Logique & Analyse **165–166** (1999), 81–106

REPRESENTATION OF j-ALGEBRAS AND SEGERBERG'S LOGICS

S.P. ODINTSOV

1. *Introduction*

This article continues investigations of paraconsistent extensions of minimal logic Lj started in [3, 4, 5]. In [3], it was stated that the class Jhn of all non-trivial extensions of Lj partition into three disjoint subclasses. These are the class Int of intermediate logics satisfying *ex contradictione quodlibet*, the class Neg of negative logics containing formula $\bot$ (or, equivalently, $\neg p$), and the class Par of proper paraconsistent extensions of Lj consisting of logics not belonging to the first two classes. The negative logics have degenerate negation in a sense that any negated formula is provable. Thus, the third class includes all non-trivial cases of paraconsistent negations. The greatest logic of this class is the logic of classical refutability Le (see, e.g. [1]).

The above mentioned decomposition of the class Jhn motivates an effort to describe the class Par in terms of classes of intermediate and negative logics, which was extensively studied. Note that the class of negative logics is definitionally equivalent to the class of positive logics. To some extent, it was done in [4]. For any logic $L \in Par$ there were defined its intuitionistic and negative counterparts, $L_{int} \in Int$ and $L_{neg} \in Neg$. It was proved that for any $L_1 \in Int$ and $L_2 \in Neg$, the class $Spec(L_1, L_2)$ consisting of all logics having L_1 and L_2 as their intuitionistic and negative counterparts, respectively, forms an interval in the lattice of logics Jhn. Moreover, intervals of the form $Spec(L_1, L_2)$ are mutually disjoined for different pairs of logics L_1 and L_2. It was not noted explicitly in [4], but it can be easily proved that the mapping $L \mapsto (L_{int}, L_{neg})$ defines a lattice homomorphism from Par onto the direct product of lattices Int and Neg. In this way, the studying of the class Par is reduced to the studying of intervals of the form $Spec(L_1, L_2)$, where $L_1 \in Int$ and $L_2 \in Neg$. We may remark also that the class Par has, in a sense, the three-dimension structure. More exactly, there are three varieties of possibilities defining a position of a logic in the class Par. We can consider logics L_{int} and L_{neg} as first and second coordinates of $L \in Par$. The third coordinate of L is its position inside the interval $Spec(L_{int}, L_{neg})$. Its is not clear yet which parameters determine this third coordinate. In this article we consider the representation of j-algebras (see also [5]), which throws light on this problem. To demonstrate that the suggested semantics for paraconsistent extensions of Lj is effective, we use it to characterize the logics considered earlier by K. Segerberg [8]. It turns out, that some of Segerberg's axioms have a mixed character, they impose non-trivial restrictions on negative counterpart of a logic L possessing such axiom and, simultaneously, on the position of L inside the interval $Spec(L_{int}, L_{neg})$. We show how to separate one restriction from the other and obtain the complete picture of interrelations between properties involved in Segerberg's axioms.

2. *Preliminaries*

In the present work, we consider logics in the propositional language $\{\wedge, \vee, \supset, \bot\}$, the negation is assumed to be an abbreviation, $\neg\varphi \rightleftharpoons \varphi \supset \bot$. As usually, by a *logic* we mean a set of formulas closed under substitution and *modus ponens*.

We adopt the following notation for propositional logics:

- Lj is the minimal or Johansson logic;
- Le is the Curry logic of classical refutability;
- Li is the intuitionistic logic;
- Lk is the classical logic;
- Ln is the negative logic;
- Lmn is the maximal negative logic;
- $\mathcal{F}$ is the trivial logic, i.e., the set of all formulas.

We recall the interrelations between the logics listed above. Intuitionistic and negative logics can be axiomatized modulo Lj as follows

$$Li = Lj + \{\bot \supset p\}, \;\; Ln = Lj + \{\bot\}.$$

The logics Lj, Li, Ln have common positive fragment equalling the positive logic. The logics Le, Lk, Lmn also have the same positive fragment, the classical positive logic, and they can be axiomatized modulo Lj, Li, and Ln, respectively, by adjoining one of the following two axioms:

$P.$ $((p \supset q) \supset p) \supset p$ (The Peirce law.)

$E.$ $p \vee (p \supset q)$ (The generalized law of excluded middle.)

As was mentioned in the introduction the class Jhn of extensions of minimal logic equals the disjoined union of three subclasses,

$$Jhn = Int \cup Neg \cup Par.$$

It turns out that logics of these subclasses form intervals in Jhn considered as a lattice of logics, and six logics defined above are the end-points of these intervals.

Proposition 2.1: [3] For any logic $L \in Par$ the following equivalences hold:

(1) $L \in Int \iff Li \subseteq L \subseteq Lk$;
(2) $L \in Neg \iff Ln \subseteq L \subseteq Lmn$;
(3) $L \in Par \iff Lj \subseteq L \subseteq Le$.

Below we give a few definitions and facts concerning the algebraic semantics for propositional logics. The detailed information can be found in [6, 7].

Let A be an algebra of the language $\{\vee, \wedge, \supset, \bot, 1\}$ ($\{\vee, \wedge, \supset, 1\}$) with an additional constant 1 for the only distinguished element. An arbitrary map $V : \{p_0, p_1, \ldots\} \to$ A from the set of propositional variables to the universe of A is called an A-*valuation.* Each A-valuation extends naturally to the set of all formulae. A formula φ is *true* in A, or is an *identity* of A, and we write $A \models \varphi$, if $V(\varphi) = 1$ for any A-valuation V.

Obviously, the set $LA = \{\varphi | A \models \varphi\}$ of formulae is a logic, which we call a *logic of* A. A *logic of the class of algebras* $\mathcal{K}$ is the intersection of logics of algebras in $\mathcal{K}$,

$$L\mathcal{K} = \bigcap \{LA | A \in \mathcal{K}\}.$$

The algebra A is a *model for* a logic L if $L \subseteq LA$. If also $L = LA$, we say that A is a *characteristic model for* L. Every logic in Jhn has a characteristic model ([7, Ch. III, Sec. 3]).

By a j-*algebra* we mean an algebra $A = \langle \mathrm{A}, \wedge, \vee, \supset, \bot, 1\rangle$ such that the algebra $\langle \mathrm{A}, \wedge, \vee, \supset, 1\rangle$ is an implicative lattice and the constant $\bot$ is interpreted as an arbitrary element of the universe A.

A *Heyting algebra* is a j-algebra with the least element $\bot$. A *negative algebra* is a j-algebra with the greatest element $\bot$, i.e., $\bot = 1$.

We call a *Peirce algebra* an implicative lattice satisfying the identity P (or, equivalently, E). A *Peirce-Johansson algebra* or, shortly, pj-*algebra* (*negative Peirce algebra*, *Boolean algebra*) is a j-algebra (respectively, negative algebra, Heyting algebra) satisfying the identity P.

All classes of algebras defined above form varieties, which define the following logics.

- Lj is the logic of the variety of j-algebras.
- Li is the logic of the variety of Heyting algebras.
- Ln is the logic of the variety of negative j-algebras.

- Le is the logic of the variety of pj-algebras.
- Lk is the logic of the variety of Boolean algebras.
- Lmn is the logic of the variety of negative Peirce algebras.

For a j-algebra $A = \langle \mathrm{A}, \vee, \wedge, \supset, \bot, 1 \rangle$, we put $\mathrm{A}^{\bot} \rightleftharpoons \{b \in \mathrm{A} \mid b \geq \bot\}$ and $\mathrm{A}_{\bot} \rightleftharpoons \{b \in \mathrm{A} \mid b \leq \bot\}$. The set $\mathrm{A}^{\bot}$ is obviously closed under the operations of A, and we can define a j-subalgebra $A^{\bot}$ of A with the universe $\mathrm{A}^{\bot}$. Obviously, $A^{\bot}$ is a Heyting algebra. Except for the case $\bot = 1$, the set $\mathrm{A}_{\bot}$ forms a sublattice, but not a j-subalgebra of A, because $\mathrm{A}_{\bot}$ is not closed under the implication. However, the operation $x \supset_{\bot} y \rightleftharpoons (x \supset y) \wedge \bot$ turns $\mathrm{A}_{\bot}$ into a j-algebra with unit element $\bot$, i.e., into a negative algebra. Denote this negative algebra by $A_{\bot}$.

We will call the just defined Heyting algebra $A^{\bot}$ and negative algebra $A_{\bot}$ the *upper* and *lower algebras associated* with a j-algebra A.

For arbitrary j-algebra A and Heyting algebra B such that they have disjoined universes we will denote by $A \oplus B$ a j-algebra obtained by identifying the greatest element of A and the least element of B. Of course, the contradiction of A will play the role of $\bot$ in the resulting algebra $A \oplus B$.

Now we associate with an arbitrary extension L of minimal logic a pair of logics one of which is intermediate and the other is negative. These logics will be called intuitionistic and negative counterparts of L. It will be done so that upper (lower) algebras of models of L will be models of intuitionistic (negative) counterpart of L.

We define the following translation. If $\varphi(p_0, \ldots, p_n)$ is a formula in propositional variables $p_0, \ldots, p_n$, then $I(\varphi) \rightleftharpoons \varphi(p_0 \vee \bot, \ldots, p_n \vee \bot)$.

For L extending Lj, we define

$$L_{int} \rightleftharpoons \{\varphi \mid L \vdash I(\varphi)\},\ L_{neg} \rightleftharpoons \{\varphi \mid L \vdash \bot \supset \varphi\}.$$

It can be easily seen that L_{int} and L_{neg} are logics. We call L_{int} and L_{neg} *intuitionistic* and *negative counterparts* of the logic L, respectively.

