Logique & Analyse **165–166** (1999), 139–160

MINIMIZING AMBIGUITY AND PARACONSISTENCY*

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Abstract
Ambiguity-adaptive logics offer a natural and rich formal solution to (possibly) inconsistent theories. Three views on 'ambiguities' result in three different, intuitively correct ambiguity-adaptive logics with interesting applications. We present their semantics and proof theory, and illustrate that ambiguity-adaptive logics form an excellent alternative to paraconsistent logics that focus on the characteristics of the logical constants.

1. *Introduction*

Inconsistencies frequently arise due to the ambiguity of our vocabulary. I give some examples. The word "chair" refers to a thing to sit on, and to a chairman. The name Albert Einstein is used for indicating Albert Einstein as a grown up, but also for indicating him at the age of three. "It's raining" is said when it is pouring and when some raindrops are falling. Due to the fact that one expression is used to refer to different 'realities', we obtain contradictions like: "A chair has four legs, but a chair does not have four legs." "Einstein is a physicist, but Einstein is not a physicist." "It rains" says one person, "it does not rain" says another person.

How should logicians deal with such 'stupid' inconsistencies? Classical logicians say that the formalization should be more precise. Paraconsistent logicians say that we should use a logic that allows for inconsistencies. In this paper a third approach, that meets the purposes of both other views, and establishes interesting results, is proposed. The first question is not about the logic, but about the non-logical terms. We want to reach the optimum between two (unrealistic) ideals, viz. (i) the non-logical terms should be precise enough, and (ii) our language should be compact (in order to keep it

*Research for this paper was supported by subventions from Ghent University and from the Fund for Scientific Research – Flanders, and indirectly by the Flemish Minister responsible for Science and Technology (contract BIL98/73).

understandable). We should also take into account that the required precision is dependent on the actual text or theory we are dealing with.

The classical view is right in that the formalization of our premises could be more precise. For instance: if we formalize the word "chair", we should make use of at least two different predicates. Same thing for Einstein as a grown up, and Einstein at the age of three, and for expressions as "it rains". Nevertheless, this view does not give an answer to the question "Which (occurrences of) non-logical terms should be precise enough?". Obviously we can not make all occurrences of non-logical terms as precise as possible—this would be an enterprise with no limits. Should we use a new predicate whenever we meet the word "chair"? Should we use a new constant for indicating Einstein at each of his moments?[1] Should we make use of an infinite number of names for sentences in order to formalize the expression "it rains"? The answer to each of these questions is obviously not yes, for then, we would never know what these expressions refer to. Communication becomes impossible if our expressions do not have a certain degree of generality. We should not only be worried about being precise enough, we should also think about concise theories and smooth communication. We have to start talking before we have reached the most precise formulations. At this point, I agree with the paraconsistent logicians: we better accept that our language cannot be absolutely precise and will always give birth to inconsistencies. But I disagree with most paraconsistent logicians in that the negation-rules of CL are not desired rules. Where our language is precise enough, we are better off using rules as Disjunctive Syllogism, Modus Tollens and Reductio ad Absurdum.

The question is: is there a formal way to choose between concision and precision?[2] The here presented ambiguity-adaptive logics offer a splendid

[1] If not, we keep on meeting stupid inconsistencies, like "Einstein (01/01/1930) is older than Einstein (31/12/1929)" and "Einstein is not older than Einstein".

[2] This question is situated within the larger problem of the relation between language and reality. If, for simplicity's sake, we consider the set of units of reality $\mathcal{R}$, and the set of units of language $\mathcal{L}$, we can define two 'border'-relations between $\mathcal{R}$ and $\mathcal{L}$, namely (i) a function Ω that maps all members of $\mathcal{R}$ to the same unit of language, *e.g.* the expression "ω", $\Omega : \mathcal{R} \rightarrow \mathcal{L} : x \rightarrow \omega$, and (ii) a bijection $\mathsf{P} : \mathcal{R} \leftrightarrow \mathcal{L}$, that maps every unit of reality on one unit of language, and vice versa. The function Ω has the advantage that our vocabulary would become very concise—one expression refers to all units of reality. The bijection P makes our language as precise as possible. In spoken or written language, there is a double tendency: towards more precision and towards more concision and generality. If we are interested in maximizing precision, we tend to be as close as possible to the bijection P: (i) different units of reality get a different name; (ii) units of reality that are subject to change, get a different name whenever they change. If we are interested in smooth and concise communication, we need expressions that make abstraction of difference and change, and hence we need units of language that refer to different or changing units of reality. Without this kind of generalization, communication would be impossible: all of our words would be unique and

solution. They consider a given text or theory as maximally concise and detect and isolate those (occurrences) of expressions that need to be more precise.

Within the field of formal logics we can only detect ambiguity if some inconsistency surfaces. As to avoid inconsistencies, we could consider each occurrence of a non-logical constant (henceforth NLC) in our premises as a new NLC.[3] This way, we meet the ideal of being as precise as possible, and we can use all rules of Classical Logic (henceforth CL). The drawback of this approach is that we can no longer make the derivations we wish to make. For instance, if we consider the premises p and $p \supset q$, and we refuse to identify the two occurrences of p, we can no longer derive q. The solution I propose, can be resumed as follows: we use CL, and we identify two occurrences of one and the same NLC unless (and until) this would lead to the derivation of an inconsistency.

The first step of the here presented approach is to suspect all occurrences of an NLC in the premises to have a different meaning. Formally this is done by giving every occurrence of an NLC in the premises a different superscript.[4] This reading of the premises is called the *maximally ambiguous interpretation*. The second step exists in re-identifying the different occurrences of one and the same NLC, *unless and until this leads to the derivation of an inconsistency*.[5] If a set of premises is not inconsistent,[6] this approach will—obviously—give the same result as CL applied to the original interpretation of the premises. If a set of premises is inconsistent, our approach will detect and isolate ambiguous (occurrences of) NLC.

would refer to unique (states of) things, and hence, none of our words would be understood by someone else. If we want our language to be concise and general, the function Ω can be considered as an ideal. The obvious drawback of this ideal is that we never know for sure what the word "ω" refers to.

[3] Inconsistencies deriving from premises as $(\forall x){\sim}(x{=}x)$ cannot be weeded out with this approach.

[4] In [3] Bryson Brown presented a similar approach to paraconsistency: if you derive $p\&{\sim}p$, choose which one of these two p's you wish to preserve, and replace the other by p'.

[5] The latter sentence is typical for adaptive logics. For the adaptive part, I am indebted to Diderik Batens. By creating ambiguity-adaptive logics, I do nothing but completing the wide range of corrective adaptive (or abnormality-adaptive) logics. Diderik Batens dealed with all logical abnormalities, I happened to have the idea to think of non-logical abnormalities.

[6] Γ is inconsistent iff $\Gamma \vdash_{CL} A\&{\sim}A$ for some A.

There are two strong reasons to elaborate ambiguity-adaptive logics: (i) Ambiguity-adaptive logics are very close to 'natural' reasoning in inconsistent situations. In a lot of situations inconsistencies arise from the ambiguous meaning of non-logical terms. People who meet an inconsistency will not weaken their logic, they will suspect some words to have a double meaning. (ii) The application requires nothing but CL. Where other abnormality-adaptive logics oscillate between a weakening of CL and CL itself, ambiguity-adaptive logics oscillate between CL applied to a maximally ambiguous interpretation of the premises and CL applied to the normal interpretation of the premises.

