det^{γ}P, for γ a limit, by

$$\vee(\&(\mathrm{Det}^{\alpha}\ \mathrm{P})\beta \leq \alpha < \lambda)0 < \beta < \lambda$$

(Thus Det^{λ} P says that P is determinate all the way up to —but without reaching— λ from some point β or other below.)

We can describe P as 'quasi$^{\alpha}$-determinate' when $\mathrm{Det}^{\alpha+1}$P holds and as hyper$^{\alpha}$-indeterminate with $\sim\mathrm{Det}^{\alpha+1}$P holds (quasi-determinacy and hyper-indeterminacy being quasi^1-determinacy and hyper^1-indeterminacy). It may be hyper-indeterminate whether or not φu in which case it will be indeterminate whether or not $\Box\varphi$u and though in such a case $\Box\Box\varphi$u will be false via :–

1	(1) $\Box\Box\psi$	Hyp.
2	(2) $\Box\psi$	Hyp.
—	(3) $\Box\psi \rightarrow$ Det ψ	Theorem
2	(4) Det ψ	2,3 $\rightarrow$E
2	(5) Det $\psi \vee \sim$Det ψ	4 $\vee$I
—	(6) $\Box\psi \rightarrow$ (Det $\psi \vee \sim$Det ψ)	5 $\rightarrow$I
1	(7) Det Det ψ	1,6 Trans.
—	(8) $\Box\Box\psi \rightarrow \mathrm{Det}^2\psi$	7 $\rightarrow$I[29]
	(9) $\sim\mathrm{Det}^2\psi \rightarrow\sim\Box\Box\psi$	8 Contrap.

[29] This proof may be generalised to show $\Box^{\alpha}\varphi \rightarrow \mathrm{Det}^{\alpha}\varphi$ with the obvious definition of $\Box^{\alpha}$.

there may be a hyper^2-indeterminate m for which ❑❑φx is indeterminate and so on.

Still, though indeterminacy may never be completed expunged by iterating the definitely operator applied to a concept φx —there is always some x or other such that ∼Det❑$^\lambda\varphi x$— nonetheless it will never be the case that an object is both definitely φ and definitely not φ. There can be blobs which are not definitely yellow nor definitely red (pace Williamson)[30] but any blob which is on the definitely red/not definitely red boundary (however fuzzy it is) will not be on the definitely yellow/not definitely yellow boundary. Or at any rate, such a principle holds for indeterminacy in the current naïve set theory framework for we have ⊢∼ (❑P & ❑∼P):[31]

1	(1) ❑P & ❑∼P	Hyp.
1	(2) ❑P	1 &E
1	(3) ❑∼P	1 &E
1	(4) P → ∼T	3 Contrap.[32]
5	(5) ∼T	Hyp.
5	(6) ⊥	5 MM[33]
—	(7) ∼T → ⊥	6 →I
1	(8) P → ⊥	4,7 Trans.
1	(9) T → ⊥	2,8 Trans.
—	(10) T	Hyp.
1	(11) ⊥	9, 10 →E
—	(12) ∼ (❑P & ❑∼P)	11 ∼I

Our focus, now, will be on the concepts of definite provability and definite unprovability, that is ❑(⊢ φ) and ❑(Not: ⊢ φ). By our V completeness

[30] *Vagueness* (London: Routledge, 1994).

[31] In fact we can make do with the weaker ⊢ ❑P → ∼❑∼P.

[32] Eliding a double negation elimination.

[33] This move is one from a wff of the form ∼&$(\mathrm{B}_i)[i \in \mathrm{I}]$ to one of the form ∨$(\sim\mathrm{B}_i)[i \in \mathrm{I}]$.

result, the classes of determinately provable wffs and of determinately unprovable wffs are represented by two terms, let us say Bew and NBew respectively. So we have:[34]

REPRESENTABILITY (REP):

a) ❏($\vdash$ P) then $\vdash$ Bew $\underline{\mathrm{P}}$.
b) ❏(Not: $\vdash$ P) then $\vdash$ NBew $\underline{\mathrm{P}}$

(writing, more familiarly, Bew x for $x \in$ Bew). By the result $\vdash\sim$ (❏P & ❏$\sim$P) we know that for every member of x of U and any term t which stands in the intended interpretation for x,

Not: ❏($\vdash$ t) & ❏(Not: $\vdash$ t).

holds in the intended interpretation. So by metatheoretic completeness:

Lemma: $\forall$P, $\vdash\sim$ (Bew $\underline{\mathrm{P}}$ & NBew $\underline{\mathrm{P}}$).

Hence we can show that the following soundness result is provable in L for the sub-sector $\Pi \subseteq$ L such that $\forall$P $\in \Pi$, ❏($\vdash$ P) or ❏(Not: $\vdash$ P).

SOUNDNESS: $\forall$P $\in \Pi$, $\vdash$ Bew $\underline{\mathrm{P}} \rightarrow$ P.[35]

Proof:

1	(1) ❏($\vdash$ P) or ❏(Not: $\vdash$ P).	Hyp.
2	(2) ❏($\vdash$ P)	Hyp.
2	(3) $\vdash$ P	2 $\rightarrow$E (from T $\vdash$ T)
2	(4) $\vdash$ Bew $\underline{\mathrm{P}} \rightarrow$ P	3 $\underline{\rightarrow \mathrm{I}}$
5	(5) ❏(Not: $\vdash$ P).	Hyp.
5	(6) $\vdash$ NBew $\underline{\mathrm{P}}$	5 REP
—	(7) Bew $\underline{\mathrm{P}} \vdash$ Bew $\underline{\mathrm{P}}$	$\underline{\text{Hyp.}}$

[34] In what follows, ⌜Det φ⌝ is used both as a metalinguistic abbreviation for ⌜necessarily φ or it is not the case that φ⌝ and as an abbreviated name for an object language operator T $\rightarrow$ ($\varphi \vee \sim\varphi$), φ a schematic variable and similarly ❏φ will function both as abbreviation and name of object language operator. English operators will often be used for metalinguistic conjunction, disjunction etc. so that for example, ⌜it is not the case that P is provable⌝ will be represented by ⌜Not: $\vdash$ P⌝. The single turnstile '$\vdash$' is thus another piece of metalinguistic notation.

[35] Soundness is thus only demonstrable for wffs which are definitely provable or definitely not provable: but why should we be concerned about soundness for any other wffs:– dodgy characters all!

5	(8) Bew $\underline{P} \vdash$ Bew $\underline{P}$ & NBew $\underline{P}$	6,7 $\underline{\&\mathrm{I}}$
—	(9) $\vdash\sim$ (Bew $\underline{P}$ & NBew $\underline{P}$)	Lemma
5	(10) Bew $\underline{P} \vdash \underline{P}$	8,9 $\underline{\sim \mathrm{E}}$
5	(11) $\vdash$ Bew $\underline{P} \to$ P	10 $\underline{\to \mathrm{I}}$
1	(12) $\vdash$ Bew $\underline{P} \to$ P	1, 4, 11 $\vee$E

In proofs such as the above, rules which are underlined are mentioned in the metalanguage not used: more fully we are appealing to clauses in the recursive definition of proofhood. Thus at step (10) we appeal to the conditional:

If Bew $\underline{P} \vdash \underline{P}$ then $\vdash$ Bew $\underline{P} \to$ P.

which is derivable from the recursive definition of proofhood by using $\forall$I's then $\vee$I to derive the right hand side of the appropriate instance (instantiating with $\ulcorner$Bew $\underline{P} \to$ P$\urcorner$) then $\to$E to get the conclusion. Note that we could widen our language to include non-mathematical sentences so that P can contain the intensional conditional subject to the intensional proof rules. For there are no uses of the intensional $\to$ rule in the proof and the $\to$E at line three is legitimate for the intensional conditional too.

As a corollary of the above theorem we can see that there is a consistency proof for the theory of L from within that theory, i.e. $\vdash\sim$Bew $\underline{\perp}$

Proof:

1	(1) Bew $\underline{\perp}$	Hyp.
—	(2) Bew $\underline{\perp} \to \perp$	Theorem
1	(3) $\perp$	1,2 $\to$E
—	(4) $\sim$ Bew $\underline{\perp}$	3 $\sim$I

In the converse direction, meta-theoretic completeness ensures that there is an object language completeness proof for members of Δ the set of all wffs all of whose sub-formulae are either definitely true or definitely untrue.

COMPLETENESS: For all P $\in \Delta \vdash$ P $\to$ Bew $\underline{P}$

Proof:

1	(1) P is true $\vee$ P is untrue	Hyp.
2	(2) P is true	Hyp.
2	(3) $\vdash$ P	2 Metacompleteness
2	(4) $\vdash$ Bew $\underline{P}$	3 REP
2	(5) $\vdash$ (P $\to$ Bew $\underline{P}$)	3 $\underline{\to \mathrm{I}}$
6	(6) P is untrue	Hyp.