In the following proposition we list some simple properties of the notions introduced above.

Proposition 2.2: [4]

(1) *For any* $L \in Par$, *we have* $L_{int} \in Int$, $L_{neg} \in Neg$, *and the following equalities hold*

$$L_{int} = L + \{\bot \supset p\} \text{ and } L_{neg} = L + \{\bot\}.$$

(2) $L \in Int$ *if and only if* $L \neq \mathcal{F}$, $L = L_{int}$, *and* $L_{neg} = \mathcal{F}$.
(3) $L \in Neg$ *if and only if* $L \neq \mathcal{F}$, $L = L_{neg}$, *and* $L_{int} = \mathcal{F}$.
(4) *If* $Lj \subseteq L^1 \subseteq L^2$, *then* $L^1_{int} \subseteq L^2_{int}$ *and* $L^1_{neg} \subseteq L^2_{neg}$.
(5) *If* $L \subseteq L_1 \in Int$, *then* $L_{int} \subseteq L_1$.

(6) *If* $L \subseteq L_1 \in Neg$, *then* $L_{neg} \subseteq L_1$.

We can see, in particular, that L_{int} is the least intermediate logic containing L, and L_{neg} is the least negative logic with the same property.

Proposition 2.3: [4] *Let* $L \in Jhn$. *The following assertions are true.*

(1) *If* A *is a model for* L, *then* $A^{\perp} \models L_{int}$ *and* $A_{\perp} \models L_{neg}$.

(2) *If* A *is a characteristic model for* L, *then* $LA^{\perp} = L_{int}$ *and* $LA_{\perp} = L_{neg}$.

Further, consider the classes of logics with the given intuitionistic and negative counterparts. For $L_1 \in Int$ and $L_2 \in Neg$, we define

$$Spec(L_1, L_2) \rightleftharpoons \{L \supseteq Lj \mid L_{int} = L_1,\ L_{neg} = L_2\}.$$

We define also the logic

$$L_1 * L_2 \rightleftharpoons Lj + \{I(\varphi),\ \bot \supset \psi \mid \varphi \in L_1,\ \psi \in L_2\},$$

which we will call a *free combination* of L_1 and L_2.

Proposition 2.4: [4] *Let* $L_1 \in Int$ *and* $L_2 \in Neg$. *Then*

$$Spec(L_1, L_2) = [L_1 * L_2, L_1 \cap L_2].$$

The next proposition allows to write axioms for $L_1 * L_2$ relative to Lj given the axiomatics of L_1 relative to Li and of L_2 relative to Ln.

Proposition 2.5: [4] *Let* $L_1 \in Int$, $L_1 = Li + \{\varphi_i \mid i \in I\}$ *and* $L_2 \in Neg$, $L_2 = Ln + \{\psi_j \mid j \in J\}$. *Then*

$$L_1 * L_2 = Lj + \{\ I(\varphi_i),\ \bot \supset \psi_j \mid i \in I,\ j \in J\ \}.$$

A semantical characterization of a free combination of logics is given in the following

Proposition 2.6: [4] *Let* $L_1 \in Int$, $L_2 \in Neg$, *and let* A *be an arbitrary* j*-algebra. Then* $A \models L_1 * L_2$ *if and only if* $A^{\perp} \models L_1$ *and* $A_{\perp} \models L_2$.

As we can see from Proposition 2.4 the class of Lj-extensions decomposes into a union of disjoint intervals

$$Jhn = \bigcup\{Spec(L_1, L_2) \mid L_1 \in Int,\ L_2 \in Neg\}.$$

In this way, the investigation of the class of Lj-extensions is reduced to the problem what is the structure of the interval $Spec(L_1, L_2)$ for the given intermediate logic L_1 and negative logic L_2.

Further, note that upper points of intervals $Spec(L_1, L_2)$ also form an interval in the lattice Jhn, namely, $[Le', Le]$, where Le' is the logic axiomatized modulo Lj via one of the two following axioms

E'. $\bot \vee (\bot \supset p)$.

D'. $(p \vee \bot \supset q \vee \bot) \supset ((p \supset q) \vee \bot)$.

Proposition 2.7: [4] *Let A be a j-algebra. A is a model for Le' if and only if one of the following equivalent conditions holds.*

1. *The mapping $\varepsilon(x) = x \vee \bot$ defines an epimorphism of j-algebra A onto Heyting algebra $A^{\bot}$.*
2. *The mapping $\lambda(x) = (x \vee \bot, x \wedge \bot)$ determines an isomorphism of j-algebras A and $A^{\bot} \times A_{\bot}$.*

Corollary 2.8: [4] *Let $L \in Jhn$. Then $Le' \subseteq L \subseteq Le$ if and only if $L = L_1 \cap L_2$, where $L_1 \in Int$ and $L_2 \in Neg$.*

Of course, if under the assumptions of the last corollary we have $L = L_1 \cap L_2$, then $L_1 = L_{int}$ and $L_2 = L_{neg}$.

Consider the following substitutional instance of the Peirce law:

P'. $((\bot \supset p) \supset \bot) \supset \bot = \neg\neg(\bot \supset p)$.

We call the logic $Lg \rightleftharpoons Lj + \{P'\}$ *Glivenko's logic*. In [8, p. 46], it was mentioned that Glivenko's logic is the weakest one in which $\neg\neg\varphi$ is derivable whenever φ is derivable in classical logic. Unfortunately, this work contains neither the proof of this assertion, nor any further reference. In [4], we gave a proof of this statement based on the following characterization of models for Lg.

For a Heyting algebra A, we denote by ∇_A its filter of dense elements and by $\mathcal{R}(A)$ the Boolean algebra of its regular elements. Recall that $\nabla_A = \{a \in A \mid \neg\neg a = 1\}$, $\mathcal{R}(A) = \{a \in A \mid a \vee \neg a = 1\}$, and that the Boolean algebra of regular elements is isomorphic to quotient algebra of A with respect to the filter of dense elements, $\mathcal{R}(A) \cong A/\nabla_A$.

Proposition 2.9: [4] *Let A be a j-algebra. Then A is a model for Lg if and only if $\bot \vee (\bot \supset a) \in \nabla_{A^{\bot}}$ for any $a \in A$.*

The logic Lg provides an example of a logic different from the end-points of the interval $Spec(Lg_{int}, Lg_{neg}) = Spec(Li, Ln)$.

In conclusion of this section we say a few words about the Kripke style semantics for minimal logic and its extensions. More detailed information can be found in [8].

We call *Kripke j-frame*, or simply *j-frame*, a triple $\mathcal{W} = \langle W, \sqsubseteq, Q\rangle$, where W is a set of possible worlds, $\sqsubseteq$ is an accessibility relation such that $\langle W, \sqsubseteq\rangle$ is an ordinary Kripke frame for intuitionistic logic, i.e., a partially ordered set, and $Q \subseteq W$ is a cone with respect to $\sqsubseteq$, which we will call the cone of *innormal worlds*. Worlds lying out of Q are called *normal*. As usual, *valuation* v of a j-frame $\mathcal{W}$ is a mapping from the set of propositional variables to the set of cones of the ordering $\langle W, \sqsubseteq\rangle$. *Model* $\mu = \langle \mathcal{W}, v\rangle$ is a pair consisting of a j-frame and its valuation. We say also in this case that μ is a model on $\mathcal{W}$.

For a model $\mu = \langle \mathcal{W}, v\rangle$, where $\mathcal{W} = \langle W, \sqsubseteq, Q\rangle$, an element $x \in W$, and a formula φ, we define the relation $\mu \models_x \varphi$ by induction on the structure of formulas in a way similar to that for ordinary Kripke frames for intuitionistic logic. The only exception is the case of the constant $\bot$:

$$\mu \models_x \bot \quad\Leftrightarrow\quad x \in Q.$$

We will read $\mu \models_x \varphi$ as "a formula φ is true at a world (or at a point) x in a model μ". As usually, we say that a formula φ is *true on a model* $\mu = \langle \mathcal{W}, v\rangle$, $\mu \models \varphi$, if for all $x \in W$ the relation $\mu \models_x \varphi$ holds. A formula φ is *true on a j-frame* $\mathcal{W}$, $\mathcal{W} \models \varphi$, if it is true on a model $\langle \mathcal{W}, v\rangle$ for an arbitrary valuation v of the j-frame $\mathcal{W}$. A formula φ is *valid on the class* $\mathcal{K}$ of Kripke j-frames if $\mathcal{W} \models \varphi$ for any j-frame $\mathcal{W} \in \mathcal{K}$.

We say that a j-frame $\mathcal{W}$ is a *model for* a logic $L \in Jhn$, $\mathcal{W} \models L$, if $\mathcal{W} \models \varphi$ for all $\varphi \in L$. For a class of j-frames $\mathcal{K}$ we put $L\mathcal{K} \rightleftharpoons \{\varphi \mid \forall \mathcal{W} \in \mathcal{K}\ (\mathcal{W} \models \varphi)\}$.

We say that $L \in Jhn$ is *characterized* by the class of j-frames $\mathcal{K}$ if $L = L\mathcal{K}$.

We will call a j-frame $\mathcal{W} = \langle W, \sqsubseteq, Q\rangle$ *normal* if $Q = \emptyset$, i.e., if all worlds of this frame are normal. It is clear that normal j-frames can be identified with ordinary Kripke frames for intuitionistic logic. We call a j-frame $\mathcal{W} = \langle W, \sqsubseteq, Q\rangle$ *innormal* if $Q = W$, i.e. if all worlds are innormal. Finally, a j-frame $\mathcal{W} = \langle W, \sqsubseteq, Q\rangle$ will be called identical if the accessibility relation $\sqsubseteq$ coincides with the identity relation on W, $\sqsubseteq = id_W$. Let Nor denote the class of all normal j-frames, Inn the class of all innormal j-frames, and Id the class of all identical frames.

