A CANONICAL MODEL FOR S2

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It is of course far too late in the day to prove the completeness of S2. It is even too late (see [3]) to give a Henkin completeness proof for S2. Nevertheless there is something which seems not yet to have been done, and that is to define a canonical model for S2 using maximal consistency *in S2* as the criterion for worldhood. What happens in S2 is this [2, pp. 274-278]: the worlds divide into normal worlds and non-normal worlds. Each normal world is reflexive, and the definition of truth is standard except that for L we have that $L\alpha$ is true in a world iff that world is normal and α is true in all worlds accessible to it. Validity in S2 is defined as truth in every normal world in every S2 model. If we define validity as truth in *every* world (whether normal or not) we get a definition of validity for Lemmon's E2 (see [2, p. 302f]). Now what usually happens in giving a completeness proof for S2 (see, e.g. [3, p. 251]) is that we take the worlds to be all sets which are maximal consistent *in E2* and let the normal ones be those which contain $L(p \supset p)$.

The purpose of this note is to define a canonical model for S2 directly, without making reference to E2 or to any other such system. The normal worlds in this canonical model are the maximal S2-consistent sets of wff. The non-normal worlds have the property that $L\alpha$ is false for every α, and this means that the only notion of consistency required is PC-consistency; so the non-normal worlds are all maximal PC-consistent sets of wff containing $\sim L\alpha$ for every wff α.

1. Some facts about S2

We shall assume the facts about S2 which are listed in [2] on pp. 230-233 and 248-252, together with their obvious consequences. In this section we list as a lemma facts we shall use in later sections.

Lemma 1

1.1 $\vdash_{S2} \alpha \rightarrow \vdash_{S2} L(p \supset p) \dashv \alpha$

To prove 1.1 we first note that if α has the form $L\beta$ then we have $\vdash_{S2} (p \supset p) ⥽ \beta$, by TS2.3 on p. 232 of [2], and so by BR (*loc. cit.*) we have $\vdash_{S2} L(p \supset p) ⥽ L\beta$. So 1.1 holds for any wff of the form $L\beta$.

We prove 1.1 by induction on the proof of α in S2. It is convenient to use Lemmon's basis, set out in [2, p. 248f]. By RP1.1 (p. 247), the axioms of this basis all have the form of a wff β such that $L\beta$ is also a theorem. So 1.1 holds both for $L\beta$ and therefore also for β. Further, 1.1 holds for anything obtained by BR. Also it is easy to see that modus ponens preserves the lemma.

1.2 $\vdash_{S2} \alpha \supset \beta \rightarrow \vdash_{S2} \alpha ⥽ (L(p \supset p) \supset \beta)$

Proof:

From 1.1 we have $\vdash_{S2} \alpha \supset \beta \rightarrow \vdash L(L(p \supset p) \supset (\alpha \supset \beta))$ which, by PC, gives $\vdash_{S2} L(\alpha \supset (L(p \supset p) \supset \beta))$ as required.

1.3 $\vdash_{S2} L(p_1 \wedge \ldots \wedge p_n)\ _6\ (Lp_1 \wedge \ldots \wedge Lp_n)$
1.4 $\vdash_{S2} Lq ⥽ L(p \supset p)$
1.5 $\vdash_{S2} ((L(p \supset p) \vee q) \wedge (L(p \supset p) \supset q))\ _6\ q$

1.3 - 1.5 are easily proved theorems of S2.

2. Validity in S2

An S2 frame $\mathscr{F}$ is a triple $\langle W, R, N \rangle$ in which $N \subseteq W$ and $R \subseteq W^2$ is reflexive over N. An S2 model $\mathscr{M}$ is a pair $\langle \mathscr{F}, V \rangle$ in which $\mathscr{F}$ is an S2 frame and V is a function from propositional variables such that $V(p) \subseteq W$. Truth in a model is defined in the standard way (see [1, p. 66f]) except that, for $x \in W$,

$$\mathscr{M} \models_x L\alpha \text{ iff } x \in N \text{ and for every } y \text{ such that } xRy,\ \mathscr{M} \models_x \alpha.$$

α is said to be *valid* in a model $\mathscr{M}$ iff, for every $x \in N$, $\models_x \alpha$. α is valid on a frame $\mathscr{F}$ iff α is valid in every model based on $\mathscr{F}$ (i.e. every model $\mathscr{M}$ such that for some V, $\mathscr{M} = \langle \mathscr{F}, V \rangle$).