6	(7) $\vdash \sim$P	6 Metacompleteness
6	(8) $\vdash$ P $\rightarrow$ Bew $\underline{\text{P}}$	7 Hyp. $\sim$ E, $\rightarrow$ I
1	(9) $\vdash$ P $\rightarrow$ Bew $\underline{\text{P}}$	1,5,8 $\vee$E

4. *Superparadox?*

It might be objected that these results are rather trivial since they are not soundness and completeness results on the relations between proof and truth, defined metatheoretically for the object language L. Rather they are demonstrations that we can prove the soundness and completeness of L, from within L (i.e. from within naïve set theory expressed in that language) itself. As we have seen, we can prove metatheoretically soundness and completeness results for a narrow fragment of L, the class Δ. But the most interesting feature of L is the fact that it provides for expression of indeterminate propositions which classical logicians officially abjure (whilst having unofficial resort to them) and if we are to be happy reasoning with such propositions we need at least soundness proofs for the sentences of the language which express them, at any rate where it is determinate whether or not the sentence is provable. But in the last section we did not prove, in our informal metatheory, soundness and completeness results for the sectors Π and Δ, rather we showed in the metatheory that, e.g. in $\Pi \subset$ L we could prove soundness for Π.

Notoriously, soundness proofs have a boot-strapping quality about them but is this not taking boot-strapping too far? Could we not show in similar boot-strapping fashion that in a language L_P containing Prior's connective 'tonk', soundness could be proved if we can utilise the tonk rules themselves in the language? However we know in advance that the 'tonk' rules are unsound so that proofs using those rules are worthless. We do not know that about the naïve rules (whatever classicists might *think* they know).

Moreover standard soundness proofs prove soundness for an object language theory θ in object language L from within a *stronger* theory θ* in metalanguage M at least as rich as O. For example, we prove soundness for ZFC from the theory ZFC plus 'there exists at least one inaccessible'. For all their technical value, such proofs are philosophically worthless. They are akin to a 'super-naturalised epistemology' in which one proves the reliability of perception by physical organisms of the ordinary physical world by positing a mysterious interaction between brain activity and Cartesian souls mediated by an intermediary link I. If such an epistemologist 'explained' how the link I worked by postulating a link I^{-1} between brain and I and a link I^{1} between I and the soul, 'explained' the link operation of the link I^{-1} by postulating a link I^{-2} between brain and I^{-1} and so on, no one would think this explanatory regress was virtuously circular. I suggest that proving the soundness of a theory θ by resort to a stronger theory θ* which one can only show to be sound by appeal to a yet stronger theory θ**, and so on and so on, is as epistemologically empty as the supernaturalised epistemologists explanation of how perception gives us knowledge of the world. Genuine naturalised epistemology —assuming our best theory of the world and then attempting to show how, *according to that very theory itself*, we could come to reliably believe it— is far from a trivial task. Similarly, proving the soundness of $\Pi \subset$ L, from within Π, is far from a trivial matter, as we have seen; we had to “warp up” to infinitary logic to effect it, after all.

As remarked re the comparison with the ‘tonk’ case, the neo-classical system is not obviously trivial. The logic, as we saw in Part I, has been tested for soundness in an approximate simulation, via the gappy Lukasiewiczian semantics, of the full non-gappy naïve set theoretic framework. But the bootstrapping nature of all soundness proofs does mean that the existence of a soundness proof for L in L does not rule out with certitude the possibility that the system is incoherent. It is fairly uncontentious now that one cannot find certitude even in mathematics, not for complex and powerful mathematical systems at any rate. None of this, however, obviates the need to make it at least reasonably plausible that the system is coherent by showing that it is not vulnerable at those stress points which have proved fatal to other systems by generating paradoxes which turn into irresoluble antinomies.

One of the most common ways to generate paradox in a formal system is via self-referential sentences such as liar sentences. To do this in a formal language, we need a fixed point or diagonalisation lemma assuring us that for any canonical predicate φ there is a sentence φ* such that:

$$\vdash \varphi^* \leftrightarrow \varphi\underline{\varphi^*} :$$

How might we attempt to prove the fixed point lemma in the neo-classical framework? Let us consider the set of one-place open formulae with a fixed free variable x. These are the results of substituting the fixed variable x (variables being terms not occurring in L) for a singular term of L. Since the set 1-Wff of such formulae is a subset of V, every member of 1-Wff has a canonical name. Provably, in L —from second-order comprehension— we get the existence of a ternary relation Rabc representing [a is the result of substituting b for a fixed free variable x in a one-place open formula c] such that

$\mathrm{Rabc} \leftrightarrow\sim (\text{1-Wff c \& ST}\mathrm{b})\ \&\ \mathrm{a} = \mathrm{b} \vee$

$$\begin{aligned}&\text{1-Wff c \& STb \& } \exists \mathrm{F} \in \Omega^2, \mathrm{c} = (\mathrm{F}x \text{ \& } \mathrm{a} = \mathrm{Fb}) \vee ...\\ &\quad \vee \exists \mathrm{S} \in \Omega^3, \exists \mathrm{d} \in \mathrm{ST}, (\mathrm{c} = \mathrm{S}x\mathrm{d} \text{ \& } \mathrm{a} = \mathrm{Sbd}) \vee\\ &\quad \vee (\mathrm{c} = \mathrm{Sd}x \text{ \& } \mathrm{a} = \mathrm{Sdb}) \vee\\ &\quad \vee \exists y \exists z \exists \lambda (\text{Conj } y \text{ \& Conj } z \text{ \& ln } y = \text{ln } z = \lambda \text{ \& }\\ &\quad \mathrm{a} = y \text{ \& } \mathrm{c} = z \text{ \& } \forall i \in \lambda, \forall v, w, ((v = y_i \text{ \& } w = z_i)\\ &\quad \to \mathrm{R}(v\mathrm{b}w) \vee ...)\end{aligned}$$

with similar clauses for $\vee$, & and $\to$. So Rabc holds if i) either c is not a one-place open formula (1-Wff) or b is not a singular term (ST); or ii) that is not so and c is an atom in which case we specify the replacement in brute fashion; or iii) c is a complex sentence, for example a conjunction, in which case we specify that a is a wff of the same length (ln = λ), that is we can index both conjuncts by an ordinal λ, and the corresponding conjuncts (y_i, z_i) under the mapping are such that a's conjunct is the result of applying the R operation to c's.

Now consider formulae such as $\sim \varphi\epsilon y(\mathrm{R}(y(\epsilon z\mathrm{CAN}xz)x))$ for any primitive second-order ternary predicate R. We will be particularly interested, of course, in $\sim$Bew $\epsilon y(\mathrm{R}(y(\epsilon z\mathrm{CAN}xz)x))$. For each R, the 1-wff will have a canonical name; fix on one particular R and suppose $\sim$Bew $\epsilon y(\mathrm{R}(y(\epsilon z\mathrm{CAN}\, xz)x))$ has canonical name t. Consider next the wff

$$\sim \text{Bew } \epsilon y(\mathrm{R}(y(\epsilon z\mathrm{CANt}z)\mathrm{t}))$$

Suppose that under the assignment to R in the intended interpretation the above biconditional defining Rabc is true so that the wff 'says of itself that it is unprovable':– the result of substituting a (and therefore, by functionality, *the*) canonical name of the referent of t into $\sim$Bew $\epsilon y(\mathrm{R}(y(\epsilon z\mathrm{CAN}xz)x))$ is just the wff $\sim$Bew $\epsilon y(\mathrm{R}(y(\epsilon z\mathrm{CANt}z)\mathrm{t}))$. If the identity sentence:

$$\underline{\sim \text{Bew } \epsilon y(\mathrm{R}(y(\epsilon z\mathrm{CANt}z)\mathrm{t}))} = \epsilon y(\mathrm{R}(y(\epsilon z\mathrm{CANt}z)\mathrm{t}))$$

belongs to the classical sector of the language then by Classical Completeness:

$$\vdash \underline{\sim \text{Bew } \epsilon y(\mathrm{R}(y(\epsilon z\mathrm{CANt}z)\mathrm{t}))} = \epsilon y(\mathrm{R}(y(\epsilon z\mathrm{CANt}z)\mathrm{t}))$$

hence by =E

$$\vdash \sim \text{Bew } \epsilon y(\mathrm{R}(y(\epsilon z\mathrm{CANt}z)\mathrm{t})) \leftrightarrow \sim \text{Bew } \underline{\sim \text{Bew } \epsilon y(\mathrm{R}(y(\epsilon z\mathrm{CANt}z)\mathrm{t}))}$$

or, abbreviating $\sim$ Bew $\epsilon y(\mathrm{R}(y(\epsilon z\mathrm{CANt}z)\mathrm{t}))$ by G, we have an instance (GL) of the fixed point lemma:

$\vdash$ G $\leftrightarrow$ $\sim$Bew $\underline{\text{G}}$.

If, however, G is either definitely provable or definitely unprovable so that G $\in \Pi$ then antinomy looms in the metatheory via the following proof:

1	(1) $\vdash$ G	Hyp.
1	(2) $\vdash$ Bew $\underline{\text{G}}$	1 REP
1	(3) $\vdash\sim$Bew($\underline{\text{G}}$).	1 GL Proof Theory
1	(4) $\vdash \perp$	2,3 Proof Theory
1	(5) $\perp$	5, MetaSoundness[36]
—	(6) Not: $\vdash$ G	5 $\sim$I
—	(7) $\vdash\sim$Bew $\underline{\text{G}}$	6 REP.
—	(8) $\vdash$ G	7 GL Proof Theory
—	(9) $\vdash$	6,8 $\sim$E

The conclusion which must be drawn, of course, is either that

$$\underline{\sim \text{Bew } \epsilon y(\text{R}(y(\epsilon z\text{CANt}z)\text{t}))} = \epsilon y(\text{R}(y(\epsilon z\text{CANt}z)\text{t}))$$

does not belong to Δ, where R is interpreted in the intended interpretation as the substitution function, that is the identity of the Gödelian sentence is an indeterminate matter, or else that G $\notin \Pi$.

