THE COMPLETENESS OF KW AND K1.1

M.J. CRESSWELL

In this paper I give fairly simple completeness proofs for two normal modal logics (1), KW and K1.1.

KW is K with the addition of

W $L(Lp \supset p) \supset Lp$

and K1.1 is K with the addition of

J1 $L(L(p \supset Lp) \supset p) \supset p$

Both these logics contain the theorem

4 $Lp \supset LLp$

and K1.1 contains, in addition, the theorem (2)

T $Lp \supset p$.

Thus KW is an extension of K4 and K1.1 is an extension of S4. These systems have been extensively discussed by Boolos, in [2], and others, who interpret *L* in KW to mean 'it is provable that' and in K1.1 to mean 'it is provable that and true that'. KW is characterized by strict finite partial orderings, and K1.1 by finite partial orderings. It is easy to see that **W** and **J1** are valid on these classes of frames respectively, but what completeness proofs exist are somewhat complicated. Segerberg in [7] has completeness proofs via filtrations,

(1) This paper assumes familiarity with the standard terminology of modal logic. In particular I follow [3], but see also [2] and [7].

(2) For the derivation of **4** in KW see [2, p. 20]. Boolos calls this system G after Gödel. Segerberg [7, p. 86] calls it K4W because he treats **4** as an axiom. For the derivation of **T** and **4** in K1.1 see [1]. Boolos and Segerberg call **J1**, **Grz** and K1.1, S4Grz. The names **J1** and K1.1 are due to Sobociński, see [6, p. 266].

and these proofs are reproduced by Boolos in Chapters 7 and 13 of [2]. Gabbay in [5, pp. 124-134] has proofs by the method of 'selective filtration'. The only proof I am aware of which does not use filtrations is one for KW that Boolos presents in Chapter 8 of [2], which is based on truth trees. The aim of the present paper is to use a technique developed in [4], involving a special sort of finite canonical model, to give simple completeness proofs for KW and K1.1, without appealing to filtrations. First take the case of KW.

Given a finite set P of propositional variables and a natural number m, we define the P/m/KW-model $\mathscr{M}$ as follows. Let Φ_m be the set of all wff made up from the variables in P and having modal degree no higher than m. Then $\mathscr{M}$ is the pair $\langle \mathscr{F}, V \rangle$ in which $\mathscr{F} = \langle W, R \rangle$ is defined as follows:

W is the class of all maximal KW-consistent (3) sets of wff from Φ^m. (By principles of K, W is obviously finite.)

For any x, $y \in W$, we define xRy iff

(i) For every $L\alpha \in x$, both $L\alpha$ and α are in y

and (ii) There is some β such that $L\beta \notin x$ and $L\beta \in y$.

Finally $x \in V(p)$, for any $p \in P$, iff $p \in x$.

THEOREM 1 Where $\mathscr{M}$ is the P/m/KW-model, then for any $\alpha \in \Phi_m$ and $x \in W$,

$$\mathscr{M} \models_x \alpha \Leftrightarrow \alpha \in x$$

PROOF:

The theorem obviously holds for the variables in P and is preserved by the truth functors. So consider some $L\alpha \in \Phi_m$.

(a) Suppose $L\alpha \in x$. Then for any y such that xRy, by condition (i) in the definition of R, we have $\alpha \in y$. So, by the induction hypothesis, $\mathscr{M} \models_y \alpha$. Since this is so for every y such that xRy, we have $\mathscr{M} \models_x L\alpha$.

(b) Suppose $L\alpha \notin$ x. We want to shew that there is some y such that $\alpha \notin y$ and xRy. Suppose there were no such y. This would mean that, where Γ is the set $\{\gamma : L\gamma \in x\} \cup \{L\gamma : L\gamma \in x\}$, then $\Gamma \cup \{\sim\alpha, L\alpha\}$ is

(3) This means that a subset x of Φ_m is in W iff (i) for any $\alpha \in \Phi_m$ either α or $\sim\alpha \in x$, and (ii) there is no finite subset $\{\alpha_1,\ldots,\alpha_n\}$ of x such that $\sim(\alpha_1 \wedge \ldots \wedge \alpha_n)$ is a theorem of KW.

inconsistent. (For if it were consistent then any maximal consistent extension y would satisfy conditions (i) and (ii) for $x\mathrm{R}y$.)