In the field of abnormality-adaptive logics, two strategies are well elaborated, namely, the reliability strategy and the minimal abnormality strategy. The idea behind the minimal abnormality strategy is that given the fact that we tolerate some logical abnormality (e.g. inconsistency or ambiguity), we should not tolerate more abnormal cases than required in order to safeguard a the theory from triviality. I think it is only natural that we hang on to the minimal abnormality strategy when we are dealing with ambiguity. It is not our purpose to indicate as much as possible expressions that might be ambiguous; to the contrary: if we can render the premises consistent by weeding out one ambiguous expression, there is no need to assume that the premises contain two ambiguous expressions.

2. *About ambiguities*

Let me first clear out that I only consider *primitive propositions, predicates and names for individuals, i.e,* the three kinds of NLC that are used in first order predicative logics. Moreover, I do not study ambiguities that are due to different grammatical functions of one and the same expression.[7] For short, I only consider these cases in which two occurrences of one propositional, individual or predicative constant have the same grammatical function, but a different meaning.[8]

[7] For instance, I am not dealing with the following example; "I am surrounded by animals. My cat is an animal. Hence I am surrounded by my cat." Bernardo Martelli (Bologna) mentioned me the following Italian example: "La vecchia porta la sbarra." In a first reading "vecchia" is a noun, "porta" is a verb, and "sbarra" is a noun, in the second reading "vecchia" is an adjective, "porta" is a noun, and "sbarra" is a verb. My approach can only deal with this ambiguity by roughly formalizing the whole sentence as one propositional constant p and saying that occurrence 1 of p is not the same as occurrence 2 of p.

[8] I mentioned three examples in the introduction.

If we talk about an ambiguous constant C, we need (at least) two occurrences C^1 and C^2 of the constant, that have a different meaning, in order to meet an ambiguity. We restrict our attention to a formal approach of ambiguities, and hence, ambiguities can only be detected if an inconsistency arises. The general scheme of the ambiguities I study is, for a set of premises Γ, an NLC C occurring at least twice in Γ and a well-formed formula A:

Γ, "$C^1{=}C^2$"$\vdash_{CL} A\&{\sim}A$

Obviously, if we want to avoid the derivation of an inconsistency, we have to question the identification of occurrence 1 of C (C^1) and occurrence 2 of C (C^2). We may consider three kinds of causes of the ambiguity at hand, each of which requires a specific approach.

1. We have in mind a specific intended meaning of C.
Hence either C^1 or C^2 does not have the intended meaning. The formal solution in this case exists in saying that either $C^1{\neq}C$ or $C^2{\neq}C$. In natural languages we will replace either C^1 or C^2 by another expression. For instance, if we talk about furniture, and especially about chairs, we better not say that the chair of our department needs a new table, but, e.g. the chairman of our department.[9]

2. We do not have in mind a specific intended meaning of C.
In this case it does not make sense to say that either $C^1{\neq}C$ or $C^2{\neq}C$. The solution at hand is as follows: the fact that C seemingly has (at least) two different meanings indicates that we should question every other occurrence C^i of C. For any C^i we have either $C^i{=}C^1$ & $C^i{\neq}C^2$ or $C^i{=}C^2$ & $C^i{\neq}C^1$. In natural languages we will replace every occurrence of C by a either C^1 or C^2, which are both new expressions. For instance, instead of "Einstein", we may talk about "Einstein as a kid" and "Einstein as a scientist".

3. We doubt whether C is a good expression at all.
In this case the solution may be very drastic: every occurrence is to be replaced by a new expression and all derivations based on the identification of occurrences of C are ruled out. Technically we have, if $C^1{\neq}C^2$, then for all i, j, $C^i{\neq}C^j$. Consider for instance a moral theory that turns out to be inconsistent. This might be due to the use of a predicate (e.g. "being good") that is so vague and ambiguous that the theory would be much better if the predicate did not occur at all.

[9] This view is very close to the preservationist's view, as established in [3].

3. *Minimal abnormality: one ambiguity logic, three ambiguity-adaptive logics*

The three mentioned approaches result in three sets of abnormalities:[10]

(1) $\mathcal{A}^1 = \{C^i \mid C^i \neq C\}$
(2) $\mathcal{A}^2 = \{\langle C^i, C^j \rangle \mid C^i \neq C^j\}$
(3) $\mathcal{A}^0 = \{C \mid C^i \neq C \text{ for some } C^i, C^j\}$

The idea behind adaptive logics based on the minimal abnormality strategy is very intuitive: with respect to a given set of formulas, we should not allow for more abnormalities than necessary in order to keep the set of formulas classically consistent. The abnormalities at hand are ambiguities of NLC. We established that there are (at least) three kinds of abnormalities, and hence we may consider three ambiguity-adaptive logics based on the minimal abnormality strategy, resp. AAL^1, AAL^2 and AAL^0.[11] These logics are based on the application of CL to a maximally ambiguous interpretation of the premises, and a selection of the minimally ambiguous models of this interpreted set of premises.

3.1. *The maximally ambiguous interpretation*

Let $\mathcal{L}$ be the language of CL, containing $\supset$, $\sim$, $\&$, $\vee$, $\equiv$, $\forall$, $\exists$, $=$ and the members of $\mathcal{S}$, $\mathcal{P}^r$, $\mathcal{C}$, $\mathcal{V}$. $\mathcal{S}$ is the set of sentential letters. $\mathcal{P}^r$ is the set of letters for predicates of rank r ($r \geq 1$). $\mathcal{C}$ is the set of letters for individual constants. $\mathcal{V}$ is the set of letters for individual variables.

Let $\mathcal{L}^{\mathcal{I}}$ be obtained from $\mathcal{L}$, by replacing $\mathcal{S}, \mathcal{P}^r, \mathcal{C}$ by respectively $\mathcal{S}^{\mathcal{I}}, \mathcal{P}^{r\mathcal{I}}$, $\mathcal{C}^{\mathcal{I}}$. For $i = 1, 2, ...$, $P^i \in \mathcal{S}^{\mathcal{I}}$ iff $P \in \mathcal{S}$, $\pi^i \in \mathcal{P}^{r\mathcal{I}}$ iff $\pi \in \mathcal{P}^r$, $\beta^i \in \mathcal{C}^{\mathcal{I}}$ iff $\beta \in \mathcal{C}$.

Where $C \in \mathcal{S} \cup \mathcal{P}^r \cup \mathcal{C}$, C is called an NLC and we define $\mathcal{I}(C) = \{C^i \in \mathcal{S}^{\mathcal{I}} \cup \mathcal{P}^{r\mathcal{I}} \cup \mathcal{C}^{\mathcal{I}}\}$. $C^i \in \mathcal{I}(C)$ is called an *indexed* NLC. Let the normal set $\mathcal{W}$

[10] The superscripts 1, 2 and 0 refer to the number of meanings we are willing to ascribe to C; approach 1 results in $\mathcal{A}^1$, approach 2 results in $\mathcal{A}^2$ and approach 3 results in $\mathcal{A}^0$.

[11] Within the wide field of adaptive logics, it would be more appropriate to call these logics resp. AAL^1_2, AAL^2_2 and AAL^0_2. The subscript 2 refers to the minimal abnormality strategy, whereas the subscript 1 usually refers to the reliability strategy. See *e.g.* [1]. As I think that the minimal abnormality strategy is more appropriate when we are dealing with ambiguities, I only consider this second strategy. For aesthetic reasons, I omit the subscript 2 in this paper.