The latter is clearly the preferable option since if the diagonalisation lemma fails in general then of course the usual self-referential antinomies are not derivable. But since self-reference is clearly part of normal English, it would be an inadequate formalisation which was not able to represent such paradoxical reasoning as occurs in natural language. However diagonalisation might hold fairly generally but not in full generality. After all, 'This sentence is not provable' is arguably not paradoxical in English because 'proof' in natural language is an informal notion, perhaps an incomplete one, at any rate one for which we have no reason to think representation theorems hold in full generality. On the other hand, 'this sentence is not true' *is* paradoxical because the naïve interderivabilities —from $\ulcorner$*s* is true$\urcorner$ conclude P and from P conclude $\ulcorner$*s* is true$\urcorner$— where substituends for '*s*' name substituends for 'P', are, it is plausible to suppose, part of our ordinary concept of truth. So we certainly would want the liar to be representable in any formalisation of our informal mathematical and philosophical language. But there is no

[36] Here we assume that it is determinate whether or not $\vdash \perp$ so that the soundness theorem applies.

reason to think that adding

$$\text{Det}\ \underline{\sim \text{True}\ \epsilon y(\text{R}(y(\epsilon z\text{CANt}z)\text{t}))} = \epsilon y(\text{R}(y(\epsilon z\text{CANt}z)\text{t}))$$

to AXDET yields antinomy because the fixed point:

$$\text{True}\ \lambda \leftrightarrow \sim \text{True}\ \underline{\lambda}$$

is not neo-classically inconsistent.

Of course there is always the threat of inconsistency in the system via 'Revenge Liars'. These typically work by taking the concept introduced in an attempt to resolve paradox and building it into strengthened liars. Thus gap approaches are vulnerable to sentences of the form:

$$\gamma :\sim \text{True}\underline{\gamma} \vee \text{Gappy}\ \underline{\gamma}$$

hierarchical approaches to the appropriate variant of

$$\eta : \forall\alpha \sim \text{True-at-level-}\alpha\ \underline{\eta}.$$

Since indeterminacy has been the key concept used to block the derivation of contradiction from neoclassical naïve set theory, the standard Revenge attack on the theory outlined here would look to sentences such as

$$\imath :\sim \text{True}\ \underline{\imath} \vee \sim \text{Det True}\ \underline{\imath}$$

to cause trouble.

Informally, the argument would go thus: suppose sentence $\imath$ is untrue. Then it is either untrue or indeterminate (i.e. indeterminate whether or not it is true), hence true. By *consequentiae mirabilis*, $\imath$ is provably true. By the naïve truth rule it is therefore either untrue or indeterminate; but we have ruled out the first disjunct. Hence it is provably indeterminate. But how can a sentence be both provably true and provably indeterminate? Moreover, the Revenger continues, you cannot wriggle out the above problem by appeal to indeterminacy. To be sure, the preceding reasoning —the use of *consequentiae mirabilis* in particular— is classical. But to block it the neoclassicist has to aver that $\imath$ is indeterminate. But if it is indeterminate, it is true: how can one be committed fully both to its truth and to its indeterminacy?

Let us look at this reasoning more carefully. First of all note that from the Ant/Det Rule we have i) Det P $\dashv\vdash$ Det $\sim$P and that ii) if $\vdash \varphi \leftrightarrow \psi$ then $\vdash$

Det $\varphi \leftrightarrow$ Det ψ[37] and so, by contraposition, $\vdash$$\sim$Det $\varphi \leftrightarrow\sim$ Det ψ. Fact ii) generalises:– a simple inductive proof establishes that if $\vdash\ \varphi \leftrightarrow \psi$ then $\vdash$ Det$^{\alpha}$ $\varphi \leftrightarrow$ Det$^{\alpha}$ ψ and (by contraposition) $\vdash$$\sim$Det$^{\alpha}$ $\varphi \leftrightarrow\sim$ Det$^{\alpha}$ ψ (for the limit case we use in each direction $\vee$E and &E followed by $\leftrightarrow$E from the premiss, then &I, $\vee$I and finally $\rightarrow$I). In particular, using the Tarskian biconditionals derivable from the naïve rules, we have, for all α and all sentences P

Lemma: $\vdash$ X $\Rightarrow$ Det$^{\alpha}$ P iff $\vdash$ X $\Rightarrow$ Det True$^{\alpha}$ $\underline{\text{P}}$ iff $\vdash$ X $\Rightarrow$ Det$^{\alpha}$ $\sim$True $\underline{\text{P}}$.

Bearing these facts in mind, we can formalise the above argument as:

1	(1) $\sim$True $\underline{\imath}$	Hyp.
1	(2) $\sim$True $\underline{\imath} \vee \sim$Det True $\underline{\imath}$	1 $\vee$I
1	(3) True $\underline{\imath}$	2 Naïve Truth
4	(4) Det $\imath$	Hyp.
4	(5) Det $\sim$True $\underline{\imath}$	4 Lemma
1,4	(6) $\perp$	1,3 [4] $\sim$E
4	(7) True $\underline{\imath}$	6 $\sim$I
4	(8) $\sim$True $\underline{\imath} \vee \sim$Det True $\underline{\imath}$	7 Naïve Truth
1,4	(9) $\perp$	1,7 $\sim$E
10	(10) $\sim$Det True $\underline{\imath}$	Hyp.
4	(11) Det True $\underline{\imath}$	4 Lemma
4,10	(12) $\perp$	10,11 $\sim$E
13	(13) Det Det $\imath$	Hyp.
4, 13	(14) $\perp$	8,9, 12 [13] $\vee$E

This proof shows that the combined assumptions (4) and (13), that is Det $\imath$ and Det Det $\imath$, are inconsistent and from this we can conclude that Det $\imath \notin$ AXDET for if it was axiomatic so would be Det Det $\imath$ which we have just seen to be impossible.

However, the Revenger needs to derive a contradiction in the neo-classical system outright, thus far all she has is the joint incompatibility of two determinacy wffs. Moreover the obvious development of the above proof towards outright contradiction is blocked by the disjointness restrictions (DR) we place on determinacy premisses in our rules. For instance, in $\sim$E:

[37] The proof uses transitivity. From our premiss $\vdash \varphi \leftrightarrow \psi$ we get $\vdash (\varphi\vee \sim \varphi) \rightarrow (\psi\vee \sim \psi)$ so from T $\rightarrow (\varphi\vee \sim \varphi)$ we get T $\rightarrow (\psi\vee \sim \psi)$ by transitivity, the converse direction being exactly symmetrical.

X	(1) A	Given
Y	(2) $\sim$A	Given
Z_i, $i \in \mathrm{I}$	(3.i) Det Q	Given, $\forall \mathrm{Q} \in \mathrm{X} \cap \mathrm{Y}$
X,Y, $\bigcup_{i\in \mathrm{I}} \mathrm{Z}_i$	(4) C	1,2, [3.i, $i \in \mathrm{I}$], $\sim$E

the determinacy restriction DR is: $\bigcup_{i\in \mathrm{I}} \mathrm{Z}_i \cap (\mathrm{X} \cup \mathrm{Y}) = \emptyset$. Similarly in $\rightarrow$E, we restrict the rule:

X_i	(1.i) A_i	Given, $i \in \mathrm{I}$
Y	(2) $\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}$	Given
Z_j	(3.j) Det C_j	$\forall \mathrm{C}_j \in \Delta j \in \mathrm{J}$
$\bigcup_{i\in \mathrm{I}} \mathrm{X}_i$, Y, $\bigcup_{j\in \mathrm{J}} \mathrm{Z}_j$	(4) B	1.i $i \in \mathrm{I}$, 2, [3.j, $j \in \mathrm{J}$], $\rightarrow$E

where Δ is the set of all wffs which either belong to $\bigcup_{i\in \mathrm{I}} \mathrm{X}_i \cap \mathrm{Y}$ or else belong to $\mathrm{X}_i \cap \mathrm{X}_k$ for some $i \neq k$, by adding the conditions that $\bigcup_{j\in \mathrm{J}} \mathrm{Z}_j \cap (\bigcup_{i\in \mathrm{I}} \mathrm{X}_i \cup \mathrm{Y}) = \emptyset$ and that $\bigcup_{j\in \mathrm{J}} \mathrm{Z}_j \cap (\mathrm{X}_i \cup \mathrm{X}_k) = \emptyset$ for $i \neq k$.