We define also the following classes of j-frames. Let $\mathcal{W} = \langle W, \sqsubseteq, Q\rangle$ be a j-frame. We say that $\mathcal{W}$ is *separated* if

$$\forall x, y \in W((x \notin Q \wedge y \in Q) \Rightarrow x \sqsubseteq y.$$

And we say that $\mathcal{W}$ is *closed* if

$$\forall x, y \in W((x \notin Q \wedge y \in Q) \Rightarrow \neg(x \sqsubseteq y).$$

Denote by Sep the class of all separated j-frames and by Cl the class of all closed j-frames.

The classes of j-frames defined above characterize the following logics.

Proposition 2.10: [8]

(1) *The logic Lj is characterized by the class of all j-frames.*
(2) *The logic Li is characterized by the class Nor.*
(3) *The logic Ln is characterized by the class Inn.*
(4) *The logic Le is characterized by the class Id.*
(5) *The logic Lk is characterized by the class $Nor \cap Id$.*
(6) *The logic Lmn is characterized by the class $Inn \cap Id$.*
(7) *The logic $Ljl' \rightleftharpoons Lj + \{(p \supset \bot) \vee (\bot \supset p)\}$ is characterized by the class Sep.*
(8) *The logic Le' is characterized by the class Cl.*

The given below characterization of Glivenko's logic was obtained by P. Woodruff [9] and used later by R. Goldblatt [2] to prove the decidability of Lg.

A j-frame $\mathcal{W} = \langle W, \sqsubseteq, Q \rangle$ is called *dense* if

$$\forall x \in W(x \notin Q \Rightarrow \exists y \sqsupseteq x \forall z \sqsupseteq y(z \notin Q)).$$

The class of all dense j-frames will be denoted by Den.

Proposition 2.11: [9] *The logic Lg is characterized by the class Den.*

3. *Representation of j-algebras*

In this section we give a convenient representation of j-algebras, which allow to describe classes of models for logics different from the end-points of intervals of the form $[L_1 * L_2, L_1 \cap L_2]$, where $L_1 \in Int$ and $L_2 \in Neg$. We can see from the results cited in the previous section that an intersection $L_1 \cap L_2$ of intermediate and negative logics is characterized by the class of all direct products of the form $A \times B$, where A is a Heyting algebra being a model for the logic L_1 and B is a negative algebra modelling L_2. Indeed, according to Corollary 2.8 the intersection $L_1 \cap L_2$ extends the logic Le', in which case any model A of $L_1 \cap L_2$ is isomorphic to the direct product $A^{\perp} \times A_{\perp}$ by Proposition 2.7. It remains to note that $L_1 = (L_1 \cap L_2)_{int}$ and $L_2 = (L_1 \cap L_2)_{neg}$ in view of Proposition 2.4, and so by Proposition 2.3, we have $A^{\perp} \models L_1$ and $A_{\perp} \models L_2$.

At the same time, the free combination of logics, $L_1 * L_2$, is characterized by the class of all j-algebras A such that the upper algebra $A^{\perp}$ is a model for the logic L_1 and the lower algebra $A_\perp$ models L_2 (see Proposition 2.6). At this point the following question arises. If a Heyting algebra B and negative algebra C are given, what is the difference between an arbitrary j-algebra A with the condition $A^\perp \cong B$ and $A_\perp \cong C$ and the direct product of algebras $B \times C$? Proposition 2.9 allows us to assume that the elements of the form $\perp \vee (\perp \supset a)$, where $a \in A_\perp$ will play a special role in the structure of a j-algebra A.

Proposition 3.1: *Let A be an arbitrary j-algebra and the mapping $f_A : A_\perp \to A^\perp$ is given by the rule $f_A(x) = \perp \vee (\perp \supset x)$. Then the following two conditions are met.*

1. *The mapping $f_A : A_\perp \to A^\perp$ is a semilattice homomorphism preserving the meet $\wedge$ and the greatest element, $f_A(\perp) = 1$.*
2. *The image of the embedding $\lambda_\perp : A \to A^\perp \times A_\perp$, where $\lambda_\perp(x) = (x \vee \perp, x \wedge \perp)$, is the following*

$$\lambda_\perp(A) = \{(x, y) \mid x \leq f_A(y),\ x \in A^\perp,\ y \in A_\perp\}.$$

Proof. 1. For brevity, we omit the lower index in the denotation f_A. We have $f(\perp) = \perp \vee (\perp \supset \perp) = 1$. Further,

$$f(y_1) \wedge f(y_2) = (\perp \vee (\perp \supset y_1)) \wedge (\perp \vee (\perp \supset y_2)) =$$
$$= \perp \vee ((\perp \supset y_1) \wedge (\perp \supset y_2)) = \perp \vee (\perp \supset y_1 \wedge y_2) = f(y_1 \wedge y_2).$$

We have thus verified that f is a semilattice homomorphism preserving the unit element.

2. If $a \in A$, then $(a \vee \perp, a \wedge \perp) \in \lambda_\perp(A)$ and the inequality $a \vee \perp \leq \perp \vee (\perp \supset a \wedge \perp)$ holds. The latter can be checked, for example, by proving in Lj the formula $p \vee \perp \supset \perp \vee (\perp \supset p \wedge \perp)$. Thus, the inclusion

$$\lambda_\perp(A) \subseteq \{(x, y) \mid x \leq f(y),\ x \in A^\perp,\ y \in A_\perp\}$$

is proved. Now, let $x, y \in A$, $x \geq \perp$, $y \leq \perp$, and $x \leq \perp \vee (\perp \supset y)$. We show that there exists an element $a \in A$ such that $x = a \vee \perp$ and $y = a \wedge \perp$. Put $a = x \wedge (\perp \supset y)$, then $a \vee \perp = (\perp \vee x) \wedge (\perp \vee (\perp \supset y)) = x \wedge (\perp \vee (\perp \supset y)) = x$ and also $a \wedge \perp = x \wedge (\perp \supset y) \wedge \perp = \perp \wedge (\perp \supset y) = y$. The inverse inclusion is also checked.

The proposition is proved.

As we can see from the above proposition, every j-algebra A determine a triple $(A^\perp, A_\perp, f_A)$ consisting of Heyting algebra, negative algebra, and semilattice homomorphism. Now, let us take a triple $(B, C, f : C \to B)$, where B is an arbitrary Heyting algebra, C an arbitrary negative algebra, and f an arbitrary semilattice homomorphism from C to B preserving the meet and the greatest element. Starting from this triple we try to construct a j-algebra A, the upper and lower algebras of which are isomorphic to B

and C, respectively, and the mapping f_A is induced in a natural way by the homomorphism f.

Define a lattice $B \times_f C$ as a sublattice of the direct product $B \times C$ with the universe

$$|B \times_f C| \rightleftharpoons \{(x, y) \mid x \in B,\ y \in C,\ x \leq f(y)\}.$$

This is really a sublattice, because the mapping f preserves the meet and, hence, the ordering, which easily imply the relation $f(y_1 \vee y_2) \leq f(y_1) \vee f(y_2)$. From the latter immediately follows that the set $|B \times_f C|$ is closed under componentwise lattice operations on the direct product of the lattices. As we can see from the proposition below, this lattice can be considered as a j-algebra.

Proposition 3.2: *Let B, C, f, and $A \rightleftharpoons B \times_f C$ be as above. The lattice A has a natural structure of a j-algebra, where the pseudocomplement operation is given by the rule*

$$(x_1, y_1) \supset (x_2, y_2) = ((x_1 \supset x_2) \wedge f(y_1 \supset y_2),\ y_1 \supset y_2),$$

whereas the unit element and the contradiction in A satisfy the following relations: $1_A = (1_B, \bot_C)$ and $\bot_A = (\bot_B, \bot_C)$. Moreover, the following holds: $B \cong A^{\bot}$, $C \cong A_{\bot}$, and these isomorphisms are given by the rules $x \mapsto (x, \bot_C)$, $x \in B$, and $y \mapsto (\bot_B, y)$, $y \in C$, respectively. Finally, for all $y \in C$, we have $(f(y), \bot_C) = \bot_A \vee (\bot_A \supset (\bot_B, y)) = f_A((\bot_B, y))$.

Proof. First, we check that the pseudocomplement operation is well-defined. Let $b_1, b_2 \in B$, $c_1, c_2 \in C$, $b_1 \leq f(c_1)$, and $b_2 \leq f(c_2)$. The element $(b_1, c_1) \supset (b_2, c_2)$, if it is defined, must be greatest among the elements $(x, y),\ x \leq f(y)$, such that $(b_1, c_1) \wedge (x, y) \leq (b_2, c_2)$. This is equivalent to the relations $x \leq (b_1 \supset b_2) \wedge f(y)$ and $y \leq c_1 \supset c_2$. Taking into account that f preserves the ordering we immediately obtain that the element $((b_1 \supset b_2) \wedge f(c_1 \supset c_2),\ c_1 \supset c_2)$ is the desired pseudocomplement.

All other relations, except the last, are trivial. Check the last relation using the obtained formula for pseudocomplement. We have

$$\bot_A \vee (\bot_A \supset (\bot_B, y)) = (\bot_B, \bot_C) \vee ((\bot_B, \bot_C) \supset (\bot_B, y)) =$$
$$(\bot_B, \bot_C) \vee (1_B \wedge f(\bot_C \supset y), \bot_C \supset y) =$$
$$(\bot_B, \bot_C) \vee (f(y), y) = (f(y), \bot_C).$$

The proof is complete.