In [5] I presented the logic $ACL2$, which is a predecessor of AAL^1. As opposed to AAL^1, $ACL2$ was defined to be applied to a maximal ambiguous interpretation of the premises, which is —from a logical point of view— quite unusual. Diderik Batens suggested me to define these logics in the same style as *e.g.* Jaskowski's Discussive logics. In [2] Diderik Batens deals with a variant of AAL^2.

of well-formed CL-formulas (henceforth wffs) in the language $\mathcal{L}$ be defined as usual and let $\mathcal{W}^{\mathcal{I}}$ be defined in the language $\mathcal{L}^{\mathcal{I}}$ in the same way. In what follows, the language of CL will be $\mathcal{L}^{\mathcal{I}}$.[12] The syntax and the semantics of CL are as usual. Where $\Gamma \subset \mathcal{W}$, let $\mathcal{I}(\Gamma)$ be such that $\Gamma^I \in \mathcal{I}(\Gamma)$ iff

(i) $\Gamma^I \subset \mathcal{W}^{\mathcal{I}}$,

(ii) each element of $\mathcal{S}^{\mathcal{I}} \cup \mathcal{P}^{r\mathcal{I}} \cup \mathcal{C}^{\mathcal{I}}$ occurs at most once in Γ^I, and

(iii) deleting the superscripts from the elements of $\mathcal{S}^{\mathcal{I}} \cup \mathcal{P}^{r\mathcal{I}} \cup \mathcal{C}^{\mathcal{I}}$ that occur in Γ^I, results in Γ.

It can be shown that all $\Gamma^I \in \mathcal{I}(\Gamma)$ give an equivalent result, and hence it is justified to restrict our attention to one paradigmatic $\Gamma^I \in \mathcal{I}(\Gamma)$. The simplest convention for a set of premises in an actual proof, is to replace the i-th occurrence of an NLC C in Γ by C^i. If, for instance, p has seven occurrences in Γ, the interpreted set of premises $\Gamma^I \in \mathcal{I}(\Gamma)$ will contain $p^1, ..., p^7$, in that order. In what follows the name Γ^I will always refer to this specific member of $\mathcal{I}(\Gamma)$. Where $A \in \mathcal{W}$, let $\mathcal{I}(A)$ be such that $A^I \in \mathcal{I}(A)$ iff

(i) $A^I \in \mathcal{W}^I$.

(ii) deleting the superscripts from the elements of $\mathcal{S}^{\mathcal{I}} \cup \mathcal{P}^{r\mathcal{I}} \cup \mathcal{C}^{\mathcal{I}}$ that occur in A^I, results in A.[13]

Definition 1: $\Gamma \vdash_{AL} A$ *iff there is some* $A^I \in \mathcal{I}(A)$ *such that* $\Gamma^I \vdash_{CL} A^I$

Definition 2: *Where* $C \in \mathcal{S}$, $C^i {\neq} C^j =_{\mathsf{df}} \sim(C^i \equiv C^j)$

Definition 3: *Where* $C \in \mathcal{P}^r$, $C^i {\neq} C^j =_{\mathsf{df}} \sim(\forall\alpha_1)...(\forall\alpha_n)(C^i\alpha_1...\alpha_n \equiv C^j\alpha_1...\alpha_n)$

Definition 4: *Where* $C \in \mathcal{C}$, $C^i {\neq} C^j =_{\mathsf{df}} \sim(C^i {=} C^j)$

3.2. *Semantics of* AAL^1

Definition 5: *Where* M *is a* CL*-model,* $\mathsf{A}^1(\mathsf{M}) = \{C^i \mid \mathsf{v}_{\mathsf{M}}(C^i {\neq} C^\omega) = 1\}$.

ω is an arbitrary index, and C^ω does not occur in Γ^I.

[12] For short: $\mathcal{L}$ contains only NLC without indices, $\mathcal{L}^{\mathcal{I}}$ contains only NLC with indices.

[13] Some indexed NLC may occur more than once in A^I.

Definition 6: *A CL-model* M *is minimally 1-ambiguous with respect to* Γ^I, *iff* M *is a* CL*-model of* Γ^I*, and there is no* CL*-model* M' *of* Γ^I*, such that* $\mathsf{A}^1(\mathsf{M}') \subset \mathsf{A}^1(\mathsf{M})$.

Definition 7: M *is an* AAL^1*-model of* Γ^I *iff* M *is minimally 1-ambiguous with respect to* Γ^I.

Definition 8: $\Gamma \models_{AAL^1} A$ *iff some* $A^I \in \mathcal{I}(A)$ *is true in all* AAL^1*-models of* Γ^I.

3.3. *Semantics of* AAL^2

Definition 9: *Where* M *is a* CL*-model,* $\mathsf{A}^1(\mathsf{M}) = \{\langle C^i, C^j\rangle \mid \mathsf{v}_{\mathsf{M}}(C^i \neq C^j) = 1\}$.

Definition 10: *A* CL*-model* M *is minimally 2-ambiguous with respect to* Γ^I, *iff* M *is a* CL*-model of* Γ^I*, and there is no* CL*-model* M' *of* Γ^I*, such that* $\mathsf{A}^2(\mathsf{M}') \subset \mathsf{A}^2(\mathsf{M})$.

Definition 11: M *is an* AAL^2*-model of* Γ^I *iff* M *is minimally 2-ambiguous with respect to* Γ^I.

Definition 12: $\Gamma \models_{AAL^2} A$ *iff some* $A^I \in \mathcal{I}(A)$ *is true in all* AAL^2*-models of* Γ^I.

3.4. *Semantics of* AAL^0

Definition 13: *Where* M *is a* CL*-model,* $\mathsf{A}^0(\mathsf{M}) = \{C \mid \mathsf{v}_{\mathsf{M}}(C^i \neq C^j) = 1,$ *for some* $i, j\}$.

Definition 14: *A* CL*-model* M *is minimally 0-ambiguous with respect to* Γ^I, *iff* M *is a* CL*-model of* Γ^I*, and there is no* CL*-model* M' *of* Γ^I*, such that* $\mathsf{A}^0(\mathsf{M}') \subset \mathsf{A}^0(\mathsf{M})$.

Definition 15: M *is an* AAL^0*-model of* Γ^I *iff* M *is minimally 0-ambiguous with respect to* Γ^I.

Definition 16: $\Gamma \models_{AAL^0} A$ *iff some* $A^I \in \mathcal{I}(A)$ *is true in all* AAL^0*-models of* Γ^I.

3.5. *Examples*

The examples given in this section establish the difference between and the characteristics of the three ambiguity-adaptive logics. We consider Γ_a, Γ_b, Γ_c and Γ_d, such that:

$\Gamma_a = \{p \vee q, q \supset p\}$
$\Gamma_b = \Gamma_a \cup \{q \supset r, q \supset \sim r, q\}$
$\Gamma_c = \Gamma_b \cup \{r, \sim r\}$
$\Gamma_d = \Gamma_c \cup \{\sim p\}$

Hence Γ^I_a, Γ^I_b and Γ^I_c are respectively:

$\Gamma^I_a = \{p^1 \vee q^1, q^2 \supset p^2\}$
$\Gamma^I_b = \Gamma_a \cup \{q^3 \supset r^1, q^4 \supset \sim r^2, q^5\}$
$\Gamma^I_c = \Gamma_b \cup \{r^3, \sim r^4\}$
$\Gamma^I_d = \Gamma_c \cup \{\sim p^3\}$

(1) Applying AAL^1 to the examples.