Nonetheless the neo-classicist cannot rest content with using such restrictions to block antinomy since they can be dropped without violating soundness if we are prepared to complicate the rules still further. For example, in $\sim$E we can add a line 4.j $j \in \mathrm{J}$ and altered conclusion (5):

W_j, $j \in \mathrm{J}$	(4.j) Det R	Given, $\forall \mathrm{R} \in \bigcup_{i\in \mathrm{I}} \mathrm{Z}_i \cap (\mathrm{X} \cup \mathrm{Y})$
X, Y, $\bigcup_{i\in \mathrm{I}} \mathrm{Z}_i$, $\bigcup_{j\in \mathrm{J}} \mathrm{W}_j$	(5) C	1,2, [3.i, $i \in \mathrm{I}$, 4.j $j \in \mathrm{J}$], $\sim$E

where for each 4.j, $\mathrm{R} \notin \mathrm{W}_j$. This would still preserve soundness[38] as would similar amendments to $\vee$E, $\rightarrow$E and $\rightarrow$I. Thus for $\rightarrow$E we amend the rule to:

[38] Proof: the truth preservation direction is straightforward; in the falsity preservation direction, suppose C is false and all of X,Y, $\bigcup_{i\in \mathrm{I}} \mathrm{Z}_i$, $\bigcup_{j\in \mathrm{J}} \mathrm{W}_j$ but P are true. If P is in X but not Y or vice versa the proof is as before. If P is in both then we must have $\mathrm{Z}_i \Rightarrow$ Det P as a premiss. If all of Z_i are true then P is determinate and so false for if it is true then the IH ensures both A and $\sim$A are true, which is absurd. If not all of the Z_i are true then $\mathrm{P} \in \mathrm{Z}_i$ as well as $\mathrm{X} \cap \mathrm{Y}$ so that we have as a premiss $\mathrm{W}_j \Rightarrow$ Det P and such that $\mathrm{P} \notin \mathrm{W}_j$; hence all of W_j are true so P is determinate and hence false, as required.

X_i	(1.i) A_i	Given, $i \in I$
Y	(2) $A_{i\ i\in I} \rightarrow B$	Given
Z_j	(3.j) Det C_j	$\forall C_j \in \Delta$
$\bigcup_{k\in K} W_k$	(4.k) Det D_k	$\forall D_k \in \Gamma$
$\bigcup_{i\in I} X_i$, Y, $\bigcup_{j\in J} Z_j$, $\bigcup_{k\in K} W_k$	(4) B	1.i $i \in I$, 2, [3.j, $j \in J$, 4.k, $k \in K$] $\rightarrow$E

where $\Gamma = \bigcup_{j\in J} Z_j \cap \Delta$ and where $D_k \notin W_k$.

With these new, more complex but more liberal, rules, we can then continue the previous proof involving $\imath$:

4,13	(14) $\perp$	8,9, 12 [13] $\vee$E
13	(15) $\sim$Det $\imath$	14 $\sim$I
13	(16) $\sim$Det True $\underline{\imath}$	15 Lemma
13, T	(17) $\imath \vee \sim \imath$	16, $\vee$I $\times$2, Exp.
18	(18) Det Det Det $\imath$	Hyp.
13,18	(19) Det $\imath$	17 [18] $\rightarrow$I
13, 18	(20) $\perp$	15, 19 [18] $\sim$E
18	(21) $\sim$Det Det $\imath$	20, $\sim$I

So from Det $\imath$ we can prove $\imath$ (apply naïve truth to line 7) and from Det Det $\imath$ together with Det $\imath$ we can generate a contradiction so that Det Det $\imath$ entails $\sim$Det $\imath$ (line 15) and hence $\imath$. But we cannot prove $\sim$Det $\imath$ (and so $\imath$) outright; we cannot prove both $\imath$ and its indeterminacy but only prove it on the assumption Det Det $\imath$. And to go from this to Det $\imath$ we need the further assumption of Det Det Det $\imath$, (Det$^3\imath$) and the use of our more liberal rule (at line 20). So overall we have only shown that Det$^3\imath$ entails $\sim$Det$^2\imath$ (line 21). However this result is now stable in that $\sim$Det Det $\imath$ leads to no further problems, we cannot prove $\imath$ from it by $\vee$I, for instance, since the right disjunct of $\imath$ is equivalent to $\sim$Det $\imath$, not $\sim$Det Det $\imath$.

The upshot is that the strengthened liar which says of itself “I am untrue or indeterminate” is hyper-indeterminate, that is it is not determinate whether or not it is determinate, so that we must be agnostic on whether or not it is indeterminate thereby blocking antinomy. But the intrepid Revenger will doubtless now attempt to construct a hierarchy of strengthened Liars of the form:

$$\sigma^{\alpha} :\sim \text{True } \underline{\sigma^{\alpha}} \vee (\vee[\sim \text{Det}^{\beta} \text{ True } \underline{\sigma^{\alpha}}]\beta < \alpha)$$

So σ^{α} says of itself that it is either untrue, or indeterminate or hyper-indeterminate or hyper-hyper-indeterminate and so on for each level of indeterminacy below hyper$^{\alpha}$-indeterminacy.

At each stage α of this hierarchy, however, the neo-classicist can block antinomy by showing how the generalisation of the intuitive argument we started from becomes, in neo-classical format, merely a harmless proof of the hyper$^{\alpha}$-indeterminacy of the sentence σ^{α}, i.e. (in the limit case in particular) $\vdash\sim\text{Det}^{\alpha}\ \sigma$; see Appendix II: Hyperindeterminacy.

But now the Revenger scents blood: this move reeks, she may say, of the stench of countless failed hierarchical resolutions of the antinomies.[39] A painful dilemma threatens, in all such cases. Either the concept involved in the solution and in the strengthened liars, in this case 'hyper-indeterminacy *to some arbitrary ordinal degree*', can be expressed in the object language or not. If not, then the object language lacks the expressive power of the natural language in which we give the solution, it is not semantically closed. But it is the avoidance of antinomy in natural language which interests us:– that some significantly weaker language is free of paradox is irrelevant, we wish to model in a formal language this key feature of natural language that, at any time, we can express paradox in the language and also devise solutions to the paradoxes using it. Hence the need for a closed semantic theory:– our formalised idealisation of natural language theorising and reasoning about truth must be able to include the very theorising and reasoning which forms part of the solution to the paradoxes. If, on the other hand, the concept of hyper-indeterminacy to an arbitrary degree is expressible in our formal language L, then will we not be able to use it to generate an ultimate super-duper strengthened liar? For what is to stop us generalising on the parameter here, and introducing a liar whose indeterminacy clause is not tied to some fixed ordinal $\alpha \in$ ON but refers to indeterminacy at every ordinal level, that is what is to stop us introducing an ultimate liar which says of itself that it is either untrue or hyper$^{\alpha}$-indeterminate for *every* ordinal level α?

The second horn of the dilemma is certainly the one we should tackle, that is we should not deny that our formal language lacks the power to express the concepts involved in the solution. For the language will contain the one-place predicate ($\vee[\sim \text{Det}^{\alpha}\ \text{True}\ x]\alpha <$ ON) which says of x that it is hyper-indeterminate at some ordinal level. The existence of this predicate in the language then enables us to create an Ultimate Liar of the form:

$$\upsilon :\sim \text{True}\ \underline{\upsilon} \vee (\vee[\sim \text{Det}^{\alpha}\ \text{True}\ \underline{\upsilon}]\alpha < \text{ON})$$

[39] Cf. Graham Priest, *Beyond the Limits of Thought* (Cambridge, England: Cambridge University Press, 1995) Section Three.

The Revenger will want to argue that our old escape route is no longer open to us. Where our strengthened liar expressed, in its right disjunct, hyper-indeterminacy to some degree $\beta <$ ON, we concluded that the liar in question is hyper$^{\beta}$-indeterminate and there the matter rested. But, the revenger will say, we cannot block paradox by that route in the case of the Ultimate Liar, for in this case there is nowhere to go, no higher-degree of (harmless, non-paradox-inducing) hyperindeterminacy to prove.

However the claim that there is no higher degree of indeterminacy is itself *simply* indeterminate. For ON $\in$ ON is simply indeterminate and that in turn induces an indeterminacy about the length of υ and of Det$^{\text{ON}}\upsilon$, and in particular an indeterminacy regarding whether the latter has itself as one of its own disjuncts. If we could prove contradiction from either the assumption that it has itself as its own disjunct, or a contradiction from the assumption that it has not, then we would be able to refute either the first or the second assumption. And then we would be in trouble, for we would be able to decide ON $\in$ ON from whence disaster follows. Fortunately both alternatives are coherent, see the Appendix II: Hyperindeterminacy.

Of course in our natural language reasoning about strengthened liars we never utter infinitely long sentences; so how does this indeterminacy in the syntax of our formal language, the indeterminacy as to whether Det$^{\text{ON}}\upsilon$ has itself as its own disjunct or not, block the paradoxical reasoning in natural language? Each actual strengthened liar we can utter will be of some finite degree of hyper-indeterminacy and so antinomy is blocked as before —by showing (on the assumption of further degrees of quasi-determinacy) that the liar is hyper-indeterminate to a higher degree than that specified in the liar itself. Attempts by a Revenger to convict this solution of falling into further paradoxes lead to a generalisation of the solution to arbitrary degrees, including transfinite degrees, of indeterminacy. The Revenger quite rightly attempts to press the point once more, appealing only to the very concepts we introduce in an attempt to avoid contradiction, namely the notion of being indeterminate to any of the degrees mentioned in the solution. But now we have moved inevitably but irreversibly away from actual concrete utterances to theoretical constructs, to artificial formal languages which are themselves mathematical entities; only by doing so can we talk of transfinite iterations of the Det operator. And where the mathematical framework is naïve set theory, we can use the full resources of this theory to show how, in the idealised language, antinomy does not result even though the language itself is capable of expressing, according to our formal semantics, the generalisation of our intuitive notions.

It remains true, of course, that the above, if successful, merely blocks *some* possible routes to showing the incoherence of the formal system outlined. No soundness proof, whether in a semantically closed or semantically open language, can rule out with Cartesian certainty the epistemic possibility that the system of the object language (a fortiori any stronger system in a soundness proof in a semantically open language) is inconsistent. But once again, such Cartesian certainty is not to be had even in mathematics, whether of the standard or the naïve variety. One can only look at each threat in turn.