As we can see from the above considerations, to define the class of j-algebras characterizing some extension L of minimal logic we must choose a class of Heyting algebras and a class of negative algebras isomorphic, respectively, to upper and lower algebras associated with models of the logic L. In this way we fix intuitionistic and negative counterparts of the logic L.

Moreover, to determine the place of L inside the interval $[L_{int} * L_{neg}, L_{int} \cap L_{neg}]$, we must distinguish in one or another way the class of admissible homomorphisms from negative algebras into Heyting ones. In no restrictions are imposed on the class of homomorphisms, we obtain the free combination of intermediate and negative logics characterized by selected classes of Heyting and negative algebras (see Proposition 2.6). If we admit only homomorphisms identically equal to the unit element, we obtain the intersection $L_{int} \cap L_{neg}$. Indeed, a j-algebra $A \times_f B$ coincides with the direct product $A \times B$ if and only if $f(y) = 1$ for all $y \in B$.

It is interesting to consider logics different from intersections and free combinations of intermediate and negative logics, i.e. logics lying inside intervals of the form $Spec(L_1, L_2)$. Numerous examples of such logics will be treated in the next section.

4. *Segerberg's logics and their semantics*

In this subsection using the representation for j-algebras obtained above we describe the algebraic semantics for logics studied previously by K. Segerberg [8], who characterized these logics in terms of Kripke semantics. Except for the logic Lj itself K. Segerberg [8] considered logics obtained by adding to Lj one or several axioms from the list below.

$I.$ $\bot \supset p$
$K.$ $\neg p \vee \neg\neg p$
$X.$ $p \vee \neg p$
$L.$ $(p \supset q) \vee (q \supset p)$
$E.$ $p \vee (p \supset q)$
$L'.$ $\neg p \vee (\bot \supset p) = (p \supset \bot) \vee (\bot \supset p)$
$E'.$ $\bot \vee (\bot \supset p)$
$Q.$ $\bot$
$L_N.$ $(p \supset q \vee \bot) \vee (q \supset p \vee \bot)$
$L^{Q_1}.$ $\bot \supset (p \supset q) \vee (q \supset p)$
$L^{Q_2}.$ $(\bot \supset (p \supset q)) \vee (\bot \supset (q \supset p))$
$E^{Q_1}.$ $\bot \supset p \vee (p \supset q)$
$E^{Q_2}.$ $(\bot \supset p) \vee (\bot \supset (p \supset q))$
$P'.$ $\neg\neg(\bot \supset p)$

We may combine these axioms, which gives rise to the big number of new logics. Some of these logics have traditional denotation, for example $Li = Lj + \{I\}$, and others have not. Due to this fact we need some notational convention.

If some logic is obtained from the logic already having a denotation, say L, by adding some axiom denoted by a capital letter, say X, then the denotation of this new logic will be obtained by joining the corresponding small letter to the existing denotation, $Lx \rightleftharpoons L + \{X\}$. Of course, in this way one logic may obtain different denotations. According to this convention we have, for example, $Lji = Li$, $Lje = Le$, $Ljq = Ln$, $Lix = Ljix = Lk$, and finally, $Ljp' = Lg$.

We say a few words about the way of arising axioms from the above list. Kripke semantics for extensions of Lj was described in Section 2. Recall that any j-frame is divided into two parts consisting of normal worlds and of innormal worlds, respectively. The first axiom I distinguishes the class of j-frames in which all worlds are normal. The next two axioms are well-known the law of excluded middle X and the weak law of excluded middle K. These axioms impose some restrictions on the accessibility relation only in the normal part of a j-frame. It must be identical in the case of X and directed in the case of K. The Dummett linearity axiom L and the generalized law of excluded middle E define properties of accessibility relation in the whole frame. A j-frame satisfying L is linear, whereas in a j-frame satisfying E the accessibility relation is identical. The next two axioms, L' and E', are partial cases of L and E, respectively. They do not impose any restrictions on either normal or innormal part of a j-frame, but they define the way, in which the cone of innormal worlds is situated in the whole frame (see Proposition 2.10).

The interrelations between logics obtained by joining to Lj one or several from the axioms reviewed up to this moment are presented in Figure 1.

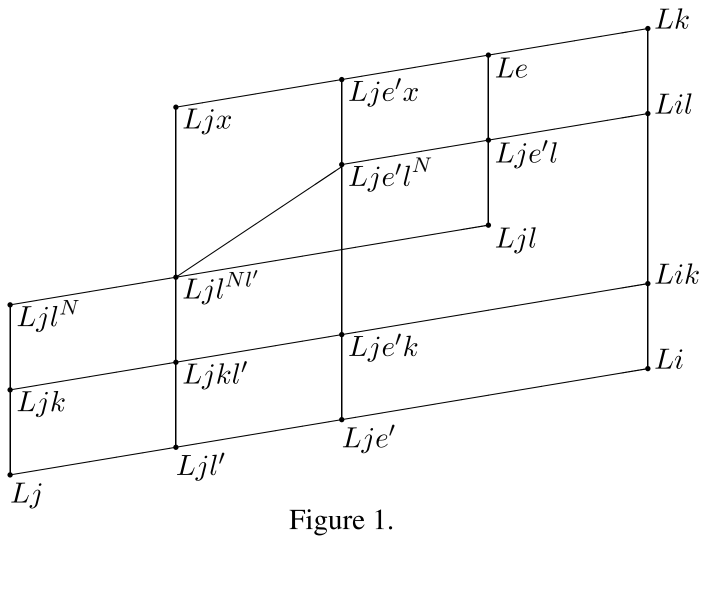


Figure 1.

Note that this diagram (as well as the diagram presented in Figure 2 below) respects only an ordering but not the lattice structure of Jhn. All logics presented at the diagram are distinct, and a logic L_1 is contained in a logic L_2 if and only if there is a path leading from L_1 to L_2 which at every point is either rising or horizontal and directed to the right.

To explain the explicit irregularities of the above diagram K. Segerberg put into consideration some new axioms, which are not so natural as axioms considered up to this moment. "But as long as we cannot account for the irregularities in the above diagram, we cannot claim to understand the situation fully" [8, p. 41].

As we can see from the above, the axiom X can be considered as a relativization of the axiom E to the normal part of a j-frame. Indeed, the axiom E imposes restriction to be identical on the accessibility relation, whereas X imposes essentially the same restriction "to be identical" but on the accessibility relation restricted to the normal part of a j-frame. The next six axioms in the list are the axiom Q distinguishing the class of innormal j-frames and relativizations of axioms E and L to the normal or to the innormal part of a j-frame. The axiom L_N is a restriction of L to normal worlds. The axioms L_1^Q and L_2^Q are variants of relativization of L to the innormal part of a j-frame. Relativizing E to innormal words K. Segerberg also suggests two variants, E_1^Q and E_2^Q.

The last axiom in the list, P' is similar to E' and L' because it restricts only the way of combination of normal and innormal parts of a j-frame (Proposition 2.11). This axiom, as well as axioms L_1^Q and E_1^Q lie out of the main line of considerations of [8].

If we exclude from the above list axioms P', L_1^Q, and E_1^Q, the logics which can be constructed via adjoining to Lj the other axioms from the list form the beautiful diagram presented at Figure 2. The logics of Figure 1 are depicted at this diagram by colored circles. The way in which these logics are situated at Figure 2 explains the irregularities of the previous diagram. Only several points on the diagram are endowed with the name of the corresponding logics. The other logics are obtained via a combination of axioms of explicitly designated logics and one can easily reconstruct which logic corresponds to one or another point on the diagram. For example, the non-designated logics lying on the horizontal line ended with Lik are the following: $Ljke'$, $Ljke'l_2^Q$, $Ljke'e_2^Q(= Le)$ (from left to right). We note also the equality $Ljl = Ljl'l^Nl_2^Q$. As we will see the equality $Ljl = Ljl'l^Nl_1^Q$ does not hold. So using axiom L_1^Q instead of L_2^Q results in a diagram of logics, which is not so regular as that of Figure 2. This explains the choice of K. Segerberg between variants of relativization of the axiom L to innormal worlds.

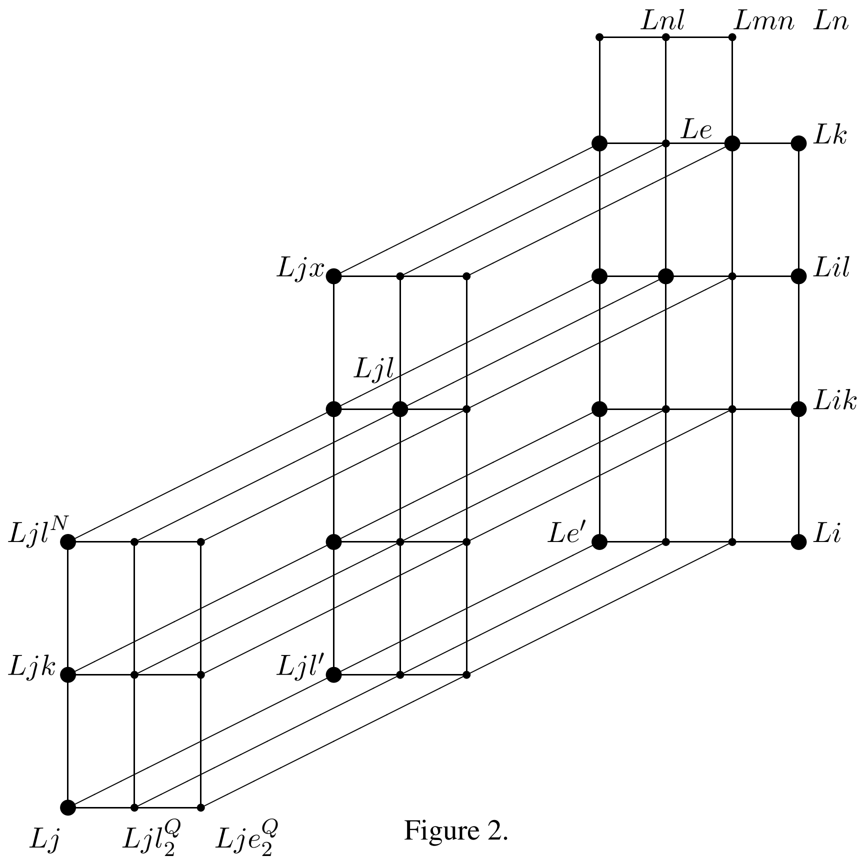


Figure 2.