The ambiguity-adaptive logic AAL^1 focuses on the question whether all occurrences of an NLC have the intended meaning. The set of abnormalities $\mathsf{A}^1(\mathsf{M})$ of an AAL^1-model M of a set of premises exists of specific occurrences of NLC. An occurrence C^i of an NLC C is a member of $\mathsf{A}^1(\mathsf{M})$ if and only if the model would not be a model of the premises if it did not verify $C^i \neq C^\omega$. The use of C^ω results in the fact that $C^i \in \mathsf{A}^1(\mathsf{M})$ does not imply that some other C^j needs to be a member of $\mathsf{A}^1(\mathsf{M})$. AAL^1 isolates the specific ambiguous occurrences of an NLC; from all other occurrences of the same NLC, AAL^1 derives all classical consequences, as we can see in this example. All CL-models of Γ^I_a verify:

$$p^1 \vee \; p^1 \neq p^\omega \; \vee \; p^2 \neq p^\omega \; \vee \; q^1 \neq q^\omega \; \vee \; q^2 \neq q^\omega \qquad (1)$$

For all AAL^1-models M of Γ^I_a: $\mathsf{A}^1(\mathsf{M}) = \emptyset$, and hence $\mathsf{v}_\mathsf{M}(p^1) = 1$ and hence

$$\Gamma_a \models_{AAL^1} p$$

All CL-models of Γ^I_b verify (1), and

$$r^1 \; \vee \; q^3 \neq q^\omega \; \vee \; q^5 \neq q^\omega \qquad (2)$$
$$\sim q^3 \; \vee \; q^3 \neq q^\omega \; \vee \; q^4 \neq q^\omega \; \vee \; r^1 \neq r^\omega \; \vee \; r^2 \neq r^\omega \qquad (3)$$
$$q^3 \neq q^\omega \; \vee \; q^4 \neq q^\omega \; \vee \; q^5 \neq q^\omega \; \vee \; r^1 \neq r^\omega \; \vee \; r^2 \neq r^\omega \qquad (4)$$

Hence all AAL^1-models of Γ^I_b verify one of the disjuncts of (4), and hence, the models that verify $q^3 \neq q^\omega$ or $q^5 \neq q^\omega$ do not necessarily verify r^1. Neither do the models that verify $q^3 \neq q^\omega$, $q^4 \neq q^\omega$, $r^1 \neq r^\omega$ or $r^2 \neq q^\omega$ necessarily verify

$\sim q^3$. Still, none of these models verifies $p^1 \neq p^\omega$, $p^2 \neq p^\omega$, $q^1 \neq q^\omega$ or $q^2 \neq q^\omega$, and hence they all verify p^1. Hence:

$\Gamma_b \models_{AAL^1} p \qquad \Gamma_b \not\models_{AAL^1} r$
$\qquad\qquad\qquad\quad \Gamma_b \not\models_{AAL^1} \sim q$

All CL-models of Γ_c^I verify (1)–(4) and

$$r^3 \neq r^\omega \quad \vee \quad r^4 \neq r^\omega \tag{5}$$

Hence all AAL^1-models of Γ_c^I verify one of the disjuncts of (3) and one of the disjuncts of (4) which has no surprising influence on the consequences:

$\Gamma_c \models_{AAL^1} p \qquad \Gamma_c \not\models_{AAL^1} \sim q$
$\Gamma_c \models_{AAL^1} r$

It is instructive for the reader to check the correctness of:

$\Gamma_d \not\models_{AAL^1} p$
$\Gamma_d \not\models_{AAL^1} \sim q$

(2) Applying AAL^2 to the examples.

The ambiguity-adaptive logic AAL^2 first detects couples of occurrences $\langle C^i, C^j \rangle$ of NLC, the members of which cannot be identified with one another without making the premises inconsistent. Next the logic focuses on the question whether the other occurrences of the same NLC can be identified with either C^i or C^j. The drawback of a formal approach is that one cannot tell from looking at C^k whether $C^k = C^i \& C^k \neq C^j$ or $C^k = C^j \& C^k \neq C^i$. Hence, as soon as $\langle C^i, C^j \rangle \in \mathsf{A}^2(\mathsf{M})$ for some AAL^2-model M, there will be AAL^2-models that verify $C^k \neq C^i$ and there will be AAL^2-models that verify $C^k \neq C^j$, and so on; hence as soon as $\langle C^i, C^j \rangle \in \mathsf{A}^2(\mathsf{M})$ for some AAL^2-model M, there will be a AAL^2-model that verifies $C^k \neq C^l$ for any k, l! This is a result that this ambiguity-adaptive logic *should* obtain. The decision whether $C^k = C^i \& C^k \neq C^j$ or $C^k = C^j \& C^k \neq C^i$ is an informal decision, and cannot be made by means of a formal logic.

All CL-models of Γ_a^I verify:

$$p^1 \vee \; p^1 \neq p^2 \;\; \vee \;\; q^1 \neq q^2 \tag{6}$$

For all AAL^2-models M of Γ_a^I: $\mathsf{A}^2(\mathsf{M}) = \emptyset$, and hence $\mathsf{v}_{\mathsf{M}}(p^1) = 1$ and hence:

$\Gamma_a \models_{AAL^2} p$

All CL-models of Γ_b^I verify (6) and

$$r^1 \quad \vee \quad q^3 \neq q^5 \tag{7}$$

$\sim q^3 \;\vee\; q^3 \neq q^4 \;\vee\; r^1 \neq r^2$ (8)
$q^3 \neq q^4 \;\vee\; q^3 \neq q^5 \;\vee\; r^1 \neq r^2$ (9)
$q^3 \neq q^4 \;\vee\; q^4 \neq q^5 \;\vee\; r^1 \neq r^2$ (10)
$q^3 \neq q^5 \;\vee\; q^4 \neq q^5 \;\vee\; r^1 \neq r^2$ (11)

Hence all AAL^2-models of Γ_b^I verify exactly one of the disjuncts of (9), (10) and (11). Every model that verifies $q^3 \neq q^4$ either verifies $q^1 \neq q^3$ or $q^1 \neq q^4$, and also verifies either $q^2 \neq q^3$ or $q^2 \neq q^4$. Some of them verify $q^1 \neq q^2$. For instance: the model that verifies $q^3 \neq q^4$, $q^1 \neq q^3$, $q^2 \neq q^4$ and $q^1 \neq q^2$ and falsifies $q^1 \neq q^4$ is an AAL^2-model of Γ_b^I. Thus we get a much weaker result in comparison with AAL^1:

$\Gamma_b \not\models_{AAL^2} p$
$\Gamma_b \not\models_{AAL^2} r$
$\Gamma_b \not\models_{AAL^2} \sim q$

All CL-models of Γ_c^I verify (8)–(13) and

$r^3 \neq r^4$ (12)

Hence all AAL^2-models of Γ_c^I verify one of the disjuncts of (9), (10), (11) and (12) which has no surprising influence on the consequences. Indeed, not all AAL^2-models of Γ_c^I that verify (12) verify $r^1 \neq r^2$; and all of these models verify 2 members of $\{q^3 \neq q^4, q^3 \neq q^5, q^4 \neq q^5\}$

(3) Applying AAL^0 to the examples.