But if $\Gamma \cup \{\sim\alpha, L\alpha\}$ is inconsistent then there are some $L\beta_1,\ldots,L\beta_n$, $L\gamma_1,\ldots,L\gamma_k$ such that $\{\beta_1,\ldots,\beta_n,L\gamma_1,\ldots,L\gamma_k,\sim\alpha,L\alpha\}$ is inconsistent in KW.

So

$$\vdash_{\mathrm{KW}} (\beta_1 \wedge \ldots \wedge \beta_n \wedge L\gamma_1 \wedge \ldots \wedge L\gamma_k) \supset (L\alpha \supset \alpha)$$

so $$\vdash_{\mathrm{KW}} L(\beta_1 \wedge \ldots \wedge \beta_n \wedge L\gamma_1 \wedge \ldots \wedge L\gamma_k) \supset L(L\alpha \supset \alpha)$$

so, since KW contains **4**,

$$\vdash_{\mathrm{KW}} (L\beta_1 \wedge \ldots \wedge L\beta_n \wedge L\gamma_1 \wedge \ldots \wedge L\gamma_k) \supset L(L\alpha \supset \alpha)$$

so, by **W**,

$$\vdash_{\mathrm{KW}} (L\beta_1 \wedge \ldots \wedge L\beta_n \wedge L\gamma_1 \wedge \ldots \wedge L\gamma_k) \supset L\alpha$$

but $L\beta_1,\ldots,L\beta_n,L\gamma_1,\ldots,L\gamma_k$ are all in x. So $L\alpha \in x$, which contradicts the fact that $L\alpha \notin x$.

This proves the theorem.

We have now simply to shew that $\mathscr{F}$ is a strict partial ordering, since we know that W is finite. This means we must shew that R is transitive and irreflexive. Condition (ii) ensures that R is irreflexive, since it requires a β such that $L\beta \notin x$ and $L\beta \in y$.

For transitivity, it is easy to see that if $x\mathrm{R}y$ and $y\mathrm{R}z$, then if $L\alpha \in x$, $L\alpha \in y$ and so $L\alpha$ and $\alpha \in z$. So condition (i) holds. And if there is some $L\beta \in \Phi_{\mathrm{m}}$ such that $L\beta \notin x$ and $L\beta \in y$, then, by condition (i), $L\beta \in z$, and so condition (ii) holds.

THEOREM 2 KW is complete for strict finite partial orderings.

PROOF:

Let α be any non-theorem of KW. Then, where m is the modal degree of α and P the set of variables in α, by Theorem 1, α fails in some world in the P/m/KW-model $\mathscr{M}$. But the frame of $\mathscr{M}$ is a finite strict partial ordering.

For K1.1 the proof is analogous except that we use the P/m+1/K1.1 model when aiming to falsify a given non-theorem α of degree m. This means that W is the set of all maximal K1.1-consistent sets x of wff from Φ_{m+1}. As before V(p), for $p \varepsilon$ P, is defined so that $x \varepsilon$ V(p) iff $p \varepsilon x$. The main difference is the definition of R:

xRy iff either $x = y$ or $x\Sigma y$,

where $x\Sigma y$ iff

(i) For all $L\alpha \varepsilon \Phi_{m+1}$, if $L\alpha \varepsilon x$ then $L\alpha \varepsilon y$, and

(ii) There is some $L\beta \varepsilon \Phi_m$ such that $L\beta \not\varepsilon x$ and $\beta \varepsilon x$ and

$L(\beta \supset L\beta) \varepsilon y$.