At this diagram there are only four intermediate logics, namely, the logics lying on the vertical line from Li to Lk. The three negative logics on the diagram are those lying on the horizontal line from Ln to Lmn. All the other logics on the diagram belong to the class Par. They form a three-dimension figure, dimensions of which, as we can see later, correspond to the three parameters, which determine a position of a paraconsistent logic in the class Par.

Now we turn to the algebraic semantics of Segerberg's logics, the studying of which allow to make clear the above mentioned correspondence between the three dimensions of the diagram and the structure of the class Par.

Recall that a *Stoune* algebra is a Heyting algebra satisfying the identity K. Let A be a Heyting (negative) algebra. We call A a *Heyting (negative) l-algebra* if $A \models (p \supset q) \vee (q \supset p)$.

Proposition 4.1: *Let A be an arbitrary j-algebra. The following equivalences hold.*

(1) $A \models Ljk$ *if and only if* $A^{\perp}$ *is a Stone algebra.*
(2) $A \models Ljx$ *if and only if* $A^{\perp}$ *is a Boolean algebra.*
(3) $A \models Ljl'$ *if and only if* $f_A(A_{\perp}) \subseteq \mathcal{R}(A^{\perp})$.
(4) $A \models Ljl$ *if and only if* $A^{\perp}$ *and* $A_{\perp}$ *are* l*-algebras,* $f_A(A_{\perp}) \subseteq \mathcal{R}(A^{\perp})$*, and for all* $y_1, y_2 \in A_{\perp}$ *we have* $f_A(y_1 \supset y_2) \vee f_A(y_2 \supset y_1) = 1$.
(5) $A \models Lg$ *if and only if* $f_A(A_{\perp}) \subseteq \nabla(A^{\perp})$.
(6) $A \models Ljl^N$ *if and only if* $A^{\perp}$ *is an* l*-algebra.*
(7) $A \models Ljl_1^Q$ *if and only if* $A_{\perp}$ *is an* l*-algebra.*
(8) $A \models Ljl_2^Q$ *if and only if* $A_{\perp}$ *is an* l*-algebra and for all* $y_1, y_2 \in A_{\perp}$ *we have* $f_A(y_1 \supset y_2) \vee f_A(y_2 \supset y_1) = 1$.
(9) $A \models Lje_1^Q$ *if and only if* $A_{\perp}$ *is a negative Peirce algebra.*
(10) $A \models Lje_2^Q$ *if and only if* $A_{\perp}$ *is a negative Peirce algebra and for all* $y_1, y_2 \in A_{\perp}$ *we have* $f_A(y_1) \vee f_A(y_1 \supset y_2) = 1$.

Proof. 1. Let $A \models Ljk$. We represent A in the form $A^{\perp} \times_{f_A} A_{\perp}$, take an arbitrary element $(x, y) \in A$ and compute

$$((x,y) \supset (\perp,\perp)) \vee (((x,y) \supset (\perp,\perp)) \supset (\perp,\perp)) = (x \supset \perp, y \supset \perp)\vee$$
$$\vee((x \supset \perp, y \supset \perp) \supset (\perp,\perp)) = (\neg x, \perp) \vee ((\neg x, \perp) \supset (\perp,\perp)) =$$
$$= (\neg x, \perp) \vee (\neg\neg x, \perp) = (\neg x \vee \neg\neg x, \perp) = (1, \perp).$$

The letter identity is satisfied if and only if the identity $\neg x \vee \neg\neg x = 1$ is true on $A^{\perp}$, i.e., if and only if $A^{\perp}$ is a Stoune algebra.

2. This item also can be proved via a direct computation.

3. Let $(x, y) \in A^{\perp} \times_{f_A} A_{\perp}$. The direct computation show

$$((x,y) \supset (\perp,\perp)) \vee ((\perp,\perp) \supset (x,y) = ((x \supset \perp) \vee f(y), \perp).$$

Hereafter we omit the lower index in the denotation of the mapping f_A if it does not lead to a confusion. As we can see, L' is an identity of A if and only if for all $x \in A^{\perp}$, $y \in A_{\perp}$, $x \leq f(y)$, the equality $(x \supset \perp) \vee f(y) = 1_{A^{\perp}}$ holds. In particular, we have $(f(y) \supset \perp) \vee f(y) = \neg f(y) \vee f(y) = 1$, i.e., each element of the form $f(y)$ is regular. The inverse implication immediately follows from the above and the fact that implication is descending with respect to the first argument. Indeed, if for some $y \in A_{\perp}$ we have $(f(y) \supset \perp) \vee f(y) = 1_{A^{\perp}}$, then for all $x \in A^{\perp}$, $x \leq f(y)$ we have also $(x \supset \perp) \vee f(y) = 1_{A^{\perp}}$.

4. Assume that $A \models Ljl$. In this case, the upper algebra $A^{\perp}$, as a subalgebra of A, is an l-algebra. The inclusion $f_A(A_{\perp}) \subseteq \mathcal{R}(A^{\perp})$ holds by item 3,

because L' is a substitutional instance of L. Further, recall that implication $\supset_\perp$ of $A_\perp$ is defined via implication $\supset$ of A as $x \supset_\perp y = (x \supset y) \wedge \perp$. Calculate

$$(x \supset_\perp y) \vee (y \supset_\perp x) = ((x \supset y) \wedge \perp) \vee ((y \supset x) \wedge \perp) =$$
$$((x \supset y) \vee (y \supset x)) \wedge \perp = 1 \wedge \perp = \perp.$$

Thus, $A_\perp$ is also an l-algebra. To check the last of the conditions listed in this item take arbitrary elements $y_1, y_2 \in A_\perp$ and represent them in the form $(\perp, y_1)$ and $(\perp, y_2)$. We have

$$(1, \perp) = ((\perp, y_1) \supset (\perp, y_2)) \vee ((\perp, y_2) \supset (\perp, y_1)) =$$
$$(f(y_1 \supset y_2) \vee f(y_2 \supset y_1), (y_1 \supset y_2) \vee (y_2 \supset y_1)),$$

in particular, $f(y_1 \supset y_2) \vee f(y_2 \supset y_1) = 1$.

Prove the inverse implication. Let $A^\perp$ and $A_\perp$ be l-algebras, and let $f_A(A_\perp) \subseteq \mathcal{R}(A^\perp)$ and for all $y_1, y_2 \in A_\perp$ we have $f_A(y_1 \supset y_2) \vee f_A(y_2 \supset y_1) = 1$. Take $(x_1, y_1), (x_2, y_2) \in A$ and using formula for implication calculate

$$((x_1, y_2) \supset (x_2, y_2)) \vee ((x_2, y_2) \supset (x_1, y_1)) = (h, (y_1 \supset y_2) \vee (y_2 \supset y_1))$$

Second component of the last pair equals $\perp = 1_{A_\perp}$, because $A_\perp$ is an l-algebra, whereas the first component has the following form:

$$h = ((x_1 \supset x_2) \vee (x_2 \supset x_1)) \wedge (f(y_1 \supset y_2) \vee (x_2 \supset x_1)) \wedge$$
$$((x_1 \supset x_2) \vee f(y_2 \supset y_1)) \wedge (f(y_1 \supset y_2) \vee f(y_2 \supset y_1)).$$

From our assumptions we immediately infer that first and last conjunctive terms of the last expression are equal to the unit element. In this way, we obtain that satisfiability of the identity L on A is equivalent to the condition:

$$\text{for all } (x_1, y_1), (x_2, y_2) \in A, (x_1 \supset x_2) \vee f(y_2 \supset y_1) = 1_{A^\perp}.$$

Taking into account the facts that implication is descending in the first argument and ascending in the second and that $x \leq f(y)$ for all $(x, y) \in A$, we obtain the chain of inequalities $(x_1 \supset x_2) \vee f(y_2 \supset y_1) \geq (x_1 \supset \perp) \vee f(\perp \supset y_1) \geq (f(y_1) \supset \perp) \vee f(y_1) = \neg f(y_1) \vee f(y_1) = 1$. The letter equality holds due to the condition that every element of the form $f(y)$ is regular.

Items 5–10 can be checked via a direct computation.

The proof is complete.

In the corollary below, the symbol $\cup^*$ denotes the join operation in the lattice of logics Jhn.

Corollary 4.2: (1) $Ljk = Lik * Ln$.
(2) $Ljx = Lk * Ln$.

(3) *For all* $L_1 \in Int$ *and* $L_2 \in Neg$, *the following equality holds*
$$(L_1 * L_2)p' \cup^* (L_1 * L_2)l' = L_1 \cap L_2.$$
In particular, $Le' = Lg \cup^* Ljl'$.