In the case of AAL^0, the abnormalities are non-indexed NLC. This has an important influence on the sets $\mathsf{A}^0(\mathsf{M})$ of the AAL^0-models. If $C \in \mathsf{A}^0(\mathsf{M})$, and M verifies, *e.g.* $C^i \neq C^j \vee D^k \neq D^l$, there is no need that $D \in \mathsf{A}^0(\mathsf{M})$. This characteristic results in the fact that AAL^0 is more sensitive to 'inconsistent extensions' of the premises than AAL^1 and AAL^2. For all AAL^0-models M of Γ_a^I: $\mathsf{A}^2(\mathsf{M}) = \emptyset$, and hence

$\Gamma_a \models_{AAL^0} p$

For all AAL^0-models M of Γ_b^I: $\mathsf{A}^0(\mathsf{M}) = \{q\}$ or $\mathsf{A}^0(\mathsf{M}) = \{r\}$. Hence we get the following result:

$\Gamma_b \not\models_{AAL^0} p$
$\Gamma_b \not\models_{AAL^0} r$
$\Gamma_b \not\models_{AAL^0} \sim q$

For all AAL^0-models M of Γ_c^I: $\mathsf{A}^0(\mathsf{M}) = \{r\}$. Hence:

$\Gamma_c \models_{AAL^0} p$ $\qquad \Gamma_c \not\models_{AAL^0} \sim q$
$\Gamma_c \models_{AAL^0} r$

For all AAL^0-models M of Γ_d^I: $\mathsf{A}^0(\mathsf{M}) = \{p, r\}$ or $\mathsf{A}^0(\mathsf{M}) = \{q, r\}$. Hence:

$\Gamma_c \models_{AAL^0} r$ $\quad$ $\Gamma_c \not\models_{AAL^0} p$

$\Gamma_c \not\models_{AAL^0} \sim q$

(4) Overview

	$\models p?$				$\models \sim q?$			$\models r?$		
	Γ_a^I	Γ_b^I	Γ_c^I	Γ_d^I	Γ_b^I	Γ_c^I	Γ_d	Γ_b^I	Γ_c^I	Γ_d^I
AAL^1	yes	yes	yes	no	no	no	no	no	yes	yes
AAL^2	yes	no	no	no	no	no	no	no	yes	yes
AAL^0	yes	no	yes	no	no	no	no	no	yes	yes

It is also interesting to notice that the three logics derive both $p{\neq}p \vee q{\neq}q$ and $r{\neq}r$ from Γ_d. Moreover they all prevent the derivation of p and $\sim q$ because of the fact that $\sim p$ and q are premises. The fact that both r and $\sim r$ are derivable is due to the fact that the two of them are premises.

4. *Proof theory of AAL^0*

4.1. *Final derivability*

I start with defining a new rule, which is typical for CL when applied to a maximally ambiguous interpretation of a set of premises.

Definition 17: *The rule DR: to derive $A \vee DA^0\{C_1, ..., C_n\}$ from $A \vee C_1^i \neq C_1^j \vee ... \vee C_n^k \neq C_n^l$ for any non-indexed NLC C_i and any indexes $i, j, ..., k, l$.*

Definition 18: *$DA^0(\Sigma)$ is a minimal DA^0-consequence of Γ iff $\Gamma^I \vdash_{CL} DA^0(\Sigma)$, and there is no Δ such that $\Delta \subset \Sigma$ and $\Gamma^I \vdash_{CL} DA^0(\Delta)$.*

Definition 19: *Φ_Γ is the set of all sets φ that (i) contain exactly one element of each minimal DA^0-consequence of Γ and that (ii) are no proper supersets of such a set.*

For instance: if $DA^0\{p, q\}$ and $DA^0\{p, r, s\}$ are the only minimal DA^0-consequences of Γ^I, then $\Phi_\Gamma = \{\{p\}, \{q, r\}, \{q, s\}\}$. The reader can easily see that every $\varphi \in \Phi_\Gamma$ is equal to some $\mathsf{A}^0(\mathsf{M})$ of an AAL^0-model.[14]

[14] Definition 13, p. 146.

Definition 20: *$\Gamma \vdash_{AAL^0} A$ iff there is a $A^I \in \mathcal{I}(A)$ and there are one or more (possibly empty) Σ_i, such that $\Gamma^I \vdash_{CL} A^I \vee DA^0(\Sigma_i)$, and for any $\varphi \in \Phi_\Gamma$, one of the Σ_i is such that $\Sigma_i \cap \varphi = \emptyset$.*

Obviously, if $\Gamma^I \vdash_{CL} A^I$ then $\Gamma \vdash_{AAL^0} A$. The following example illustrates the AAL^0-derivability relation. Consider the set Γ, containing a formalization of the following sentences:

The "liar's paradox" is the name of a sentence that is both true and false.
All true sentences are well-formed.
All false sentences are well-formed.
If something is true and false, it is a paradox.
Paradoxes are well-formed.
If a sentence is true, then it is not false.

$\Gamma = \{(\exists x)((x = l\&Sx)\&(Tx\&Fx)), (\forall x)((Tx\&Sx)\supset Wx), (\forall x)((Fx\& Sx)\supset Wx), (\forall x)((Tx\&Fx)\supset Px), (\forall x)(Px\supset Wx), (\forall x)((Tx\&Sx)\supset {\sim}Fx)\}$

It is easily seen that Γ is inconsistent. Applying CL to Γ^I results in the following proof:

(1)	$(\exists x)((x = l^1\&S^1x)\&(T^1x\&F^1x))$	Prem
(2)	$(\forall x)((T^2x\&S^2x)\supset W^1x)$	Prem
(3)	$(\forall x)((F^2x\&S^3x)\supset W^2x)$	Prem
(4)	$(\forall x)((T^3x\&F^3x)\supset P^1x)$	Prem
(5)	$(\forall x)(P^2x\supset W^3x)$	Prem
(6)	$(\forall x)((T^4x\&S^4x)\supset {\sim}F^4x)$	Prem
(7)	$(S^1l^1\&T^1l^1)\&F^1l^1$	1
(8)	$(T^2l^1\&S^2l^1)\supset W^1l^1$	2
(9)	$(F^2l^1\&S^3l^1)\supset W^2l^1$	3
(10)	$(\forall x)((T^3x\&F^3x)\supset W^3x) \vee DA^0\{P\}$	4,5
(11)	$(T^3l^1\&F^3l^1)\supset W^3l^1) \vee DA^0\{P\}$	10
(12)	$W^1l^1 \vee DA^0\{S,T\}$	7,8
(13)	$W^2l^1 \vee DA^0\{S,F\}$	7,9
(14)	$W^3l^1 \vee DA^0\{T,F,P\}$	7,10
(15)	$W^1l^1 \vee DA^0\{S,F,W\}$	13
(16)	$W^1l^1 \vee DA^0\{T,F,P,W\}$	14
(17)	$(T^4l^1\&S^4l^1)\supset {\sim}F^4l^1$	6
(18)	$(F^4l^1\&{\sim}F^4l^1) \vee DA^0\{S,T,F\}$	7,17
(19)	$DA^0\{S,T,F\}$	18

Obviously all wffs occurring in this proof are, if we omit the superscripts, AAL^0-consequences of Γ. The formula in line (19) is the only minimal

DA^0-consequence of Γ and hence $\Phi_\Gamma = \{\{T\},\{S\},\{F\}\}$. In view of (12), (15) and (16), Definition 20, and

$\{S\} \cap \{T, F, W, P\} = \emptyset$
$\{T\} \cap \{S, F, W\} = \emptyset$
$\{F\} \cap \{S, T\} = \emptyset$,

we have $\Gamma \vdash_{AAL^0} Wl$. Notice that if one of the formulas in lines (12), (15) and (16) would not be CL-derivable from Γ^I, then $\Gamma \nvdash_{AAL^0} Wl$. The derivation of (19) is to be interpreted as follows: at least one of the predicates "being true", "being false" or "being a sentence" is ambiguous (with respect to these premises).

