FINITE MATRICES FOR QUASI-CLASSICAL MODAL LOGICS

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ABSTRACT

It is shown that every finite matrix which determines a quasi-classical modal logic is equivalent to one based on a finite Boolean algebra, in which the designated elements comprise a filter.

This note adresses a lacuna in recent work on tabularity in modal logic.

A *logic* is a set of propositional formulas, in the connectives $\perp$, $\rightarrow$, $\Box$, containing all classical tautologies and closed under modus ponens and substitution; it is *classical* if it is also closed under RE (from $A \leftrightarrow B$ infer $\Box A \leftrightarrow \Box B$). The smallest classical logic is called E, and a logic is *quasi-classical* if it contains E.

A *matrix* is a structure $\mathfrak{M} = (M, 0, \supset, *, D)$ such that $(M, 0, \supset, *)$ is an algebra and $D \subseteq M$; D is a *filter* if whenever $a \supset b \in D$ and $a \in D$ then $b \in D$; $\mathfrak{M}$ is *Boolean* if $(M, 0, \supset)$ is a Boolean algebra and D is a filter. A formula A is *valid* on $\mathfrak{M}$, $\mathfrak{M} \models A$, if $h(A) \in D$ for every homomorphism h from the algebra $\mathfrak{F}$ of formulas into $(M, 0, \supset, *)$. Two matrices are *equivalent* if the same formulas are valid on each. A matrix is *characteristic for*, or *determines*, a logic if the formulas in the logic are exactly the formulas valid on the matrix. A logic is *tabular* if it has a finite characteristic matrix.

Every Boolean matrix determines a quasi-classical logic, since $\{A \mid h(A) = 1 \text{ for all } h: \mathfrak{F} \rightarrow \mathfrak{M}\}$ is a classical logic. But if $\mathfrak{M}$ is not Boolean then $\{A \mid \mathfrak{M} \models A\}$ may be any set of formulas closed under substitution–let $(M, 0, \supset, *)$ be the algebra of formulas and let D be the desired set. Every quasi-classical logic has (and every matrix determining such a logic is equivalent to) a *countable* Boolean characteristic matrix–let $(M, 0, \supset, *)$ be the algebra of formulas, modulo equivalence provable in E, and let D be the filter of equivalence classes of formulas in the logic.

The problem is this. Recent discussions of tabularity (for example [1, 2, 6, 7]), in order to utilize methods derived from universal algebra, have limited themselves to (at most) the class of Boolean matrices, without considering the possibility that a logic having no finite Boolean characteristic matrix might yet be tabular in the sense defined above. I shall prove that that cannot happen:

Theorem. Any finite matrix which determines a quasi-classical logic is equivalent to a finite Boolean matrix.

But first, three comments:

1. The lacuna is recent. Dugundji [3] proved that S5 is not tabular, in the strict sense. Scroggs [8] proved that every proper extension of S5 *is* tabular– the matrices he constructed happened to be Boolean.

2. In the cases of [1, 6, 7] the lacuna is easily resolved. For those works are limited to classical (indeed, normal) logics. By [4, Lemma 3.2], for any finite matrix $\mathfrak{M}$ there is an equivalent finite matrix $\mathfrak{M}'$ such that every rule "weakly valid" on $\mathfrak{M}$ is "strongly valid" on $\mathfrak{M}'$. if $\mathfrak{M}$ determines a classical logic then modus ponens and RE are weakly valid on $\mathfrak{M}$, and hence strongly valid on $\mathfrak{M}'$, and this means that D′ is a filter in $\mathfrak{M}'$ and equivalence modulo D′ is a congruence relation. Reducing $\mathfrak{M}'$ modulo this congruence relation yields an equivalent Boolean matrix $\mathfrak{M}''$. (And if $\mathfrak{M}$ determines a normal logic then $\mathfrak{M}''$ is isomorphic to the power-set matrix based on a finite Kripke frame (W, R).)

3. The lacuna affects only tabularity, not the finite model property. Indeed the finite model property is vacuous absent some restriction on matrices–if L is any logic and $A \notin L$ then there is a finite matrix on which every formula in L is valid but A is not [5, Theorem 2.4]. (Originally, of course, the finite model property was intended as a property not of "logics" but of "systems" comprising axioms and rules.)

Proof of Theorem. Let $\mathfrak{M} = (M, 0, \supset, *, D)$ be a finite matrix, with $m = \text{card}(M)$, determining a quasi-classical logic. Let $\mathfrak{F}(m) = (F(m), \bot, \to, \Box)$ be the algebra of formulas all of whose variables are included among $p_1, p_2, \ldots, p_m$. Let $\sim$ be the smallest equivalence relation on F(m) satisfying

i. if $E \vdash A \leftrightarrow B$ then $A \sim B$,
ii. if $h(A) = h(B)$ for all $h: \mathfrak{F}(m) \to \mathfrak{M}$ then $A \sim B$.