(4) *For every* $L_1 \in Int$ *and* $L_2 \in Neg$ *such that* $L_1 \neq Lk$, *the logics* $(L_1 * L_2)p'$ *and* $(L_1 * L_2)l'$ *are different from the endpoints of the interval* $Spec(L_1, L_2)$. *At the same time, if* $L_1 = Lk$, *we have* $(Lk * L_2)p' = Lk \cap L_2$ *and* $(Lk * L_2)l' = Lk * L_2$.

(5) $Ljl^N = Lil * Ln$.

(6) $Ljl_1^Q = Li * Lnl$.

(7) $Ljl_2^Q \in Spec(Li, Lnl)$, $Ljl_2^Q \neq Li * Lnl$, $Ljl_2^Q \neq Li \cap Lnl$.

(8) $Lje_1^Q = Li * Lmn$.

(9) $Lje_2^Q \in Spec(Li, Lmn)$, $Lje_2^Q \neq Li * Lmn$, $Lje_2^Q \neq Li \cap Lmn$.

(10) *The logic* Ljl *is a proper extension of* $(Lil * Lnl)l' = Ljl'l^N l_1^Q$.

Proof. Items 1,2,5,6,8 easily follow from Propositions 2.5 and 2.6 and suitable items of the last proposition.

3. By item 3 of Proposition 4.1, in models of the logic $(L_1 * L_2)l'$ all elements of the form $\perp \vee (\perp \supset a)$ are regular. On the other hand, in models of the logic $(L_1 * L_2)p'$ all elements of the indicated form are dense as it follows from item 5 of Proposition 4.1. Thus, in the models of the least upper bound of logics $(L_1 * L_2)p'$ and $(L_1 * L_2)l'$ elements of the form $\perp \vee (\perp \supset a)$ are regular and dense simultaneously, i.e., they all are equal to the unit element. Consequently, models of the considered least upper bound are exactly the direct products of the form $B \times C$, where $B \models L_1$ and $C \models L_2$, whence we immediately obtain the desired equality by Proposition 2.7. and Corollary 2.8.

4. The assertion of this item is true due to the fact that for any Heyting algebra A the following three conditions are equivalent: A is a Boolean algebra; the unit element is the only dense element of A; all elements of A are regular.

7. By item 8 of Proposition 4.1, the logic Ljl_2^Q belongs to the interval $Spec(Li, Lnl)$. Consider the model A of the free combination $Li * Lnl$ structured as follows. The upper algebra $A^\perp$ is arbitrary; the lower algebra $A_\perp$ is a 4-element negative Peirce algebra with universe $\{\perp, a, b, 0\}$, where $0 \leq a \leq \perp$, $0 \leq b \leq \perp$, and elements a and b are incomparable; $f(\perp) = 1$, $f(x) = \perp$ for $x \neq \perp$. Calculate
$$f(a \supset b) \vee f(b \supset a) = f(b) \vee f(a) = \perp \vee \perp = \perp,$$
which proves that Ljl_2^Q differs from the lower point of the interval $Spec(Li, Lnl)$.

We point out now a model for Ljl_2^Q different from the direct product of Heyting and negative algebras. This will prove that Ljl_2^Q does not equal to the intersection of logics Li and Ljl. Let B and C be Heyting and negative l-algebras, respectively, which are isomorphic as implicative lattices, and let $f : C \to B$ be an arbitrary lattice isomorphism. It is not hard to check that $B \times_f C$ is the desired model of Ljl_2^Q.

9. The fact that Lje_2^Q belongs to the interval $Spec(Li, Lmn)$, follows from item 10 of Proposition 4.1. Examples of j-algebras showing that Lje_2^Q differs from the endpoints of the indicated interval can be constructed in a way similar to that of item 7.

10. This item also can be proved in a way similar to that of item 7. As a counterexample showing that the indicated extension is proper we may take a j-algebra A from item 7 with additional restriction that $A^{\perp}$ is a Heyting l-algebra.

The corollary is proved.

Now we have enough information about j-algebras modelling Segerberg's axioms and we can come back to the analyses of Figure 2. We will denote by $\mathcal{N}eg$ the line passing trough the logics Ln and Lmn and by $\mathcal{I}nt$ the line passing through the logics Li and Lk. Recall that logics lying on the line $\mathcal{I}nt$ ($\mathcal{N}eg$) form an intersection of the class $\mathcal{D}$ of logics presented on Figure 2 with the class Int (respectively, with the class Neg), $\mathcal{D} \cap Int = \mathcal{I}nt$ and $\mathcal{D} \cap Neg = \mathcal{N}eg$. These lines play the role of coordinate axes for the three-dimensional part of Figure 2, which we denote by $\mathcal{P}ar$, $\mathcal{P}ar = \mathcal{D} \cap Par$. For any logic $L \in \mathcal{P}ar$ we can naturally define its projections $I(L)$ and $N(L)$ to the axes $\mathcal{I}nt$ and $\mathcal{N}eg$, respectively. For example,

$$I(Lj) = Li, N(Lj) = Ln, I(Ljl) = Lil, N(Ljl) = Lnl, I(Ljx) = Lk, N(Ljx) = Ln.$$

Using Proposition 4.1 and Corollary 4.2 we can easily check that for all logics $L \in \mathcal{P}ar$ the equalities

$$I(L) = L_{int} \text{ and } N(L) = L_{neg}$$

take place. Thus, for any line $\mathcal{L}$ on the diagram which is parallel to the line (Lj, Le'), the logics lying on this line have fixed intuitionistic and negative counterparts, say L_1 and L_2, respectively. And so we have $\mathcal{L} = \mathcal{D} \cap Spec(L_1, L_2)$.

We stated in this way that the three dimensions of the part $\mathcal{P}ar$ of Figure 2 exactly correspond to the three parameters determining a position of a logic in the class Par. One coordinate of a logic L is its intuitionistic counterpart $L_{int} \in Int$, second coordinate is its negative counterpart $L_{neg} \in Neg$, and the third coordinate corresponds to a position of L inside the interval

$Spec(L_{int}, L_{neg})$, which is determined in turn by the class of admissible semilattice homomorphisms from models of L_{neg} to models of L_{int}.

At this point we note one obvious defect of Figure 2. Let us consider the planes in the part $\mathcal{P}ar$ of the figure parallel to the plane with points Lj, Ljk, and Ljl_1^Q. There are three such planes. We denote by $\mathcal{P}j$ the plane containing the point Lj, by $\mathcal{P}l$ the plane containing the point Ljl, and, finally, by $\mathcal{P}e$ the plane containing the point Le. If we want follow to the geometrical analogues sketched above, we should expect that all logics belonging to one of the planes $\mathcal{P}j$, $\mathcal{P}l$, $\mathcal{P}e$ will define the same class of admissible homomorphisms. But this holds only for the plane $\mathcal{P}e$. For any logic $L \in \mathcal{P}e$ we have $\bot \vee (\bot \supset p) \in L$, and so $L = L_{int} \cap L_{neg}$ is the greatest point of the interval $Spec(L_{int}, L_{neg})$, which is determined by the class of homomorphisms identically equal to the unit element.

Let us consider the plane $\mathcal{P}j$. Elements of this plane are the least points in the sets of the form $\mathcal{P}ar \cap Spec(L_1, L_2)$, where $L_1 \in \{Li, Lik, Lil\}$ and $L_2 \in \{Ln, Lln, Lmn\}$. As we know from Proposition 2.4, the least point of an interval of the form $Spec(L_1, L_2)$ is a free combination $L_1 * L_2$ of logics L_1 and L_2. Moreover, according to Proposition 2.6 for free combinations all semilattice homomorphisms from models of negative counterpart to models of intuitionistic counterpart are admissible. However, only three points of $\mathcal{P}j$, namely, the logics Lj, Ljk, and Ljl^N are free combinations of theirs intuitionistic and negative counterparts (see items 1 and 5 of Corollary 4.2). Logics Ljl_2^Q and Lje_2^Q are proper extensions of free combinations $Li * Lnl$ and $Li * Lmn$, respectively, as it follows from items 7 and 9 of Corollary 4.2. What concerns the remaining four logics in $\mathcal{P}j$, we can easily modify the proofs of items 7 and 9 of Corollary 4.2 to show that the restrictions which axioms L_2^Q and E_2^Q impose on the class of admissible semilattice homomorphisms remain non-trivial even if the intuitionistic counterpart of a logic satisfies the axioms either K or L^N (see also Propositions 4.4 and 4.5 below).

In case of the plane $\mathcal{P}l$ we have a quite similar situation. Only the logics in the leftmost vertical line have the class of admissible semilattice homomorphisms with range contained in the set of regular elements of an upper algebra (see item 3 of Proposition 4.1). The other logics are characterized by narrower classes of admissible homomorphisms (see Propositions 4.4 and 4.5).

The indicated defect can easily be overcome if we replace the axioms L_2^Q and E_2^Q by L_1^Q and E_1^Q, respectively. As follows from items 7 and 9 of Proposition 4.1, these axioms do not impose any restrictions on the class of admissible homomorphisms and restrict only the class of lower algebras. So namely these axioms can be considered as an adequate relativization of the axioms L and E to the negative counterpart of a logic. After the above mentioned replacement and deleting the axiom L we obtain the diagram of logics having exactly the same configuration as that of Figure 2.