We want to show that S2 is characterized by S2 frames. It is not hard to check that S2 is sound with respect to this class. We shall prove it to be complete.

3. The canonical model of S2

A set Λ of wff (of the language of propositional modal logic) is S2-consistent iff for no finite subset $\{\alpha_1, \ldots, \alpha_n\}$ of Λ do we have $\vdash_{S2} \sim(\alpha_1 \wedge \ldots \wedge \alpha_n)$.

A set Λ of wff is maximal iff, for every wff α, either $\alpha \in \Lambda$ or $\sim\alpha \in \Lambda$.

A set Λ is PC-consistent iff, for no finite subset $\{\alpha_1, \ldots, \alpha_n\}$ of Λ, is $\sim(\alpha_1 \wedge \ldots \wedge \alpha_n)$ a PC tautology.

We now define a set Q as follows. Q is the set of all maximal PC-consistent sets of wff which contain no wff of the form $L\alpha$. That is to say Q is the class of all sets x of wff, which have the property

(i) If α is any wff then $\sim L\alpha \in x$.
(ii) For any wff α, either $\alpha \in x$ or $\sim\alpha \in x$.
(iii) x is PC-consistent.

Lemma 2: For any set Λ of wff, if there is no $x \in$ Q such that $\Lambda \subseteq x$, then there are $\beta_1, \ldots, \beta_m$ in Λ, and wff $\gamma_1, \ldots, \gamma_m$ such that

$$(\beta_1 \wedge \ldots \wedge \beta_n) \supset (L\gamma_1 \vee \ldots \vee L\gamma_m)$$

is a PC-tautology.

Proof:

Suppose $(\beta_1 \wedge \ldots \wedge \beta_n) \supset (L\gamma_1 \vee \ldots \vee L\gamma_m)$ were not a PC-tautology for any $\beta_1, \ldots, \beta_n$ in Λ and any $\gamma_1, \ldots, \gamma_m$. Then $\{\beta_1, \ldots, \beta_n, \sim L\gamma_1, \ldots, \sim L\gamma_m\}$ is PC-consistent for every $\beta_1, \ldots, \beta_n$ in Λ. Which is to say that $\Lambda \cup \{\sim L\gamma : \gamma$ is any wff$\}$ is PC-consistent. So it is contained in a maximal PC-consistent set which satisfies (i). But such a set also satisfies (ii) and (iii), and so is in Q.

We are now ready to define the canonical model of S2. Let N be the set of all maximal S2-consistent sets of wff and let W in the canonical model be N $\cup$ Q and define xRy iff $x \in$ N and, for every wff α, if $L\alpha \in x$ then $\alpha \in y$. Since $L\alpha \supset \alpha$ is a theorem of S2, then R will be reflexive over N. V in the canonical model is defined in the usual way; *viz.*, for any propositional variable, p, and $x \in$ W: $x \in$ V(p) iff $p \in x$.

Obviously $\mathcal{M}$ is an S2 model. We now suppose that $\models$ is an evaluation of the kind described in section two.

Theorem: For any wff α, where $\mathscr{M}$ is the canonical model of S2 and $x \in W$, then

$$\models_x \alpha \text{ iff } \alpha \in x$$

The proof is by induction on α. The only case which is not standard is where α has the form $L\beta$.

Suppose $L\beta \in x$, then $x \in N$, so consider any y such that xRy. By definition of R, $\beta \in y$; so, by the induction hypothesis, $\models_y \beta$. Since this is so for all such y we have $\models_x L\beta$.