Thus $A \sim B$ if and only if there is a finite chain $A = A_1 \sim A_2 \sim A_3 \sim \ldots \sim A_n = B$ such that $E \vdash A_1 \leftrightarrow A_2$, $h(A_2) = h(A_3)$ for all h, $E \vdash A_3 \leftrightarrow A_4$, etc.

Then $\sim$ is a congruence relation. For suppose $A \sim A'$ and $B \sim B'$, say $E \vdash A \leftrightarrow A'$ and $h(B) = h(B')$ for all h (the more complex cases follow easily); then $h(A \to B) = h(A) \supset h(B) = h(A) \supset h(B') = h(A \to B')$ for all h, and $E \vdash (A \to B') \leftrightarrow (A' \to B')$, so $(A \to B) \sim (A' \to B')$. Similarly, if $A \sim A'$ then $\Box A \sim \Box A'$. Now let $(M', 0, \supset, *)$ be $\mathfrak{F}(m)/\sim$; for $A \in F(m)$ let $\lfloor A \rfloor \in M'$ be the equivalence class of A; and let $D' = \{\lfloor A \rfloor \mid \mathfrak{M} \models A\}$. Then $(M', 0, \supset)$ is a Boolean algebra, since $\lfloor A \rfloor = \lfloor B \rfloor$ whenever $E \vdash A \leftrightarrow B$. Furthermore, if $A \sim A'$ and $\mathfrak{M} \models A$ then $\mathfrak{M} \models A'$ (use the definition of $\sim$); hence $\lfloor A \rfloor \in D' \leftrightarrow \mathfrak{M} \models A$, and it follows that D' is a filter and $\mathfrak{M}' = (M', 0, \supset, *, D')$ is a Boolean matrix.

And $\mathfrak{M}'$ is a finite: if $\lfloor A \rfloor \neq \lfloor B \rfloor$ then $h(A) \neq h(B)$ for some $h: \mathfrak{F}(m) \to \mathfrak{M}$; since there are just m^m such homomorphisms h, $\mathrm{card}(M') \leq m^{m^m}$. It remains to show that $\mathfrak{M}$ and $\mathfrak{M}'$ are equivalent.

Let $M = \{a_1, \ldots, a_m\}$, and suppose A is a formula, whose variables are $q_1, \ldots, q_n$, such that $\mathfrak{M} \not\models A$; say $h: \mathfrak{F} \to \mathfrak{M}$ and $h(A) \notin D$. Let $a_{i_j} = h(q_j)$ $(j = 1, \ldots, n)$ and let $A' = A\lfloor p_{i_1}, \ldots, p_{i_n}/q_1, \ldots, q_n \rfloor \in F(m)$. ($A\lfloor B_1, \ldots, B_n/q_1, \ldots, q_n \rfloor$ is the result of simultaneously replacing all occurrences of each q_j in A by B_j.) If $h'(p_i) = a_i$ $(i = 1, \ldots, m)$ then $h'(p_{i_j}) = a_{i_j} = h(q_j)$ $(j = 1, \ldots, n)$ and $h'(A') = h(A) \notin D$. Hence $\mathfrak{M} \not\models A'$, and $\lfloor A' \rfloor \notin D'$. Then $\mathfrak{M}' \not\models A'$, since if $k(p_i) = \lfloor p_i \rfloor$ $(i = 1, \ldots, m)$ then $k(A') = \lfloor A' \rfloor \notin D'$. Since A' is a substitution instance of A, it follows that $\mathfrak{M}' \not\models A$.

Conversely, suppose A is a formula in $q_1, \ldots, q_n$ and $\mathfrak{M}' \not\models A$, say $h: \mathfrak{F} \to \mathfrak{M}'$ and $h(A) \notin D'$. Let $\lfloor A_j' \rfloor = h(q_j)$ $(j = 1, \ldots, n)$, and let $A' = A\lfloor A_1, \ldots, A_n/q_1, \ldots, q_n \rfloor \in F(m)$. If $h'(p_i) = \lfloor p_i \rfloor$ $(i = 1, \ldots, m)$ then $\lfloor A' \rfloor = h'(A')$, and since $h'(A_j) = \lfloor A_j \rfloor = h(q_j)$ $(j = 1, \ldots, n)$, $h'(A') = h(A) \notin D'$. Since $\lfloor A' \rfloor \notin D'$, $\mathfrak{M} \not\models A'$, and since A' is a substitution instance of A, $\mathfrak{M} \not\models A$.

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