As we have seen above the axioms L_2^Q and E_2^Q impose restrictions on the classes of lower algebras of theirs models and simultaneously on the classes of admissible homomorphisms from the lower algebras of theirs models to the upper ones. We can separate these restrictions. As it follows from Proposition 4.1 axioms L_1^Q and E_1^Q restrict the classes of lower algebras in the same way as axioms L_2^Q and E_2^Q, respectively, and have no influence on the classes of admissible homomorphisms. On the other hand, the axioms

F_1. $\bot \vee (\bot \supset (p \supset q)) \vee (\bot \supset (q \supset p))(= \bot \vee L_2^Q)$,

F_2. $\bot \vee (\bot \supset p) \vee (\bot \supset (p \supset q))(= \bot \vee E_2^Q)$

as it follows from the next proposition will restrict the classes of admissible homomorphisms in the same way as it was done by axioms L_2^Q and E_2^Q, respectively, and will not change the classes of lower algebras.

Proposition 4.3: *Let A be an arbitrary j-algebra. The following equivalences hold.*

(1) $A \models Ljf_1$ *if and only if we have* $f_A(y_1 \supset y_2) \vee f_A(y_2 \supset y_1) = 1$ *for all* $y_1, y_2 \in A_\bot$.

(2) $A \models Ljf_2$ *if and only if we have* $f_A(y_1) \vee f_A(y_1 \supset y_2) = 1$ *for all* $y_1, y_2 \in A_\bot$.

This statement can be proved via a direct computation. It is clear that $Ljl_2^Q = Ljl_1^Q f_1$, $Lje_2^Q = Lje_1^Q f_2$, and $Ljl = Ljl'l^N l_1^Q f_1$.

Let us consider the class $\mathcal{D}_1$ consisting of logics which can be obtained by adjoining to Lj some subset of the following set of axioms

$$\{I, Q, K, X, L, E, L', E', P', L^N, L_1^Q, E_1^Q, F_1, F_2\}.$$

Obviously, $\mathcal{D} \subseteq \mathcal{D}_1$. At the same time, $\mathcal{D}$ satisfies the condition that for any $L_1 \in Int \cap \mathcal{D}$ and $L_2 \in Neg \cap \mathcal{D}$ the intersection $Spec(L_1, L_2) \cap \mathcal{D}$ is linearly ordered, in case of $\mathcal{D}_1$ this condition fails, as we can see from the propositions below.

Proposition 4.4: *Let* $L_1 \in \{Li, Lik, Lil\}$, $L_2 \in \{Ln, Lnl\}$, *and let* $L^* \rightleftharpoons L_1 * L_2$. *The set of logics* $Spec(L_1, L_2) \cap \mathcal{D}_1$ *forms an upper semilattice shown on the following semilattice diagram.*

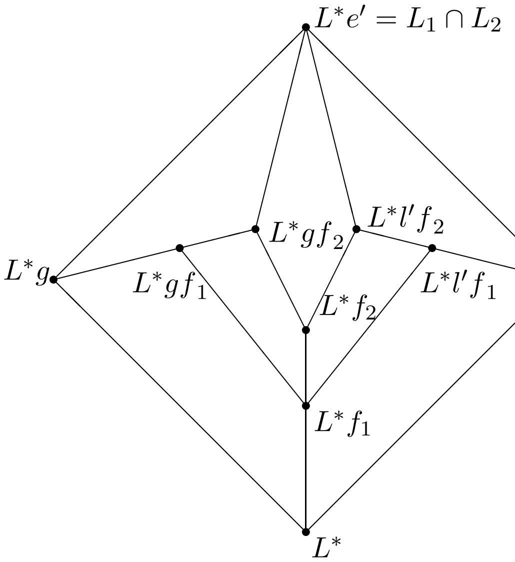


Figure 3.

In the course of proving this and subsequent propositions we will construct various j-algebras to check the interrelations between different logics. The following Heyting and negative algebras will play the role of breaks in our constructions: $\mathbf{2}$ and $\mathbf{2}'$ are two-element Heyting and negative algebras; 3^H and 3^N are three-element Heyting and negative algebras, the elements of which are linearly ordered; finally, 4^H and 4^N are four-element Heyting and negative algebras, respectively, whose implicative lattices are Peirce algebras.

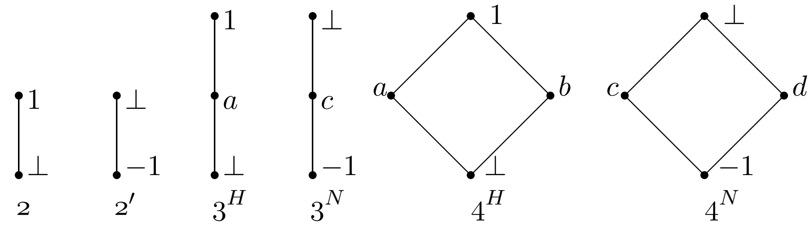


Figure 4.

For any Heyting algebra B, negative algebra C, and any constructed from them j-algebra $B \times_f C$, we will identify an element b of B (c of C) with the corresponding element $(b, \bot)$ of $(B \times_f C)^\bot$ ($(\bot, c)$ of $(B \times_f C)_\bot$).

Proof (of Proposition 4.4). First of all we note that due to our assumption $L_1 \neq Lk$. This fact together with items 3 and 4 of Corollary 4.2 implies that logics L^*g and L^*l' are different from the end-points of the interval

$Spec(L_1, L_2)$ and the least upper bound of these logics coincides with the greatest point of the interval, $L^*g \cup^* L^*l' = L^*e'$, which means, in particular, that L^*g and L^*l' are incomparable.

Let us consider the logics L^*f_1 and L^*f_2. Take an arbitrary model A for L^*f_2. Due to Proposition 4.3, we have $f_A(y_1) \vee f_A(y_1 \supset y_2) = 1$ for all $y_1, y_2 \in A_\perp$. Since $y_1 \leq y_2 \supset y_1$, we have $f_A(y_1) \leq f_A(y_2 \supset y_1)$, and also $f_A(y_2 \supset y_1) \vee f_A(y_1 \supset y_2) = 1$ for all $y_1, y_2 \in A_\perp$. In view of Proposition 4.3 the latter means that A is a model for L^*f_1, and we have in this way the inclusion $L^*f_1 \subseteq L^*f_2$.

Let us consider the j-algebra $A_1 \rightleftharpoons 3^H \times_{f_1} 3^N$, where $f_1 : 3^N \to 3^H$ is a uniquely defined implicative lattice isomorphism (see Figure 5, at which the structures of algebras constructed in this and the next proposition are presented). For any $y_1, y_2 \in A_{1_\perp}$ we have

$$f_1(y_1 \supset y_2) \vee f_1(y_2 \supset y_1) = (f_1(y_1) \supset f_1(y_2)) \vee (f_1(y_2) \supset f_1(y_1)) = 1$$

since $3^H \models (p \supset q) \vee (q \supset p)$. Thus, $A_1 \models L^*f_1$. Now we take the elements $-1, c \in 3^N$. It is clear that $f_1(-1) = \perp$ and that $f_1(c) = a$ (see Figure 4). We have

$$f_1(c) \vee f_1(c \supset -1) = f_1(c) \vee f_1(-1) = a \vee \perp = a \neq 1.$$

This means that A_1 is not a model for L^*f_2, and so the inclusion $L^*f_1 \subset L^*f_2$ is proper.

Consider the j-algebras $A_2 \rightleftharpoons 2 \times_{f_2} 4^N$, where $f_2(\perp) = 1$ and $f_2(x) = \perp$ for $x < \perp$, and $A_3 \rightleftharpoons 4^H \times_{f_3} 4^N$, where f_3 is an implicative lattice isomorphism between 4^N and 4^H. As in items 7 and 9 of Corollary 4.2, we can show that A_2 is a model of L^*, but is not a model of L^*f_1, respectively, that A_3 is a model of L^*f_2, but is not a model of L^*e'. We have thus proved that the logics L^*f_1 and L^*f_2 are different from the end-points of the interval $Spec(L_1, L_2)$.

Now we check that each of the logics L^*f_1 and L^*f_2 is incomparable with either of the logics L^*g or L^*l'. The j-algebras A_1 and A_3 are models of L^*f_1 and L^*f_2, respectively, but their are not models of L^*g, which implies that L^*g is not contained in either of the logics L^*f_1 or L^*f_2. Define a j-algebra A_4 as $3^H \times_{f_4} 4^N$, where $f_4(\perp) = 1$ and $f_4(x) = a$ for $x < \perp$. The A_4 is a model of L^*g, since the element a is dense in 3^H, but it is not a model of L^*f_1, in which case it is not also a model of L^*f_2. Indeed, for $c, d \in 4^N$ we have $f_4(c \supset d) \vee f_4(d \supset c) = f_4(d) \vee f_4(c) = a \vee a = a$. We have proved thus that the logics L^*f_1 and L^*f_2 are incomparable with L^*g.

The algebra A_2 provides a counterexample, demonstrating that either of the logics L^*f_1 or L^*f_2 is not contained in L^*l'. To state that the inverse inclusions also fail we consider the j-algebra $A_5 \rightleftharpoons 3^H \times_{f_5} 2'$, where

$f_5(-1) = a$. This is not a model of L^*l' since a is not regular, at the same time the direct calculation shows that $A_5 \models L^*f_2$. In this way L^*l' is not contained in L^*f_2, moreover, in L^*f_1.