THEOREM 3 If $\alpha \varepsilon \Phi_m$ then $\mathcal{M} \models_x \alpha$ iff $\alpha \varepsilon x$
(Note that the theorem is stated for $\alpha \varepsilon \Phi_m$, not Φ_{m+1}.)

PROOF:

The proof is by induction on the construction of α. The theorem obviously holds for the variables and is preserved by the truth functors. So consider $L\alpha \varepsilon \Phi_m$.

(a) Suppose $L\alpha \varepsilon x$. Then, for any y such that xRy, by condition (i) in the definition of R, $L\alpha \varepsilon y$. But K1.1 contains **T** and so $\alpha \varepsilon y$. So by induction hypothesis $\mathcal{M} \models_y \alpha$. Since this is so for every y such that xRy then $\mathcal{M} \models_x L\alpha$.

(b) Suppose $L\alpha \not\varepsilon x$. Then, if $\alpha \not\varepsilon x$, $\mathcal{M} \,=\!\!|_x\, \alpha$ and xRx, and so $\mathcal{M} \,=\!\!|_x\, L\alpha$. If $L\alpha \not\varepsilon x$ and $\alpha \varepsilon x$, then we shew that there is some y such that $x\Sigma y$ and $\alpha \not\varepsilon y$. Let $\Gamma = \{L\beta : L\beta \varepsilon x\}$. We shew that $\Gamma \cup \{\sim\alpha, L(\alpha \supset L\alpha)\}$ is K1.1-consistent; for if so any maximal consistent extension y will satisfy conditions (i) and (ii) for $x\Sigma y$, and will have $\alpha \not\varepsilon y$.

So suppose $\Gamma \cup \{\sim\alpha, L(\alpha \supset L\alpha)\}$ is not K1.1-consistent. Then, for some $L\beta_1,\ldots,L\beta_n \varepsilon x$ we have

$$\vdash_{K1.1} (L\beta_1 \wedge \ldots \wedge L\beta_n) \supset (L(\alpha \supset L\alpha) \supset \alpha)$$

So, since K1.1 contains S4,

$$\vdash_{K1.1} (L\beta_1 \wedge \ldots \wedge L\beta_n) \supset L(L(\alpha \supset L\alpha) \supset \alpha)$$

and so, by **J1** and S4

$$\vdash_{K1.1} (L\beta_1 \wedge \ldots \wedge L\beta_n) \supset L\alpha$$

But $L\beta_1,\ldots,L\beta_n \varepsilon x$ and so $L\alpha \varepsilon x$, which contradicts the assumption that it is not. So there is some y such that $x\Sigma y$ and $\alpha \notin y$. So $\mathscr{M} \not\models_y \alpha$ and xRy, so $\mathscr{M} \not\models_x L\alpha$.

This proves Theorem 3.

It is easy to see that $\mathscr{F}$ is a finite partial ordering. For W is finite and R can be seen to be reflexive, transitive and antisymmetrical: Reflexiveness obtains by definition; for transitivity, if either $x=y$ or $y=z$ the result is trivial, so suppose $x\Sigma y$ and $y\Sigma z$. (i) If $L\alpha \varepsilon x$ then $L\alpha \varepsilon y$ and so $L\alpha \varepsilon z$. (ii) If there is some $L\beta \varepsilon \Phi_{\mathrm{m}}$ such that $L\beta \notin x$ and $\beta \varepsilon x$ and $L(\mathrm{B} \supset L\beta) \varepsilon y$, then, by (i), $L(\beta \supset L\beta) \varepsilon z$. So $x\Sigma z$. For antisymmetry, if $x\Sigma y$, then there is some $L\beta \notin x$, $\beta \varepsilon x$ and $L(\beta \supset L\beta) \varepsilon y$; so if $y\Sigma x$ then $L(\beta \supset L\beta) \varepsilon x$ and so $\beta \supset L\beta \varepsilon x$, contradicting the consistency of x. So, if xRy and yRx, we must have $x=y$.

The completeness of K1.1 then follows as for KW.

Victoria University of Wellington M. J. CRESSWELL

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