4.2. *Dynamic proofs / provisionally derived formulas*

Adaptive logics have dynamic proofs. I use to the example from Section 3.5 to illustrate this feature of AAL^0.

(1)	$p^1 \vee q^1$	Prem
(2)	$q^2 \supset p^2$	Prem
(3)	$p^1 \vee DA^0\{p, q\}$	1,2
(4)	$q^3 \supset r^1$	Prem
(5)	$q^4 \supset {\sim}r^2$	Prem
(6)	q^5	Prem
(7)	${\sim}q^3 \vee DA^0\{q, r\}$	4,5
(8)	$r^1 \vee DA^0\{q\}$	4,6
(9)	${\sim}r^2 \vee DA^0\{q\}$	5,6
(10)	$DA^0\{q, r\}$	6,7 or 8,9
(11)	$r^3 \& {\sim}r^4$	Prem
(12)	$DA^0\{r\}$	11
(13)	${\sim}p^3$	Prem
(14)	$DA^0\{p, q\}$	7,8

If the proof had stopped at line (3), there were no DA^0-consequences, and hence p was derived. If the proof had stopped at line (10), there would be only one minimal DA^0-consequence. At this stage $\Phi_{\Gamma_b} = \{\{q\},\{r\}\}$; as $\{p, q\} \cap \{q\} \neq \emptyset$, p is not derivable at line (10); neither are ${\sim}q$, r or ${\sim}r$. If the proof had stopped at line (12), there would be only one minimal DA^0-consequence, namely $DA^0\{r\}$, and hence $\Phi_{\Gamma_c} = \{\,\{r\}\,\}$, and hence p at line (3) becomes derived again. Finally, at stage (14), there are two minimal DA^0-consequences, namely $DA^0\{r\}$ and $DA^0\{p, q\}$, and hence $\Phi_{\Gamma_d} = \{\{p, r\},\{q, r\}\}$; as $\{p, q\} \cap \{p, r\} \neq \emptyset$, p is not derivable; as $\{q, r\} \cap \{q, r\} \neq \emptyset$, ${\sim}q$ is not derivable.

Please notice that if we omit the superscripts in the proof, the usual classical rules become *conditional*, *i.e.*, we get the usual classical consequences in disjunction with a DA^0-formula.

When we are dealing with infinite sets of premises, or with 'slowly extending' sets of premises, we never know which formulas will be finally derivable. In this cases, dynamic proofs offer a splendid provisional solution. For the user's comfort we add a fifth element to each line in which sets of abnormalities may occur. These sets are subject to the Conditional Rule RC:

Definition 21: *The rule RC: from a line* (i) *to derive a line* (j)*, and vice versa:*[15]

(i)	$A^I \vee DA^0(\Sigma)$	(*line numbers*); *Rule*	Δ
(j)	A^I	(i); *RC*	$\Delta \cup \Sigma$

It is possible to define AAL^0 by means of the rule RC, the unconditional rule RU, and a marking rule RM. A proof in which only these rules are used, is called an ACL^0-proof.

Definition 22: *The rule RU: from lines* $(i_1), ..., (i_n)$*, with resp.* $A_1^I, ..., A_n^I$ *as second element and* $\Sigma_1, ..., \Sigma_n$ *as fifth element* $(n \geq 0)$*, to derive a line* (j) *with* B^I *as second element,* $(i_1, ..., i_n)$ *as third element, UR as fourth element, and* $\Sigma_1 \cup ... \cup \Sigma_n$ *as fifth element, given that* $A_1^I, ..., A_n^I \vdash_{CL} B^I$.

Definition 23: $DA^0(\Sigma)$ *is a minimal* DA^0*-formula of* Γ^I *at line* (i) *of a proof from* Γ^I*, iff* $DA^0(\Sigma)$ *is the second element of a line* (j) $(1 \leq j \leq i)$ *the fifth element of which is empty, and there is no* $\Delta \subset \Sigma$ *such that* $DA^0(\Delta)$ *is the second element of a line* (k) $(1 \leq k \leq i)$ *the fifth element of which is empty.*

Definition 24: $\Phi_{(i)}$ *is the set of all sets* φ *that* (i) *contain exactly one element of each minimal* DA^0*-formula of* Γ^I *at line* (i) *of that proof from* Γ^I *and, that* (ii) *are no proper supersets of such a set.*

Definition 25: *Line* (j) *with* A^I *as second and* Σ *as fifth element, fulfils the integrity criterion at stage* (i) *of a proof from* Γ^I*, iff* (i) $\varphi \cap \Sigma = \emptyset$ *for some* $\varphi \in \Phi_{(i)}$*, and for each* $\varphi \in \Phi_{(i)}$ *there is a line* (k) $(1 \leq k \leq i)$ *such that, where* Σ_k *is the fifth element of line* (k)*,* $\varphi \cap \Sigma_k = \emptyset$.

[15] Actually, the second direction is a derivable rule.

Definition 26: *The rule RM: If a line does not fulfil the integrity criterion at a stage (i) of a proof, then the line is market* OUT.

Definition 27: A^I *is derived at a stage* (i) *of a proof, iff* A^I *is the second element of a line that is not marked* OUT.

When applied to the same example, we get the following ACL^0-proof:

(1)	$p^1 \vee q^1$	Prem	$\emptyset$	
(2)	$q^2 \supset p^2$	Prem	$\emptyset$	
(3)	p^1	1,2	$\{p, q\}$	OUT(10) IN(12) OUT(14)
(4)	$q^3 \supset r^1$	Prem	$\emptyset$	
(5)	$q^4 \supset \sim r^2$	Prem	$\emptyset$	
(6)	q^5	Prem	$\emptyset$	
(7)	$\sim q^3$	4,5	$\{q, r\}$	OUT(10)
(8)	r^1	4,6	$\{q\}$	OUT(10) IN(12)
(9)	$\sim r^2$	5,6	$\{q\}$	OUT(10) IN(12)
(10)	$DA^0\{q, r\}$	6,7	$\emptyset$	
(11)	$r^3 \& \sim r^4$	Prem	$\emptyset$	
(12)	$DA^0\{r\}$	11	$\emptyset$	
(13)	$\sim p^3$	Prem	$\emptyset$	
(14)	$DA^0\{p, q\}$	7,8	$\emptyset$	

Definition 28: A^I *is finally derived in a line of an* ACL^0*-proof from* Γ^I*, iff it is the second element of that line and any (possibly infinite) extension of the proof can be further extended in such way that the line is unmarked (or marked* IN*).*

Theorem 1: $\Gamma \vdash_{AAL^0} A$ *iff some* $A^I \in \mathcal{I}(A)$ *is finally derived at some line of an* ACL^0*-proof from* Γ^I.