5. *Conclusion*

Naïve set theory in the neoclassical format I have outlined offers the prospect of proving all of standard mathematics but also much more. It validates the naïve interpretations that nearly all set theorists surreptitiously give to their theories when reflecting on them as any responsible mathematician should and so promises to restore coherence and good faith to the practice of set theory. The cost is a certain artificiality in reasoning beyond the determinate levels: but artificiality is better than a refusal to reason at all (a refusal very difficult to stick to in practice) or a lapse into incoherence. The theory is paraconsistent in the weak sense that it relies on a background logic in which not all classically inconsistent theories are trivial but departs form the spirit of most paraconsistent approaches by neglecting considerations of relevance and embracing $\sim$E (and related principles such as the disjunctive syllogism) unrestrictedly in their 'operational' forms, for instance as one-step natural deduction rules. It is argued that the results of doing so provide a simpler and more satisfactory recapture of standard set theory than the main paraconsistent alternative, dialetheism.

Appendix I. THE TARSKIAN EQUIVALENCES

The notion of an *s-definite* wff was defined by

> θ is s-definite iff $\psi(\underline{\theta})$ is definitely true or definitely false, in the intended interpretation, where ψ is any syntactic predicate and θ any sub-formula of φ.

Somewhat more precisely, let us say that ψ is a syntactic predicate just in case it is of the form $x = \mathrm{t}$, for any singular term t, or of the form Ξx where the latter is the object language definition of one of our syntactic concepts, x is an atom, a conjunction, negation, disjunction, consequence, antecedent, wff, one-place wff etc. It follows from Classical Completeness that any for any such syntactic claim ψ and s-definite wff φ, $\vdash \psi(\underline{\varphi})$ or $\vdash\sim \psi(\underline{\varphi})$. Any such appeal to classical completeness for s-definite wffs and syntactic claims

ψ will be cited as SDEF. Using this, we can prove

For all s-definite $\varphi, \vdash$ True $\underline{\varphi} \leftrightarrow \varphi$.

Proof: By induction on wff complexity. For atoms, suppose our atom is $\underline{\mathrm{Ft}}$ (the relational case is similar) and assume in the left-to-right direction True $\underline{\mathrm{Ft}}$. Here once again is the definition of the truth predicate:

True $x \leftrightarrow x = \underline{\mathrm{Ft}}$ & Ft $\vee\, x = \underline{\mathrm{Rtu}}$ & Rtu & ... [through all the atoms] $\vee$
$\vee$ Neg x & $\forall$ [or $\exists$] y in x, ∼True y
$\vee$ Conj x & $\forall y(y$ in $x \rightarrow$ True $y)\vee$
$\vee$ Disj x & $\exists y(y$ in x & True $y)\vee$
$\vee$ Cond x &
&(((&(Ant $\mathrm{u}_m\ x \leftrightarrow \vee(\mathrm{u}_m = \mathrm{t}_i)[i \in \mathrm{I}])[m \in \Omega]$ & True $\mathrm{t}_i)_{[i\in \mathrm{I}]} \rightarrow$
$\forall z$(Cons $z\ x \rightarrow$ True z)))$\mathrm{I} \in \mathrm{P}(\Omega)$

Provably, by SDEF, since Ft is not a non-atom,

True $\underline{\mathrm{Ft}} \vdash \underline{\mathrm{Ft}}$ & Ft $\vee\, x = \underline{\mathrm{Rtu}}$ & Rtu & ... etc.

by $\vee$E on the right hand side of the above truth definition, we prove Ft; the minor premisses, in all other cases except $\underline{\mathrm{Ft}} = \underline{\mathrm{Ft}}$, use ∼E on $\underline{\mathrm{Ft}} = \underline{\mathrm{Gu}}$ (for example) plus the provable (from SDEF) $\underline{\mathrm{Ft}} \neq \underline{\mathrm{Gu}}$, to conclude Ft. The provability of the other direction is a straightforward $\vee$I from $\underline{\mathrm{Ft}} = \underline{\mathrm{Ft}}$ & Ft.

For the inductive clauses, consider the trickiest case, that for the conditional.

Right to Left: Take any set I of simple terms such that I is not the set of exactly those terms t_i such that $\vdash$ Ant t_i $\underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}}$. Assume

(i) $\vdash$ &(Ant u_m $\underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}} \leftrightarrow \vee(\mathrm{u}_m = \mathrm{t}_i)[i \in \mathrm{I}])[m \in \Omega]$.

Let us abbreviate this particular wff φ. Given our assumption about I, for at least one t_i we have Not: $\vdash$ Ant t_i $\underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}}$; since $\underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}}$ is s-definite:

(ii) $\vdash$∼Ant t_i $\underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}}$.

From $\mathrm{u}_m = \mathrm{u}_m$, where $\ulcorner \mathrm{u}_m \urcorner = \ulcorner \mathrm{t}_i \urcorner$, then $\vee$I we get $\vdash \vee(\mathrm{u}_m = \mathrm{t}_i)[i \in \mathrm{I}]$, hence using the right to left direction of that particular conjunct of φ we contradict (ii) yielding by ∼I on (i) $\vdash\sim \varphi$ hence also

(iii) $\vdash\sim (\varphi$ & True $\mathrm{t}_k)$

for any $k \in \mathrm{I}$. We can then derive the relevant instance of the right hand conjunct of the conditional clause, namely:

(iv) $((\&(\mathrm{Ant}\ \mathrm{u}_m\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}} \leftrightarrow \vee(\mathrm{u}_m = \mathrm{t}_i)[i \in \mathrm{I}])[m \in \Omega]\&\ \mathrm{True}\ \mathrm{t}_i)_{[i\in\mathrm{I}]} \rightarrow$
$\rightarrow \forall z(\mathrm{Cons}\ z\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}} \rightarrow \mathrm{True}\ z))$

by

—	(1) $\sim(\varphi\ \&\ \mathrm{True}\ \mathrm{t}_k)$	iii
2.i	(2.i) $\varphi\ \&\ \mathrm{True}\ \mathrm{t}_i$	Hyp. $i \in \mathrm{I}$
2.$i \in \mathrm{I}$	(3) $\varphi\ \&\ \mathrm{True}\ \mathrm{t}_k$	2.k Exp.
2.$i \in \mathrm{I}$	(4) $\forall z(\mathrm{Cons}\ z\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}} \rightarrow \mathrm{True}\ z)$	1,3 $\sim$E
—	(5) iv	5 $\rightarrow$I

So we need concern ourselves only with the I* which indexes exactly those terms t_i such that $\vdash \mathrm{Ant}\ \mathrm{t}_i\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}}$. For this I* we will have by SDEF

—	(1.i) $\mathrm{t}_i = \underline{\mathrm{A}_{f(i)}}$	$i \in$ I* SDEF

where f is a function which maps one:one the simple terms indexed by I* onto the canonical names of the antecedents of $\underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}}$. This instance of (iv)–

(iv*): $((\&(\mathrm{Ant}\ \mathrm{u}_m\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}} \leftrightarrow \vee(\mathrm{u}_m = \mathrm{t}_i)[i \in \mathrm{I}^*])[m \in \Omega]$
$\&\ \mathrm{True}\ \mathrm{t}_i)_{[i\in\mathrm{I}^*]} \rightarrow \forall z(\mathrm{Cons}\ z\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}} \rightarrow \mathrm{True}\ z))$

(with φ^* thus being $\&(\mathrm{Ant}\ \mathrm{u}_m\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}} \leftrightarrow \vee(\mathrm{u}_m = \mathrm{t}_i)[i \in \mathrm{I}^*])[m \in \Omega]$) is then provable by:

2.i	(2.i) $\varphi^*\ \&\ \mathrm{True}\ \mathrm{t}_i$	Hyp. $i \in$ I*
2.i	(3.i) $\mathrm{True}\ \underline{\mathrm{A}_{f(i)}}$	2.i &E, 1.i =E
—	(4.i) $\mathrm{True}\ \underline{\mathrm{A}_{f(i)}} \leftrightarrow \mathrm{A}_{f(i)}$	IH
2.i	(5.i) $\mathrm{A}_{f(i)}$	3.i, 4.i $\leftrightarrow$E
—	(6.i) $(\varphi^*\ \&\ \mathrm{True}\ \mathrm{t}_i) \rightarrow \mathrm{A}_{f(i)}$	5.i $\rightarrow$I
7	(7) $(\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B})$	Hyp.
7	(8) $((\varphi^*\ \&\ \mathrm{True}\ \mathrm{t}_i)_{[i\in\mathrm{I}^*]} \rightarrow \mathrm{B})$	7, 6.i $i \in$ I*, Trans.[40]

[40] This application of infinitary transitivity is correct since the $f(i)$ enumerate all the antecedents.