Suppose $L\beta \notin x$. Then we have two cases to consider, either $x \in N$ or $x \in Q$. Suppose $x \in Q$. Then $=\!|_x L\beta$. Suppose $x \in N$; then we have to show that there is some $y \in W$, such that xRy and $\sim\beta \in y$. This will be the case if $\{\alpha : L\alpha \in x\} \cup \{\sim\beta\}$ is either

(i) S2-consistent

or (ii) has an extension in Q.

Suppose that neither (i) nor (ii) hold. This means that

(a) There are wff $\gamma_1, \ldots, \gamma_m$ such that $L\gamma_1, \ldots, L\gamma_m \in x$

and

$$\vdash_{S2} (\gamma_1 \wedge \ldots \wedge \gamma_n) \supset \beta$$

and

(b) by lemma 2, there are wff $\delta_1, \ldots, \delta_m$ such that $L\delta_1, \ldots, L\delta_m$ are in x, and wff $\eta_1, \ldots, \eta_k$ such that

$$(\sim\beta \wedge \delta_1 \wedge \ldots \wedge \delta_m) \supset (L\eta_1 \vee \ldots \vee L\eta_k)$$

is a PC-tautology.

From (b) we have, by PC and RP1 [2, p. 247],

$$\vdash_{S2} (\delta_1 \wedge \ldots \wedge \delta_m) \prec (L\eta_1 \vee \ldots \vee L\eta_k \vee \beta)$$

So, by using lemma 1.4, we have

$$\vdash_{S2} (\delta_1 \wedge \ldots \wedge \delta_m) \prec (L(p \supset p) \vee \beta)$$

Now from (a) we have, by lemma 1.2,

$$\vdash_{S2} (\gamma_1 \wedge \ldots \wedge \gamma_n) \prec (L(p \supset p) \supset \beta)$$

So, using lemma 1.5,

$$\vdash_{S2} (\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \delta_1 \wedge \ldots \wedge \delta_m) \prec \beta$$

So by Becker's rule [2, p. 232]

$$\vdash_{S2} L(\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \delta_1 \wedge \ldots \wedge \delta_m) \mathbin{-\!3} L\beta$$

so by lemma 1.3

$$\vdash_{S2} (L\gamma_1 \wedge \ldots \wedge L\gamma_n \wedge \mathrm{L}\delta_1 \wedge \ldots \wedge L\delta_m) \mathbin{-\!3} L\beta$$

But $L\gamma_1,\ldots, L\gamma_n, L\delta_1,\ldots, L\delta_m$ are all in x while $L\beta$ is not. So x is inconsistent contrary to hypothesis.

This completes the induction and so proves the theorem.

Corollary: S2 is complete for S2-frames.

Proof:

Suppose $\not\vdash_{S2} \alpha$, then there is some $x \in \mathrm{N}$ such that $\alpha \notin x$, so ${}_{6}|_{x}\ \alpha$. But $x \in \mathrm{N}$ and so α is not valid in this model. So if α is not an S2 theorem then it is not S2-valid. So if α is S2-valid then $\vdash_{S2} \alpha$.

This method can be extended to other Lewis systems and, as an example, we can take the case of S3. We have to shew that R is transitive. Suppose $x\mathrm{R}y$ and $y\mathrm{R}z$. Then with R defined as for S2, we must have $x \in \mathrm{N}$ and $y \in \mathrm{N}$. So suppose $L\alpha \in x$. We must prove that $\alpha \in z$. Now $\vdash_{S2} \alpha \mathbin{-\!3} ((p \supset p) \supset \alpha)$ and so, by BR, $\vdash_{S2} L\alpha \mathbin{-\!3} \mathrm{L}((p \supset p) \supset \alpha)$, so $\vdash_{S3} L\alpha \mathbin{-\!3} L(L(p \supset p) \supset L\alpha)$ so $L(L(p \supset p) \supset L\alpha) \in x$ and so $(L(\mathrm{p} \supset \mathrm{p}) \supset L\alpha \in y$. But $y \in \mathrm{N}$ and so $L(p \supset p) \in y$, and so $L\alpha \in y$, and so $\alpha \in z$.

Henkin proofs for many other non-normal modal systems will be found in [3]. In many cases proofs in the style of the present paper will also be available.

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