Theorem 2: *If* $\Gamma \vdash_{AAL^0} A$*, then, for some* $A^I \in \mathcal{I}(A)$*, it is possible to extend any proof from* Γ^I *into a proof in which* A^I *is finally derived.*[16]

5. *Proof theory of* AAL^1

Definition 29: *A* DA^1*-formula is a formula of the form* $C_1^{i1} \neq C_1^{\omega} \vee ... \vee C_n^{in} \neq C_n^{\omega}$*, abbreviated as* $DA^1\{C_1^{i1}, ..., C_n^{in}\}$.

[16] The proofs of Theorems 1 and 2 are analogous to the proofs of resp. Theorems 7.1 and 7.2 in [1].

The definitions of (1) the conditional rule RC, (2) the unconditional rule UR, (3) a minimal DA^1-formula at a stage of a proof, (4) $\Phi_{(i)}$, (5) the integrity criterion, (6) the marking rule RM, (7) an ACL^1-proof, (8) finally derived in a line of an ACL^1-proof, (9) minimal DA^1-consequence, (10) and Φ_Γ are completely analogous as for AAL^0.

Definition 30: $\Gamma \vdash_{AAL^1} A$ *iff some* $A^I \in \mathcal{I}(A)$ *is finally derived at some line of an* ACL^1*-proof from* Γ^I.

Theorem 3: $\Gamma \vdash_{AAL^1} A$ *iff there is a* $A^I \in \mathcal{I}(A)$ *such that there are one or more (possibly empty) finite sets* $\Sigma, \Sigma, \ldots \supset \mathcal{A}^1$*, such that* $\Gamma^I \vdash_{CL} A^I \vee DA^1(\Sigma_i)$, $\Gamma^I \vdash_{CL} A^I \vee DA^1(\Sigma_i)$*, ..., and for any* $\varphi \in \Phi_\Gamma$*, one of the* Σ_i *is such that* $\Sigma_i \cap \varphi = \emptyset$.

Theorem 4: *If* $\Gamma \vdash_{AAL^1} A$*, then, for some* $A^I \in \mathcal{I}(A)$*, it is possible to extend any proof from* Γ^I *into a proof in which* A^I *is finally derived.*

Consider the following example. Let Γ be a formalization of
“John is the father of Paul. John and his relatives live in Yorkshire. Robert is a brother of John. Robert is the father of Melanie, and Paul is the father of John. [If you are father or brother of someone, (s)he is one of your relatives. “Being father of” is an asymmetric relation.]”

The members of Γ^I can be found in lines (1)–(6):

(1)	$F^1j^1p^1$	Prem	$\emptyset$	
(2)	$(\exists x)((x{=}j^2\&(\forall y)((R^1yx \vee x = y) \supset Y^1y)))$		$\emptyset$	
(3)	$B^1r^1j^3$	Prem	$\emptyset$	
(4)	$F^4r^2m^1\&F^5p^2j^4$	Prem	$\emptyset$	
(5)	$(\forall x)(\forall y)((F^6xy \vee B^2xy) \supset R^2yx)$	Prem	$\emptyset$	
(6)	$(\forall x)(\forall y)(F^7xy \supset {\sim}F^8yx)$	Prem	$\emptyset$	
(7)	$R^2p^1j^1$	1,5	$\{F^1, F^6\}$	OUT
(8)	$R^1p^1j^2 \supset Y^1p^1$	2	$\emptyset$	
(9)	Y^1p^1	7,8	$\{F^1, F^6, R^1, R^2, j^1, j^2\}$	OUT
(10)	$R^2m^1r^2$	4,5	$\{F^4, F^6\}$	
(11)	Y^1j^2	2	$\emptyset$	
(12)	$j^1 = j^\omega$	—	$\{j^1\}$	OUT
(13)	$j^3 = j^\omega$	15	$\{j^3\}$	
(14)	$DA^1\{F^1, F^5, F^7, F^8, j^1, j^4, p^1, p^2\}$	1,4,6	$\emptyset$	

As long as we are not informed about the fact that John and his grandchild have the same name, there are 8 possibly ambiguous occurrences of NLC, *viz.* those in the DA^1-formula in line (14). The ACL^1-derivation of this

DA^1-formula blocks the AAL^1-derivation of the formulas Rpj and Yp. Notice that $\Gamma \vdash AAL^1 Rmr$, whereas $\Gamma \nvdash AAL^0 Rmr$. Indeed: the AAL^1-derivation of Rmr only depends on the normal behaviour of occurrences F^4 and F^6, whereas the AAL^0-derivation of Rmr depends on the normal behaviour of all occurrences of F.[17]

Also notice that, although we intuitively know that j^1, j^2 and j^3 have the intended meaning of j, whereas j^4 refers to the grandchild of j, we can derive $j^2 = j^\omega$ and $j^3 = j^\omega$, but neither $j^1 = j^\omega$ nor $j^4 \neq j^\omega$. This is normal: AAL^1 is a formal logic that does not take into account intuitive preferences.

With respect to applications, it might be interesting to consider the following consequence relation. Let $A^\omega \in \mathcal{I}(A)$ be a formula in which all NLC have the index ω.

Definition 31: $\Gamma \vdash_{AAL^\omega} A$ *iff*
(i) $\Gamma^I \vdash_{ACL^1} A^\omega$ *or*
(ii) $A \in \mathcal{I}(A)$ *and* $\Gamma^I \vdash_{ACL^1} A$*, and for any formula* $A^I \in \mathcal{I}(A)$ *obtained by replacing a superscript* $i \neq \omega$ *in* A *by* ω*,* $\Gamma^I \nvdash_{ACL^1} A^I$

Interesting about this consequence relation is that *all* possibly ambiguous occurrences of NLC *and only those* are replaced by new NLC.

A straightforward example is taken from the proof above. $\Gamma^I \vdash_{ACL^1} F^5p^2j^4$, whereas $\Gamma^I \nvdash_{ACL^1} F^\omega p^2 j^4$, $\Gamma^I \nvdash_{ACL^1} F^5 p^\omega j^4$, and $\Gamma^I \nvdash_{ACL^1} F^5p^2j^\omega$, because these formulas can only be derived in a line with respectively $\{F^5\}$, $\{j^2\}$ and $\{p^4\}$ as fifth element—conditions which are obviously overruled in view of (14). Hence $\Gamma \vdash_{AAL^\omega} F^1j^1p^1$. We also have $\Gamma \vdash_{AAL^\omega} Rmr$, and $\Gamma \nvdash_{AAL^\omega} R^2m^1r^2$.

6. *Proof theory of* AAL^2

Definition 32: *A* DA^2*-formula is a formula of the form* $C_1^{i1} \neq C_1^{j1} \vee ... \vee C_n^{in} \neq C_n^{jn}$*, abbreviated as* $DA^2\{\langle C_1^{i1}, C_1^{j1}\rangle, ..., \langle C_n^{in}, C_n^{jn}\rangle\}$.

The definitions of (1) the conditional rule RC, (2) the unconditional rule UR, (3) a minimal DA^2-formula at a stage of a proof, (4) $\Phi_{(i)}$, (5) the integrity criterion, (6) the marking rule RM, (7) an ACL^1-proof, (8) finally derived in

[17] $\Gamma \vdash_{AAL^0} DA^0\{F, j, p\}$

a line of an ACL^2-proof, (9) minimal DA^2-consequence, and (10) Φ_Γ are completely analogous as for AAL^0.

