A COMPLETENESS PROPERTY OF NEGATIONLESS INTUITIONIST PROPOSITIONAL LOGICS

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The intuitionist propositional logic for implication alone may be expressed by the following formal system: w.f.f.s defined recursively from propositional variables and "$\rightarrow$"; sole rule of inference *modus ponens* and axiom schemata

$$A \rightarrow (B \rightarrow A)$$
$$\{A \rightarrow (B \rightarrow C)\} \rightarrow \{(A \rightarrow B) \rightarrow (A \rightarrow C)\}$$

Just those w.f.f.s of intuitionist propositional logic containing implication as sole connective are theorems of this system.

There are w.f.f.s involving material implication alone which, though tautologous, are not theorems of the system. (Peirce's schema $\{(A \rightarrow B) \rightarrow A\} \rightarrow A$ provides many such instances; its addition to the system as an axiom schema provides a complete system of logic for material implication.)

It follows that the system is neither absolutely complete (the addition of a schema not all of whose instances are provable as an additional axiom schema does not result in all w.f.f.s being provable) nor complete in the sense of Post (the addition of a schema not all of whose instances are provable does not result in any proposional variable being provable).

In this paper I show that the system does, however, have the following completeness property: *the addition of any schema, not all instances of which are tautologies, as an axiom schema results in a system which is neither absolutely consistent nor consistent in the sense of Post.*

Lemma: It A is a w.f.f. containing at most one propositional variable, $\vee$, then either $\vdash A$ or both $\vdash A \rightarrow \vee$ and $\vdash \vee \rightarrow A$. The proof by induction presents no difficulties and will be left to the reader. As a corollary we have

Corollary: If A is a w.f.f. containing at most one variable then eiher $\vdash A$ or $\vdash A \to \vee$.

Theorem: If A is a schema not all instances of which are tautologies then the addition of A as an axiom schema gives a system which is not consistent in the sense of Post.

Proof: Suppose A is a schema with such a non-tautological instance B. Given an assignment of truth-values to the variables of B which results in B having the truth-value F, replace each variable of B by $\vee \to \vee$ or $\vee$ depending on whether it has the truth-value T or F respectively in the assignment. The resulting w.f.f. C is a non-tautological instance of the schema A and contains at most one variable, $\vee$. By the Corollary just established either $\vdash C$ or $\vdash C \to \vee$. Since C is not a tautology and hence not provable (as is well known, if $\vdash C$ then C is a tautology) it follows that $\vdash C \to \vee$. So if C is added as an axiom (and *a fortiori* if A added as an axiom schema) the propositional variable, $\vee$, will be provable in the revised system. By induction from the fact that any propositional variable may be proved in such a system it can be established that any w.f.f. may be proved in the system.

These results, when appropriate modifications are made for the additional connectives, will also hold for any system for which the Corollary holds and so for any of the systems formed by adding to our original any combination of the following groups 1, 2 and 3:

1 $A \to (A \vee B)$
$B \to (A \vee B)$
$(A \to C) \to \{(B \to C) \to [(A \vee B) \to C]\}$

2 $(A \cdot B) \to A$
$(A \cdot B) \to B$
$(A \to B) \to \{(A \to C) \to [A \to (B \cdot C)]\}$

3 $(A \equiv B) \to (A \to B)$
$(A \equiv B) \to (B \to A)$
$(A \to B) \to \{(B \to A) \to (A \equiv B)\}$

For each such choice we obtain a system of logic which contains just those w.f.f.s of intuitionist logic having the relevant connectives as their only connectives and which possess the following completeness property: *the addition of any non-tautological schema as an axiom schema results in a system which is not absolutely consistent.*

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