—	(9) $\mathrm{B} \rightarrow \forall z(\mathrm{Cons}\ z\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}} \rightarrow \mathrm{True}\ z)$	Lemma
7	(10) $((\varphi^* \ \&\ \mathrm{True}\ \mathrm{t}_i)_{[i\in\mathrm{I}^*]} \rightarrow$ $\forall z(\mathrm{Cons}\ z\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}} \rightarrow \mathrm{True}\ z)$	8, 9 Trans

We then prove the conditional clause in the truth definition (x instantiated by $\underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}}$) by proving the left conjunct Cond $\underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}}$ using SDEF and the right conjunct by &I over all the I sets. True $\underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}}$ then follows by $\vee$I and $\rightarrow$E. The proof of the Lemma at line (9) utilises the facts that we have from the IH $\vdash \mathrm{B} \rightarrow \mathrm{True}\ \underline{\mathrm{B}}$ and that from the definition of Cons and set comprehension on $\underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}}$ we have:

$$\vdash \forall z(\mathrm{Cons}\ z\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}} \rightarrow z = \underline{\mathrm{B}})$$

whence by transitivity we get:

—	(1) $\mathrm{True}\ \underline{\mathrm{B}} \rightarrow (z = \underline{\mathrm{B}} \rightarrow \mathrm{True}\ z)$	=E
—	(2) $\mathrm{B} \rightarrow \mathrm{True}\ \underline{\mathrm{B}}$	IH
—	(3) $\mathrm{B} \rightarrow (z = \underline{\mathrm{B}} \rightarrow \mathrm{True}\ z)$	1,2 Trans.
4	(4) B	Hyp.
4	(5) $z = \underline{\mathrm{B}} \rightarrow \mathrm{True}\ z$	3,4 $\rightarrow$E
—	(6) $\forall z(\mathrm{Cons}\ z\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}} \rightarrow z = \underline{\mathrm{B}})$	Given
4	(7) $\forall z(\mathrm{Cons}\ z\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}} \rightarrow \mathrm{True}\ z)$	5,6 Trans.
—	(8) $\mathrm{B} \rightarrow \forall z(\mathrm{Cons}\ z\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}} \rightarrow \mathrm{True}\ z)$	7 $\rightarrow$I

In the left to right direction we start out from the truth definition —TD— then:

1	(1) $\mathrm{True}\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}}$	Hyp.
1	(2) $\mathrm{Con}\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}}\ \&\ \&(\mathrm{iv})[\mathrm{I} \in \mathrm{P}\Omega]$	TD,1 $\leftrightarrow$E $\vee$E SDEF
1	(3) iv*	2 &E

Here at line (2), SDEF, as in the atomic case, ensures that we can prove that $\underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}}$ is a conditional so that by $\vee$E and $\sim$E and &E we derive the conjunction at line (2) whose right conjunct is:

$$\&(((\&(\mathrm{Ant}\ \mathrm{u}_m\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}} \leftrightarrow \vee(\mathrm{u}_m = \mathrm{t}_i)[i \in \mathrm{I}])[m \in \Omega]\ \&\ \mathrm{True}\ \mathrm{t}_i)_{[i\in\mathrm{I}]} \rightarrow \forall z(\mathrm{Cons}\ z\ \underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}} \rightarrow \mathrm{True}\ z)))\mathrm{I} \in \mathrm{P}(\Omega)$$

At line three we instantiate by the set I* which indexes exactly the simple names of the antecedents of $\underline{\mathrm{A}_{i\ i\in\mathrm{I}} \rightarrow \mathrm{B}}$ to yield (iv)*. Now from SDEF we get for any $\mathrm{u}_m \in \mathrm{I}^*$:

—	(4.j) $\underline{\mathrm{A}_j} = \mathrm{u}_m$	SDEF
—	(5.j) Ant $\underline{\mathrm{A}_j}$ $\underline{\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}}$	SDEF
—	(6.j) Ant u_m $\underline{\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}}$	4.j, 5.j =E
—	(7.j) $\vee(\mathrm{u}_m = \mathrm{t}_i)[i \in \mathrm{I}^*]$	=I $\vee$I
—	(8.j) Ant u_m $\underline{\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}} \leftrightarrow \vee(\mathrm{u}_m = \mathrm{t}_i)[i \in \mathrm{I}]$	6.j, 7.j $\rightarrow$I$\times$2, &I

Here the $\rightarrow$Is at the last line are vacuous. Consider, next, the case of those u_r which are not simple names of antecedents of our conditional. The s-definiteness of that conditional yields a similar proof of those biconditionals:

—	(9.r) $\sim$Ant u_r $\underline{\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}}$	SDEF
—	(10.r) $\&(\mathrm{u}_r \neq \mathrm{t}_i)[i \in \mathrm{I}^*]$	SDEF
—	(11.r) $\sim\vee(\mathrm{u}_r = \mathrm{t}_i)[i \in \mathrm{I}^*]$	10.r $\sim$I MM
—	(12.r) Ant u_r $\underline{\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}} \leftrightarrow \vee(\mathrm{u}_r = \mathrm{t}_i)[i \in \mathrm{I}^*]$	As 8.j

whence by &I over the u terms we prove φ^*

$$\&(\text{Ant } \mathrm{u}_m\ \underline{\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}} \leftrightarrow \vee(\mathrm{u}_m = \mathrm{t}_i)[i \in \mathrm{I}^*])[m \in \Omega]$$

and can proceed

—	(15.i) True $\underline{\mathrm{A}_{f(i)}} \leftrightarrow \mathrm{A}_{f(i)}$	IH $i \in \mathrm{I}^*$
—	(16.i) $\underline{\mathrm{A}_{f(i)}} = \mathrm{t}_i$	SDEF[41]
—	(17.i) $\mathrm{A}_{f(i)} \rightarrow$ True t_i	15/16.i =E, &E
18.i	(18.i) $\mathrm{A}_{f(i)}$	Hyp. $i \in \mathrm{I}^*$
18.i	(19.i) True t_i	17/18.i $\rightarrow$E
—	(20) φ^*	As Above
18.i	(21.i) (φ^* & True t_i)	20, 19.i &I
—	(22.i) $\mathrm{A}_{f(i)} \rightarrow$ (φ^* & True t_i)	21.i $\rightarrow$I
1	(23) ($\mathrm{A}_{f(i[i\in \mathrm{I}]} \rightarrow \forall z$(Cons z $\underline{\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}}$ $\rightarrow$ True z))	3, 22.i, $i \in \mathrm{I}^*$ Trans.
—	(23) $\forall z$(Cons z $\underline{\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}} \rightarrow$ True z) $\rightarrow$ B	Lemma[42]
1	(24) $\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}$	23,24 Trans.[43]

[41] Here, as before, f maps the indices of simple terms one:one onto the names of the antecedents of $\mathrm{A}_{i\ i\in \mathrm{I}} \rightarrow \mathrm{B}$.

[42] This Lemma proven in much the same way as the Lemma in the right to left direction.

[43] Since the set of $\mathrm{A}_{f(i)}$ and the set of A_i are one and the same.

Appendix II. HYPERINDETERMINACY

The wff:

$$\sigma :\sim \text{True}\ \underline{\sigma} \vee (\vee[\sim \text{Det}^{\beta}\ \text{True}\ \underline{\sigma}]\beta < \alpha)$$

α definitely less than ON, cannot be used to generate absurdity; from it we derive merely $\sim\text{Det}^{\alpha}\sigma$ as follows, using the liberalised determinacy restrictions and the principles established in section 4, Lemma I:

$$\vdash\ \text{X} \Rightarrow \text{Det}^{\alpha}\ \text{P iff} \vdash\ \text{X} \Rightarrow \text{Det True}^{\alpha}\ \underline{\text{P}}\ \text{iff} \vdash\ \text{X} \Rightarrow \text{Det}^{\alpha} \sim \text{True}\ \underline{\text{P}}.$$

together with the following Lemma II:

For any limit κ and $\beta < \kappa$, $\&(\text{Det}^{\mu}\sigma[\beta < \mu < \kappa]) \vdash \text{Det}(\&(\text{Det}^{\mu}\sigma[\beta < \mu < \kappa]))$.

Proof: We use here the Det ii principle of Part I, that if all conjuncts of a conjunction are determinate then the conjunction itself is determinate. This is demonstrable for the language of naïve set theory.

The lemma is then provable by:

1	(1) $\&(\text{Det}^{\mu}\sigma[\beta < \mu < \kappa])$	Hyp.
1	(2.μ) Det $\text{Det}^{\mu}\sigma$	1 &E $\beta < \mu < \kappa$
1	(3) $\text{Det}(\&(\text{Det}^{\mu}\sigma[\beta < \mu < \kappa]))$	2.μ Det ii

With these principles and the above Lemma II we go on to show $\vdash\sim\text{Det}^{\alpha}$:

1	(1) Det σ	Hyp.
2	(2) $\sim$True $\underline{\sigma}$	Hyp.
2	(3) σ	2 $\vee$I
2	(4) True $\underline{\sigma}$	3 Naïve Truth
1	(5) Det $\sim$True $\underline{\sigma}$	1 Lemma I
1,2	(6) $\perp$	2,4 [5] $\sim$E
1	(7) True $\underline{\sigma}$	6 $\sim$I
1	(8) σ	7 Naïve Truth

Thus far, we follow pretty much the same path as the formalisation of the intuitive reasoning involving $\imath$.