Definition 33: $\Gamma \vdash_{AAL^2} A$ *iff some* $A^I \in \mathcal{I}(A)$ *is finally derived at some line of an* ACL^2*-proof from* Γ^I.

Theorem 5: $\Gamma \vdash_{AAL^2} A$ *iff there is a* $A^I \in \mathcal{I}(A)$ *such that there are one or more (possibly empty) finite sets* $\Sigma, \Sigma, ... \supset \mathcal{A}^1$*, such that* $\Gamma^I \vdash_{CL} A^I \vee DA^1(\Sigma_i)$*,* $\Gamma^I \vdash_{CL} A^I \vee DA^1(\Sigma_i)$*, ..., and for any* $\varphi \in \Phi_\Gamma$*, one of the* Σ_i *is such that* $\Sigma_i \cap \varphi = \emptyset$.

Theorem 6: *If* $\Gamma \vdash_{AAL^0} A$*, then, for some* $A^I \in \mathcal{I}(A)$*, it is possible to extend any proof from* Γ^I *into a proof in which* A^I *is finally derived.*

When we look at the example given in Section 3.5, one may get the impression that AAL^2 is too weak. This weakness is due to fact that the AAL^2-ambiguities are more specified and fundamental than the AAL^0- and the AAL^1-ambiguities. Indeed, from the fact that a couple $\langle C^i, C^j \rangle$ is AAL^2-ambiguous with respect to the premises, we can imply that C^i and C^j are AAL^1-ambiguous with respect to the premises and that C is AAL^0-ambiguous with respect to the premises, *but not the other way round*. This property allows for specific strong applications. Let me first show that the apparent weakness can be overcome very easily by making proofs with the purpose to derive *least* DA^2*-consequences*.

Definition 34: *If* $DA^2(\Sigma)$ *is a minimal* DA^2*-consequence of* Γ^I*, then it is a least* DA^2*-consequence of* Γ^I *iff there is no* $DA^2(\Delta)$ *such that* $DA^2(\Delta)$ *is a minimal* DA^2*-consequence of* Γ^I *and* $DA^2(\Delta) \vdash_{CL} DA^2(\Sigma)$.

For instance, if $p^1 \neq p^2$ is a minimal DA^2-consequence of Γ^I, then $p^1 \neq p^3 \vee p^2 \neq p^3$ is also a minimal DA^2-consequence of Γ^I. In view of $p^1 \neq p^2 \vdash_{CL} p^1 \neq p^3 \vee p^2 \neq p^3$, the latter is not a least DA^2-consequence of Γ^I. It is the purpose of AAL^2 to divide the occurrences of ambiguous NLC into partitions. For instance if $p^1 \neq p^2$ is the only least DA^2-consequence in which p occurs, the intended partitions for p will be $\Pi(p^1) = \{p^i \mid p^i = p^1\}$ and $\Pi(p^2) = \{p^i \mid p^i = p^2\}$. Clearly is not the duty of a formal logic to decide for every p^i whether $p^i \in \Pi(p^1)$ or $p^i \in \Pi(p^2)$; this is the aim of the one who applies the logic in a specific situation. The least DA^2-consequences suggest which partitions one should consider. The specific situation may offer sufficient information to decide whether $p^i = p^1$ or $p^i = p^2$.

In [6] a very interesting application of AAL^2 is presented. If we only allow for 'ambiguities' with respect to the individual constants, we can consider the superscripts as time-indexes. This results in a change-adaptive logic which is very useful in all fields that deal with changing objects. This approach allows, *e.g.* to work with the following kind of formulas (t^1, t^2 are time-indexes, i, j are variables for time-indexes):

$(t^1 < i < t^2 \ \& \ t^1 < j < t^2) \supset (Pa^i \equiv Pa^j)$.
$(i < t^2 < j) \supset \sim(Pa^i \equiv Pa^j)$.

7. *Soundness and completeness*

Theorem 7: $\Gamma \vdash_{AAL^0} A$ *iff* $\Gamma \models_{AAL^0} A$.

The proof follows from $\Gamma^I \vdash_{CL} A^I$ iff $\Gamma^I \models_{CL} A^I$, and the fact that Φ_Γ is exactly the same as the set of all $\mathsf{A}^0(\mathsf{M})$ of all AAL^0-models M. An analogous remark holds for Theorems 8 and 9.[18]

Theorem 8: $\Gamma \vdash_{AAL^1} A$ *iff* $\Gamma \models_{AAL^1} A$.

Theorem 9: $\Gamma \vdash_{AAL^2} A$ *iff* $\Gamma \models_{AAL^2} A$.

8. *The logical constants of the ambiguity logic*

It is interesting to take a closer look at the behaviour of the logical constants in the ambiguity logic AL. Obviously, all CL-theorems are AL-theorems. As for instance $\vdash_{CL} ((p^1 \supset q^1) \supset p^1) \supset p^1$, we have $\vdash_{AL} ((p \supset q) \supset p) \supset p$. Also, all CL-theorems are AL-derivable from any set of premises.

The implication is not detachable. We have $A \vdash_{AL} (A \supset B) \supset B$, but we do not have $A, A \supset B \vdash_{AL} B$.

There is no contraction for the disjunction. We have $\vdash_{AL} (A \vee A) \supset A$, but we do not have $A \vee A \vdash_{AL} A$. Obviously, addition is valid.

The negation is highly paraconsistent. Although $\vdash_{AL} (A \& \sim A) \supset B$, still $A \& \sim A \nvdash_{AL} B$.

The conjunction behaves classically: $A \& B \vdash_{AL} A$; $A \& B \vdash_{AL} B$; $A, B \vdash_{AL} A \& B$.

[18] For a more detailed soundness- and completeness theorem, I refer to [5].

There is a remarkable similarity between the behaviour of the connectives of the logic AL and *e.g.* Priest's logic of paradox. In [2], Diderik Batens proves that (full) LP can be defined within a variant of AL.

9. *Concluding remarks*

The results of this paper and papers like [2], [3] and [5], shed a new light on paraconsistency. Usually the paraconsistent approach exists in weakening the strength of one or more classical connectives. I think this paper made it clear that paraconsistency is also possible without such a weakening, and a fortiori without true inconsistencies. If we consider the fact that most logical rules rely on the identification of two or more non-logical constants, it is right to say that ambiguity logics deal with a fundamental logical issue.

Where all classical and most paraconsistent logicians assume that —or act as if— all occurrences of one NLC can be identified with one another, ambiguity-adaptive logics explicitly question this assumption and offer a result that meets the purposes of both classical and paraconsistent logics.

Let AAL refer to either AAL^0, AAL^1 or AAL^2. If Γ is classically consistent, then $\Gamma \vdash_{CL} A$ iff $\Gamma \vdash_{AAL} A$. If the premises do not contain a formula in which no NLC occurs and that is inconsistent with a CL-theorem,[19] then CL derives triviality, whereas AAL does not. Moreover, 'outside the scope' of the members of the minimal DA-formulas, AAL behaves completely classically. Ambiguity-adaptive logics do not only block derivations from possibly ambiguous NLC, they also indicate which NLC (AAL^0) or specific occurrences of NLC (AAL^1) are possibly ambiguous, or which pairs of occurrences can not be identified with one another (AAL^2). The consequence relation $\vdash_{AAL^\omega}$ solves inconsistencies in a creative way: it replaces ambiguous occurrences of NLC by new NLC.

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[19] For instance $\sim(\forall x)x = x$.

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