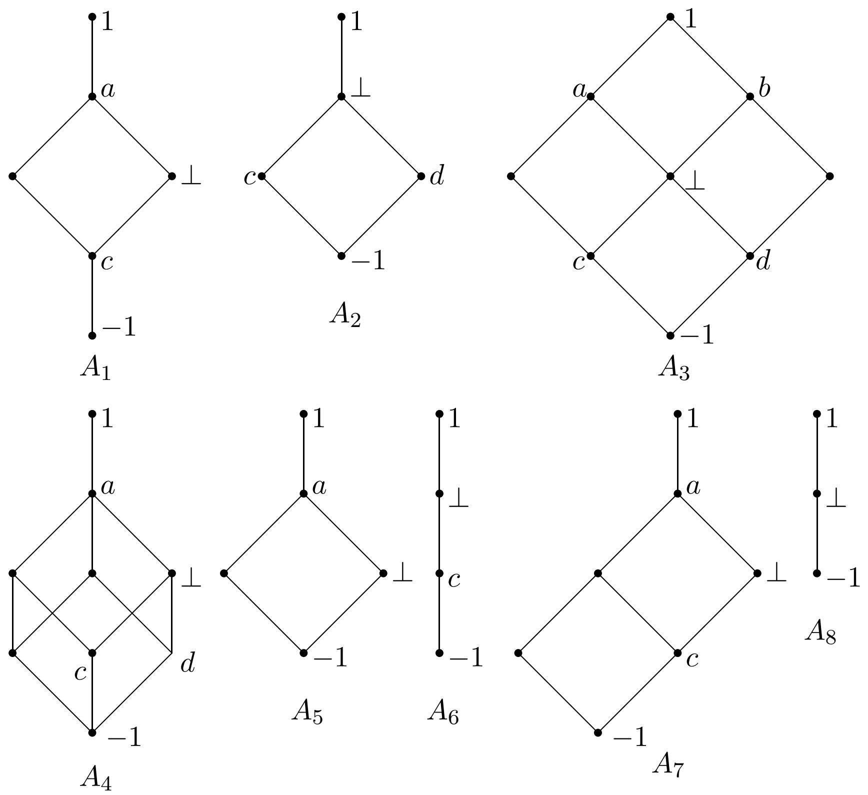


Figure 5.

The above facts about incomparability of logics imply, in particular, that L^*gf_i is a proper extension of L^*f_i and of L^*g, $i = 1, 2$, respectively, $L^*l'f_i$ is a proper extension of L^*f_i and of L^*l', $i = 1, 2$. So, it remains to verify that the inclusions $L^*gf_1 \subseteq L^*gf_2 \subseteq L^*e'$ and $L^*l'f_1 \subseteq L^*l'f_2 \subseteq L^*e'$ are proper.

The j-algebra $A_6 \rightleftharpoons 3^N \oplus 2$ ($\cong 2 \times_{f_6} 3^N$, where $f_6(x) = \bot$ for $x < \bot$) will show that the inclusion $L^*l'f_1 \subseteq L^*l'f_2$ is proper. It will be a model for L^*lf_1 since for any $y_1, y_2 \in 3^N$, either $y_1 \supset y_2 = \bot$ or $y_2 \supset y_1 = \bot$. On the other hand, $f_6(c) \vee f_6(c \supset -1) = \bot \vee \bot = \bot$. Note that the j-algebra A_3 will be, in fact, a model for $L^*l'f_2$ which is different from the direct product of 4^H and 4^N. This proves that L^*e' is a proper extension of $L^*l'f_2$.

Finally, consider the algebras $A_7 \rightleftharpoons 3^H \times_{f_7} 3^N$, where $f_7(x) = a$ for $x < \bot$, and A_5 defined above. The first of these algebras is a counterexample showing that the inclusion $L^*gf_1 \subseteq L^*gf_2$ is proper. The second algebra can be used to check that L^*e' is a proper extension of L^*gf_2.

The proposition is proved.

Proposition 4.5: *Let* $L_1 \in \{Li, Lik, Lil\}$, *and let* $L^* \rightleftharpoons L_1 * Lmn$. *The set of logics* $Spec(L_1, L_2) \cap \mathcal{D}_1$ *forms an upper semilattice shown on the following semilattice diagram.*

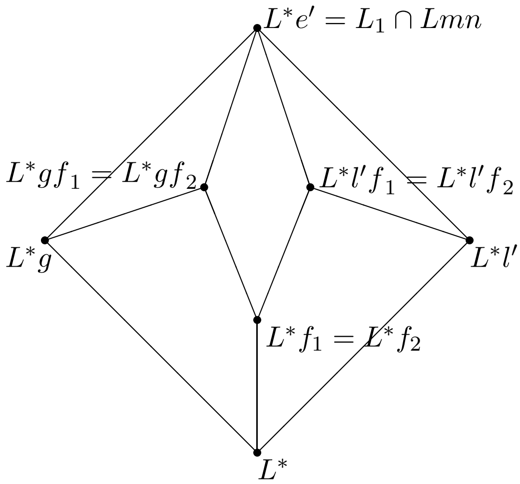


Figure 6.

Proof. As in the previous proposition we have the assumption $L_1 \neq Lk$, which implies that logics L^*g and L^*l' are different from the end points of the interval $Spec(L_1, L_2)$, are incomparable, and theirs upper bound coincides with the greatest point of the interval.

We argue to prove the equality $L^*f_1 = L^*f_2$. The inclusion $L^*f_1 \subseteq L^*f_2$ was stated above. Let us check the inverse inclusion. Take an arbitrary model A of L^*f_1, which means that $f_A(x \supset y) \vee f_A(y \supset x) = 1$ for all $x, y \in A_\bot$. According to our assumption $A_\bot$ satisfies the Peirce law, and so for any $x, y \in A_\bot$ we have $x = (x \supset y) \supset x$, on the other hand, in any j-algebra we have the identity $x \supset y = x \supset (x \supset y)$. In this way we have for any $x, y \in A_\bot$

$$f_A(x) \vee f_A(x \supset y) = f_A((x \supset y) \supset x) \vee f_A(x \supset (x \supset y)) = 1,$$

which proves the desired equality.

The lower algebras of j-algebras A_2, A_3, A_4, and A_5 defined in Proposition 4.4 are models of Lmn, and so these algebras can be used in the following reasoning. In particular, the j-algebras A_2 and A_3 can be used to check that the logic L^*f_1 lies inside the interval $Spec(L_1, Lmn)$.

With the help of the algebras A_4 and $A_8 \rightleftharpoons 2' \oplus 2$ we may show that the logics L^*f_1 and L^*g are incomparable. The algebra A_4 is a model of L^*g, but is not a model of L^*f_1. The algebra A_8 is, conversely, a model of L^*f_1, but not of L^*g.

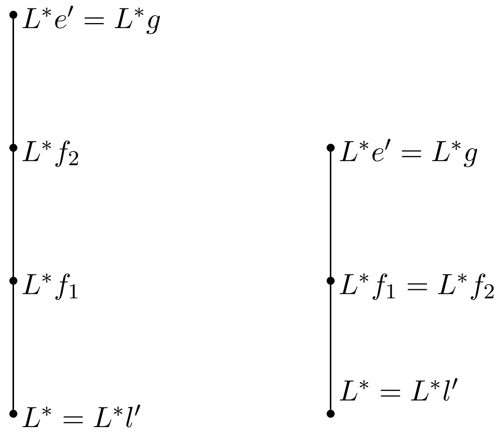
In a similar way, we can use the algebras A_2 and A_5 to check that the logics L^*f_1 and L^*l' are incomparable.

We are left to check that the following inclusions are proper: $L^*f_1g \subseteq L^*e'$ and $L^*f_1l' \subseteq L^*e'$. The suitable counterexamples are provided by algebras A_5 and A_3, respectively.

The proposition is proved.

We do not consider yet the case when intuitionistic counterpart coincides with the classical logic. It turns out that only in this case sets of the form $Spec(L_1, L_2) \cap \mathcal{D}_1$ are linearly ordered with respect to inclusion.

Proposition 4.6: *Let $L_2 \in \{Ln, Lnl, Lmn\}$, and let $L^* \rightleftharpoons Lk * L_2$. The sets of logics $Spec(Lk, L_2) \cap \mathcal{D}_1$ have the following structure.*



$L_2 \in \{Ln, Lnl\}$ $L_2 = Lmn$

Figure 7.

Proof. First, consider the case $L_2 \in \{Ln, Lnl\}$. The algebras A_6, A_8, and A_2 can be used to verify that the inclusions $L^* \subset L^* f_1$, $L^* f_1 \subset L^* f_2$, and, respectively, $L^* f_2 \subset L^* e'$ are proper.

In case $L_2 = Lmn$ we may use again the algebras A_8 and A_2, to check the corresponding relations between logics.

The proposition is proved.

Department of Logic,
Nicholas Copernicus University, Toruń
Department of Mathematical Logic,
Institute of Mathematics, Novosibirsk

REFERENCES

[1] H. B. Curry. *Foundations of mathematical logic.* McGraw-Hill Book Company, New York, 1963.
[2] R. I. Goldblatt. *Decidability of some extensions of J.* Z. für Math. Logic und Gründ. Math., 20, 1974, 203–206.
[3] S. P. Odintsov. *Maximal paraconsistent extension of Johansson logic.* Logique at Analyse, 161/162, 2000.
[4] S. P. Odintsov. *Logic of classical refutability and class of extensions of minimal logic.* Logic and Logical Philosophy, 7/8, 1999/2000.
[5] S. P. Odintsov. *Algebraic semantics and Kripke semantics for extensions of minimal logic.* Logicheskije issledovaninja, No. 2 (electronic journal, `http://www.logic.ru`), 1999 [in Russian].
[6] H. Rasiowa. *An algebraic approach to non-classical logics.* Amsterdam, North-Holland, 1974.
[7] W. Rautenberg. *Klassische und nichtclassische Aussagenlogik.* Braunschweig, Vieweg, 1979.
[8] K. Segerberg. *Propositional logics related to Heyting's and Johansson's.* Theoria, 34, 26–61(1968).
[9] P. W. Woodruff. *A note on JP'.* Theoria, 36, 1970, 183–184.