TRUTH-FUNCTIONS

Peter M. SIMONS

It is sometimes said that truth-functions are functions from propositions to propositions, again that they are functions from truth-values to truth-values, yet again that they are functions forming compound sentences (or propositions) from simpler ones. Even the names given to the classical propositional calculus vary; one hears also of sentential calculus, while the German *Aussagenkalkül* means 'statement calculus', and it has even been proposed to call it a truth-value calculus. (1) This confusion calls for clarification.

The whole terminology of truth-functions presupposes treating propositions and truth-values as abstract objects, even if this is regarded as ultimately a *façon de parler*. The sense of 'proposition' in question is the modern one originating with Bolzano's *Satz an sich* and reinvented by Frege as his *Gedanke*; propositions are objects which are timelessly true or false, the archetypal truth-bearers. This use of 'proposition' differs from that found in medieval and ancient authors, as has frequently been pointed out. (2) The truth or falsehood of a (modern) proposition is conceived as independent of which expressions are used to express it, who uses them, and under what circumstances. Propositions are thus different from both sentences (the abstract objects investigated by linguists) and from statements (concrete speech acts). But a sentence used in an assertion (making a statement) very often, perhaps with other factors, determines a proposition, and thereby a truth-value. It is not our intention here to settle the question of priority, i.e. whether a sentence or statement gets a truth-value because it expresses a proposition, or whether propositions arise by abstraction from sentences or statements which are themselves primarily to be reckoned true or false. As long as the existence of propositions and truth-values is accepted, that is enough for my present purposes.

One reason for the confusion between propositions, sentences and statements is that for truth-functional logic the differences are of little moment. If we consider 'eternal' sentences, which are true or false whoever utters them, in whatever (normal) conditions, etc., then such sentences determine a proposition alone, and we are entitled to speak of *the* proposition expressed by the sentence. Given such sentences S, a special functor 'that...' forms a name of this proposition. (3) From propositions with names like 'p', we may gain a sentence which expresses a true proposition just in case p is true by predicating a special predicate of it, namely '... is the case'. This syntactic switching from sentences to names of propositions and back is governed by two rules:

R1 for all $S : S \leftrightarrow$ (that S) is the case
R2 for all $p : p =$ that (p is the case)

Such switching was not unknown to the young Frege, before he arrived at the idea that sentences were a kind of name: in his *Begriffsschrift* (though not later) he treated the content stroke as we have treated 'that' and the judgement stroke as we have treated 'is the case'. (4) The similarity of R1 and R2, marked only by the difference between the order of bracketing and the difference between the identity predicate and a strong biconditional connective, is another reason why the difference between sentence and proposition is overlooked, and especially so when sentences are taken as names of truth-values, and identity replaces equivalence.

Connectives like 'and', 'or', 'if... then---' are *expressions* forming compound sentences. Grammatically they are functors of category s/ss. Such functorial expressions must be distinguished from the sentence-*pattern* in which they typically occur, of the form S and S', the variables here being place-holder dummies showing where sentences are inserted to make the pattern with the aid of the word 'and'. Functor expressions are sometimes called 'incomplete' or (following Frege) 'unsaturated', but this latter expression is better reserved for the sentence pattern the connective serves to form, since the pattern, but not the word, is incapable of isolated or separate existence. (5)

The expression of sentential negation in English is the logicians' 'it is not the case that...'. This has the desired category s/s, but is itself a complex expression involving the name-forming functor 'that', the special predicate 'is the case' and the usual English negation particle 'not', which modifies verbs and not sentences.

We can use connectives and our special syntactic adaptors to form expressions similar to connectives, but allowing proposition names as inputs and/or outputs. The complex expression 'that (... and ---)' has category *n/ss*, while expressions having gaps for proposition names are '... is the case and --- is the case', of category *s/nn*, i.e. a binary or relational predicate of propositions, and the expression 'that (... is the case and --- is the case)' forms a name of a proposition from the names of two propositions, and has category *n/nn*. It is this last nominal functor which may be said to express conjunction as a truth-function of propositions. If p,q are propositions, we express it here by $\wedge(p,q)$. The result of the functor is a name of a proposition, the *conjunction* of p and q. The functor (which, being the propositional equivalent of a connective, we may call a *junctor*), expresses an operation, a function from propositions to propositions, which we may call a *junction*. Conjunction is thus one particular binary junction. Both junctions and junctors are to be contrasted with connectives.

For propositions we may always ask 'Is it true or false?', or 'What is its truth-value?' Wh-questions are given answers which employ concepts I call *classifiers*. Among classifiers are the concepts of height, age, direction, number. (6) The answer to a wh-question gives a particular *value* of the general property meant by a classifier, e.g. 2m tall, 32 years old, north, six etc. Now for propositions the classifier which interests us is the concept expressed by 'the truth-value of---', which is abbreviated 'TV (---)'. Classifiers may be considered as function-like, in that they take arguments and yield values. The values are hypostatised properties. In the case of the TV-classifier they are the abstract truth-values T and F. When it is said of a particular junction that (in constrast to others) it is a *truth-function* then what is meant is that the truth-value of its output depends only or, better, at most, on the truth-values of its inputs, and not on other factors. This is an instance of a general kind of relation holding among certain classfiers which I call *regularity*. (7) For the special case of conjunction we have:

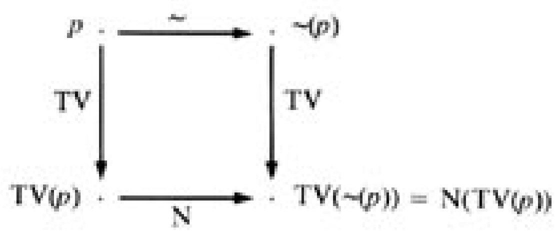
$$\forall pqrs.\mathrm{TV}(p) = \mathrm{TV}(q) \wedge \mathrm{TV}(r) = \mathrm{TV}(s) \supset \mathrm{TV}(\wedge(p,r)) = \mathrm{TV}(\wedge(q,s)).$$

Since truth-functions which are not dependent on (whose output truth-values are not covariant with the truth-values of) their inputs, such as the monadic junctions *Tautology* and *Contradiction*, are also counted as truth-functions, in the wide sense familiar from mathematics, we cannot always happily speak of *dependence*. However to the extent that functionality is captured by the notion of regularity, the terminology of truth-functions is justified. If we represent the compounding of classifiers (such as 'the truth-value of the conjunction of... and---') symbolically by '$\circ$', the same sign as used to represent functional composition, then we can say that the regularity represented above is that of $\mathrm{TV} \circ \wedge$ (itself binary) on TV. In general, for any junction $*$, it is a truth-function iff $\mathrm{TV} \circ *$ reg TV.

Because however we can indicate the effect of truth-functions with truth-tables or matrices, it is tempting to take them as functions from truth-values to truth-values. But, while there are indeed such simple functions, they are not the same as the truth-functions we have described, which are junctions, i.e. functions from propositions to propositions. The connection may be made clear again using the example of conjunction. Let K be the binary function on truth-values to truth-values given by $K(T,T) = T$, $K(F,T) = F$, $K(T,F) = F$, $K(F,F) = F$. Then

$$\forall pq.\mathrm{TV}(\wedge(p,q)) = K(\mathrm{TV}(p),\mathrm{TV}(q))$$

The two functions K and $\wedge$ are *homomorphically* related under the function TV: K is the homomorphic transform of $\wedge$ in the domain of values of TV, namely truth-values. This close relationship may explain the tendency to take truth-functions now as one, now as the other, but does not excuse it. For the monadic truth-function of negation, with junctor '$\sim$', and its homomorph N, the inversion function between the truth-values, we have precisely a commutative composition diagram



That we are entitled to take both truth-functions and their homomorphs as functions becomes clear when we consider which single-valued relations we can define as equivalent to them. If $\&pqv := \wedge(p,q) = v$, then clearly & is single-valued, and the same goes for a triadic relation defined similarly in terms of K. On the other hand it is possible also to consider the truth-functions to correspond to certain *relations* among propositions, or again, among truth-values. The latter alternative has been suggested by Dr Steen Olaf Welding,(8) though provided we keep clear as to the general priority of relations over functions there seems to be no reason to accept Dr Welding's criticisms of the functional view of truth-functions, which can, as we have demonstrated, be defended provided it is cleared of confusion. In the case of truth-functions, it so happens that we have arrived at the functional rather than the relational view first, and have perhaps found it more natural to proceed that way. But the other way is equally acceptable: for instance we can define a relation between propositions k such that p k q iff both p and q are true. It also has a homomorphic image under TV in truth-values, namely that relation which holds only from T to T and not otherwise. Since the Fregean concepts corresponding to these relations are just the functions $\wedge$ and K, Dr Welding's suggestion may be seen as a way of doing propositional logic without functions at all, reversing the Fregean procedure. As such it is legitimate, but neither the only possible viewpoint, nor the most natural or easiest to use. As elsewhere in mathematics, the use of functions makes things run smoothly, although it is theoretically dispensable.

University of Salzburg Peter M. SIMONS

NOTES

(1) ANDERSON, A.R., «An intensional interpretation of truth-values», *Mind* 81.

(2) Cf. e.g. A.N. PRIOR, *The Doctrine of Propositions and Terms*, London: Duckworth, 1976, Ch.1.

(3) Cf. ANDERSON, A.R. & BELNAP, N.D., *Entailment*, Vol. 1, Princeton: Princeton U.P., 1975, Appendix, where 'that' is called a 'subnector'.

(4) FREGE, G., *Begriffsschrift und andere Aufsätze*, ed. I.Angelelli, Hildesheim: Olms, 1964. Cf. §§2-3.

(5) Cf. my «Unsaturatedness», *Grazer Philosophische Studien 14* (1981), 73-96.

(6) The term '*Klassifikator*' comes from Grelling, K. & Oppenheim, P., «Der Gestaltbegriff im Lichte der neuen Logik», *Erkenntnis 7* (1938), 211-25. The explanation in terms of wh-questions is not theirs; they prefer to see classifiers straightforwardly as functions. Since this however commits us straight away to a large ontology of abstract objects, I prefer to leave open whether we can reduce such apparently abstractum-demanding questions like «What is the height of this?» to the simpler «How tall is this?» The use of the function-like notions of argument and value enables a unified treatment of classifiers (in terms of equality and difference of their values) to be given, but this tells us nothing about the ontological priorities.

(7) The monadic classifier f is regular with respect to the monadic classifier g iff, for a domain of values D (here implicit)

$$\forall xy(g(x) = g(y) \supset f(x) = f(y)). \text{ (We write '}f \text{ reg } g\text{'.)}$$

Since this is true whenever f is constant, or whenever g always takes different values, it does not in our view suffice to define the notion of functional *dependence*. In Grelling, K., «A logical theory of dependence», *Erkenntnis 9* (1939; unpublished because of war conditions), what we call regularity is termed 'equidependence'.

(8) WELDING, S.O., «Logic as based on truth-value relations», *Revue Internationale de Philosophie 30* (1976), 151-166.