1,2	(9) $\perp$	2,7 $\sim$E
10.1.1	(10.1.1) $\sim$Det True $\underline{\sigma}$	Hyp.
10.1.1	(10.1.2) $\sim$Det σ	10.1.1 Lemma I
1,10.1.1	(10.1.3) $\perp$	1,10.1.2 $\sim$E
10.β	(10.β) Det$^{\beta}$ σ	Hyp. $1 < \beta < \alpha$
10.β.1	(10.β.1) $\sim$Det$^{\beta}$ True $\underline{\sigma}$	Hyp.
10.β.1	(10.β.2) $\sim$Det$^{\beta}$ σ	10.β.1 Lemma I
10.β,10.β.1	(10.β.3) $\perp$	10.β, 10.β.2 $\sim$E
1, 10.β, $1 < \beta < \alpha$	(11) $\perp$	8,9,10.β.3, $0 < \beta < \alpha$ $\vee$E [10.2]
10.β, $1 < \beta < \alpha$	(12.1.1) $\sim$Det σ	11 $\sim$I

We have now applied $\vee$E to the disjunction σ, assuming Det$^{\beta}$ σ for every $\beta < \alpha$ to get a contradiction and thereby refuting Det^1 σ, line (1), discharging it.

10.β, $1 < \beta < \alpha$	(12.1.2) $\sim$Det True $\underline{\sigma}$	12.1.1 Lemma I
10.β, $1 < \beta < \alpha$, T	(12.1.3) $\sigma \vee \sim \sigma$	12.1.2 $\vee$I $\times$2, Exp.
10.β, $1 < \beta < \alpha$	(12.1.4) Det σ	12.1.3 $\rightarrow$I [10.β, $2 < \beta < \alpha$]
10.β, $1 < \beta < \alpha$	(12.1.0) $\perp$	12.1.1, 12.1.4 $\sim$E [10.β, $2 < \beta < \alpha$]
10.β, $2 < \beta < \alpha$	(12.2.1) $\sim$Det^2 σ	12.1.0 $\sim$I

The proof of $\sim$Det σ from Det$^{\beta}$ σ, $1 < \beta < \alpha$ has now been extended to a proof of $\sim$Det^2 σ from Det$^{\beta}$ σ, $2 < \beta < \alpha$.

10.β, $2 < \beta < \alpha$	(12.2.2) $\sim$Det^2 True $\underline{\sigma}$	12.2.1 Lemma I
10.β, $2 < \beta < \alpha$, T	(12.2.3) $\sigma \vee \sim \sigma$	12.2.2 $\vee$I $\times$2, Exp.
10.β, $2 < \beta < \alpha$	(12.2.4) Det σ	12.2.3 $\rightarrow$I [10.β, $3 < \beta < \alpha$]
10.β, $2 < \beta < \alpha$, T	(12.2.5) Det $\sigma \vee$ $\sim$Det σ	12.2.4 $\vee$I, Exp.
10.β, $2 < \beta < \alpha$	(12.2.6) Det^2 σ	12.2.5 $\rightarrow$I [10.β, $3 < \beta < \alpha$]
10.β, $2 < \beta < \alpha$	(12.2.0) $\perp$	12.2.1, 12.2.6 $\sim$E [10.β, $3 < \beta < \alpha$]
10.β, $3 < \beta < \alpha$	(12.3.1) $\sim$Det^3 σ	12.2.0 $\sim$I

And similarly we have extended further to a proof of $\sim$Det^3 σ from Det$^{\beta}$ σ, $3 < \beta < \alpha$.

...	...	...
10.γ, $\delta+1<\gamma<\alpha$	(12.δ+1.1) $\sim$Det$^{\delta+1}$ σ	Given
10.γ, $\delta+1<\gamma<\alpha$	(12.δ+1.2) $\sim$Det$^{\delta+1}$ True $\underline{\sigma}$	12.δ+1.1 Lemma I
10.γ, $\delta+1<\gamma<\alpha$, T	(12.δ+1.3) $\sigma\vee\sim\sigma$	12.δ+1.2 $\vee$I $\times$2, Exp.
10.γ, $\delta+1<\gamma<\alpha$, T	(12.δ+1.4) Det σ	12.δ+1.3 $\rightarrow$I [10.γ, $\delta+2<\gamma<\alpha$], Exp.
...	...	...
10.γ, $\delta+1<\gamma<\alpha$	(12.δ+1.$\epsilon+2n+2$) Det$^{\epsilon+n+1}$ σ	12.δ+1.$\epsilon+2n+1$ $\rightarrow$I [10.γ, $\delta+2<\gamma<\alpha$], Exp.
...	...	...
10.γ, $\delta+1<\gamma<\alpha$	(12.δ+1.λ) &(Det$^{\zeta}$ σ) $\zeta<\lambda$	12.δ+1.4/.6/.8 ... &I
10.γ, $\delta+1<\gamma<\alpha$	(12.δ+1.λ.λ) Det$^{\lambda}$ σ	12.δ+1.λ $\vee$I
...	...	...
10.γ, $\delta+1<\gamma<\alpha$	(12.δ+1.$\xi+2k+4$) Det$^{\delta+1}$ σ	12.δ+1.$\xi+2k+3$ $\rightarrow$I [10.γ, $\delta+2<\gamma<\alpha$]
10.γ, $\delta+1<\gamma<\alpha$	(12.δ+1.0) $\perp$	12.δ+1.1, (12.δ+1.ξ+2k+4) $\sim$E [10.γ, $\delta+2<\gamma<\alpha$]
10.γ, $\delta+2<\gamma<\alpha$	(12.δ+2.1) $\sim$Det$^{\delta+2}$ σ	(12.δ+1.0) $\sim$I

The result generalises to arbitrary successor levels $\delta+1$ in which we extend a proof of $\sim$Det$^{\delta+1}$ σ from Det$^{\beta}$ σ, $\delta+1<\beta<\alpha$ to a proof of $\sim$Det$^{\delta+2}$ σ from Det$^{\beta}$ σ, $\delta+2<\beta<\alpha$. In the above section of the proof, λ is a limit ordinal. The limit stage of the proof is slightly trickier:– κ below is a limit ordinal:

...	...	...
10.γ, $\kappa\leq\gamma<\alpha$	(12.κ.0.0) (Det$^{\mu}\sigma[0<\mu<\kappa]) \rightarrow \perp$	11 [10.γ, $\kappa<\gamma<\alpha$] $\rightarrow$I, $\kappa<\alpha$
10.γ, $\kappa\leq\gamma<\alpha$	(12.κ.β.0) (Det$^{\mu}\sigma[\beta<\mu<\kappa]) \rightarrow \perp$	12.β.0 [10.γ, $\kappa<\gamma<\alpha$] $\rightarrow$I, ($0<\beta<\kappa$)
12.κ.β.1	(12.κ.β.1) &(Det$^{\mu}\sigma[\beta<\mu<\kappa]$)	Hyp. ($0\leq\beta<\kappa$)
12.κ.β.1	(12.κ.β.1.0) Det &(Det$^{\mu}\sigma[\beta<\mu<\kappa]$)	Lemma II 12.κ.β.1
12.κ.β.1	(12.κ.β.1.μ) Det$^{\mu}\sigma$	12.κ.β.1 &E ($\beta<\mu<\kappa$)

10.γ, $\kappa \leq \gamma < \alpha$, 12.$\kappa$.$\beta$.1	(12.κ.β.2) $\perp$	12.κ.β.0, 12.κ.β.1.μ [12.κ.β.1.1.0] ($\beta < \mu < \kappa$) $\rightarrow$E
10.γ, $\kappa \leq \gamma < \alpha$	(12.κ.0) $\perp$	10.κ, 12.κ.β.2, $\beta < \kappa$, $\vee$E
10.γ, $\kappa < \gamma < \alpha$	(12.κ.1) $\sim$Det$^{\kappa}\sigma$	12.κ.2 $\sim$I

In the above section we start from the fact that at line 11 we have shown Det$^{\beta}$ σ, $1 \leq \beta < \alpha \vdash \perp$ and so if we can show the determinacy of all Det$^{\gamma}$ σ, $\kappa \leq \gamma$ we can, by $\rightarrow$I, prove Det$^{\mu}$ $\sigma[0 < \mu < \kappa]) \rightarrow \perp$ from those remaining, determinate, assumptions. But Det$^{\gamma+1}$ σ, expresses the determinacy of Det$^{\gamma}$ σ and $\gamma + 1 < \alpha$, α a limit, so we have (using our liberalised determinacy constraints) Det$^{\gamma}$ σ, $\kappa \leq \gamma < \alpha \vdash$ Det$^{\mu}$ $\sigma[0 < \mu < \kappa]) \rightarrow \perp$. Similarly at line 12.$\kappa$.$\beta$.0 we prove Det$^{\gamma}$ σ, $\kappa \leq \gamma < \alpha \vdash$ Det$^{\mu}$ $\sigma[\beta < \mu < \kappa]) \rightarrow \perp$ with the antecedent wffs of the conditional being determinacy claims at all levels below Det$^{\kappa}$ σ from some point β below. If we conjoin together these antecedent claims we get one of the disjuncts of Det$^{\kappa}$ σ which thus entails $\perp$ by $\rightarrow$E so long as it is determinate: but this wff entails its own determination by Lemma II. Finally we take the special case of the limit stage where $\kappa = \alpha$.

...	...	...
—	(12.α.0.0) (Det$^{\mu}\sigma[0 < \mu < \alpha]) \rightarrow \perp$	11 $\rightarrow$I
—	(12.α.β.0) (Det$^{\mu}\sigma[\beta < \mu < \alpha]) \rightarrow \perp$	12.β.0 $\rightarrow$I, ($0 < \beta < \kappa$)
12.α.β.1	(12.α.β.1) &(Det$^{\mu}\sigma[\beta < \mu < \alpha]$)	Hyp. ($0 \leq \beta < \alpha$)
12.α.β.1	(12.α.β.1.0) Det 12.α.β.1	Lemma II 12.α.β.1
12.α.β.1	(12.α.β.1.μ) Det$^{\mu}\sigma$	12.α.β.1 &E $\beta < \mu < \alpha$
12.α.β.1 $\beta < \alpha$	(12.α.β.2) $\perp$	12.α.β.0, 12.α.β.1.μ [12.α.β.1.1.0] ($\beta < \mu < \alpha$) $\rightarrow$E
12.α	(12.α) Det$^{\alpha}$ σ	Hyp.
12.α	(12.α.0) $\perp$	12.α, 12.α.β.2, $\beta < \alpha$, $\vee$E
—	(13) $\sim$Det$^{\alpha}$ σ	12.α $\sim$I

Here, at line (12.α.0.0) we apply $\rightarrow$I to line (11) Det$^{\beta}$ σ, $1 \leq \beta < \alpha \vdash \perp$ but this time incorporate all the assumptions into the antecedent of the succeedent wff by $\rightarrow$I giving us (Det$^{\mu}\sigma[0 < \mu < \alpha]) \rightarrow \perp$ —and more generally at the next lines (Det$^{\mu}\sigma[\beta < \mu < \alpha]) \rightarrow \perp$— as theorems so that the $\vee$E at line (12.α.0) yielding $\perp$ has only Det$^{\alpha}$ σ as its assumption, enabling us to refute Det$^{\alpha}$ σ.

So we have a somewhat stronger result here than that available for our simple strengthened liar:– in the limit case we can prove *outright* that a sentence which says it is either untrue or indeterminate to a degree up to but not including α is indeterminate to degree α. This is achieved because in the limit case the application of determinacy principles needed in rules such as $\vee$E at line 11 or $\rightarrow$I at line 12.1.4 hits a fixed point at which the determination of each of the infinitely many premisses in the intersection is itself a further premiss. Nonetheless we still block paradox —since $\sim$Det$^{\alpha}$ σ is not a disjunct of σ we cannot go on to prove σ from line 13; we conclude that σ is hyper-α-indeterminate but do not have to conclude also that it is true.

Now consider the Ultimate Liar:

$$\upsilon :\sim \text{ True } \underline{\upsilon} \vee (\vee[\sim \text{Det}^{\alpha} \text{ True } \underline{\upsilon}]\alpha < \text{ON})$$

It was argued that neither the supposition that υ contains a disjunct of the form $\sim$Det$^{\text{ON}}$ True $\underline{\upsilon}$ nor the supposition that it does not leads to contradiction. Suppose, firstly, that ON $\in$ ON so that υ does contain a disjunct of the form $\sim$Det$^{\text{ON}}$ True $\underline{\upsilon}$. It also follows that $\text{ON} = \text{ON} + 1 = \text{ON} + 2 = \text{ON} + \text{ON}$ and so on. (I will use x' for the successor $x \cup \{x\}$ of x in the following proof.) Under this supposition there exists a purported proof which is an extension of the previous proof and which ends like this:

—	(12.ON.0.0) $(\text{Det}^{\mu}\sigma[0 < \mu < \text{ON}]) \rightarrow \perp$	11 $\rightarrow$I
—	(12.ON.β.0) $(\text{Det}^{\mu}\upsilon[\beta < \mu < \text{ON}]) \rightarrow \perp$	12.β.0 $\rightarrow$I, $0 < \beta < \text{ON}$
12.ON.β.1	(12.ON.β.1) $\&(\text{Det}^{\mu}\upsilon[\beta < \mu < \text{ON}])$	Hyp. $0 \leq \beta < \text{ON}$
12.ON.β.1	(12.ON.β.1.0) Det 12.ON.β.1	Lemma II 12.ON.β.1
12.ON.β.1	(12.ON.β.1.μ) Det$^{\mu}\upsilon$	12.ON.β.1 &E $\beta < \nu < \text{ON}$
12.ON.β.1, $\beta < \text{ON}$	(12.ON.β.0) $\perp$	12.ON.β.0, 12.ON.β.1.μ ~~[12.ON.β.1.1.0]~~ $(\beta < \mu < \text{ON}) \rightarrow$E
12.ON	(12.ON) Det$^{\text{ON}}$ υ	Hyp.
12.ON	(12.ON.0) $\perp$	12.ON, 12.ON.β.0, $\beta < \text{ON}$, ~~[12.ON]~~ $\vee$E

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2002/6/11
page 339

—	(13) $\sim\mathrm{Det}^{\mathrm{ON}}\ \upsilon$	12 $\sim$I
T	(14) $\upsilon \vee \sim \upsilon$	13 $\vee$I $\times$2, Exp.
—	(15) Det υ	14 $\rightarrow$I
T	(16) Det $\upsilon \vee \sim$ Det υ	15 $\vee$I, Exp.
—	(17) $\mathrm{Det}^2\ \upsilon$	16 $\rightarrow$I
...	...	...
—	(18) $\mathrm{Det}^{\mathrm{ON}}\ \upsilon$	As 12.δ+1.4/6 etc. of previous proof.
—	(16) $\perp$	13, 18 $\sim$E

Note, though, that one of the disjuncts of $\mathrm{Det}^{\mathrm{ON}}\ \upsilon$ is the succeedent of line (12.ON.ON.1) (since ON $\in$ ON there is such a line) namely

$$\&(\mathrm{Det}^{\nu}\upsilon[\mathrm{ON} < \nu < \mathrm{ON}])$$

and this is just $\mathrm{Det}^{\mathrm{ON}}\ \upsilon$ itself (granted that ON $\in$ ON, i.e. ON $<$ ON) so that $\mathrm{Det}^{\mathrm{ON}}\ \upsilon$ has itself as one of its disjuncts. Inevitably, then, the assumption that ON $\in$ ON has as a consequence the ill-foundedness of our syntax. Thus in the application of $\vee$E at line (12.ON.0), one of the discharged disjuncts is in fact the disjunction $\mathrm{Det}^{\mathrm{ON}}\ \upsilon$ itself. But now we have an overlap between the assumption on which the major premiss $\mathrm{Det}^{\mathrm{ON}}\ \upsilon$ at line 12.ON depends, namely itself, and the assumption $\mathrm{Det}^{\mathrm{ON}}\ \upsilon$ made at line 12.ON.ON.1.ON for the minor proof on the assumed disjunct $\mathrm{Det}^{\mathrm{ON}}\ \upsilon$. Hence we require the extra premiss that this assumption $\mathrm{Det}^{\mathrm{ON}}\ \upsilon$ is determinate. But granted our assumption ON $\in$ ON, ON $=$ ON$'$ from which it follows that $\mathrm{Det}^{\mathrm{ON}+1}$ υ, that is Det $\mathrm{Det}^{\mathrm{ON}}\ \upsilon$, is identical with $\mathrm{Det}^{\mathrm{ON}}\ \upsilon$. So the additional determinacy assumption we need for the $\vee$E at line 12.ON.0 is, in fact, $\mathrm{Det}^{\mathrm{ON}}$ υ itself. Similarly the determinacy assumption 12.ON.ON.1.0 is $\mathrm{Det}^{\mathrm{ON}}\ \upsilon$ itself but so is the wff which is being declared determinate, the formula on line 12.ON.ON.1. These applications are therefore illicit; thus the $\vee$E application is incorrect not only on our original form of $\vee$E but even on the more liberal form:

X	(1) $\vee A_{i\ i \in I}$	Given
$Y_{i\ i\in I}$, A_i	(2.i) C	Given $i \in I$
$W_{j\ j\in J}$	(3.j) Det R_j	Given, $\forall R_j \in X \cap \bigcup_{i\in I} Y_i$
$\bigcup_{k\in K} Z_k$	(4.k) Det S_k	Given $\forall S_k \in \bigcup_{j\in J} W_j \cap (X \cup \bigcup_{i\in I} Y_i)$
X, $\bigcup_{i\in I} Y_i$, $\bigcup_{k\in K} Z_k$, $\bigcup_{j\in J} W_j$	(5) C	1,2.i $i \in I$, [3.j $j \in J$, 4.k $k \in K$], $\vee$E

with $S_k \notin Z_k\ \forall k \in K$. For in the present case, our determinacy assumption is

12.ON (12.ON) $\mathrm{Det}^{\mathrm{ON}}\ \upsilon$ Hyp.

which is the same as

12.ON (12.ON) Det $\mathrm{Det}^{\mathrm{ON}}\ \upsilon$ Hyp.

but this particular S_k, namely $\mathrm{Det}^{\mathrm{ON}}\ \upsilon$ does belong to $Z_k = \{\mathrm{Det}^{\mathrm{ON}}\ \upsilon\}$. Thus this purported proof of contradiction is in fact not a proof at all.

Suppose, on the other hand, that ON $\notin$ ON. Then well-foundedness is restored to the language. But now line 14 fails since $\sim\mathrm{Det}^{\mathrm{ON}}\ \upsilon$ is not a disjunct of υ. So we are able to be agnostic one whether or not $\sim\mathrm{Det}^{\mathrm{ON}}$ is a disjunct of υ just as we are agnostic on whether or not ON $\in$ ON and this simple indeterminacy in the syntax enables us to deny that the language is inconsistent, to deny that $\vdash \perp$.